The *complex* relationship of mathematics and physics

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Not so uncommon views of the math/physics relationship

In math classes: physics as an **application** of previously/abstractly defined concepts and/or as an **introduction scenario**.

In physics classes: mathematics as a **language** to express physical quantities and a **tool** to calculate.

The relationship is much more complex!

Not very useful to think generally about the math/physics relationship

3 cases illustrate a complex interplay

- Case 1: Circular polarization and complex numbers
- Case 2: Electromagnetism and quaternions
- Case 3: Hydrodynamics and complex analysis

1. Fresnel equations become complex

MÉMOIRE

SUR LA LOI DES MODIFICATIONS

QUE LA RÉFLEXION IMPRIME A LA LUMIÈRE POLARISÉE (*),

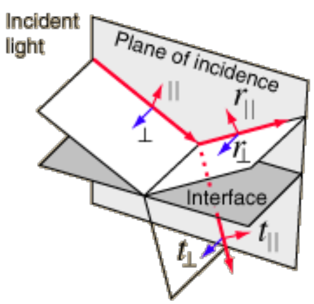
LU À L'ACADÉMIE DES SCIENCES, LE 7 JANVIER 1823.

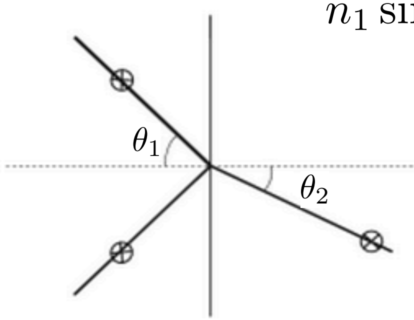
$$r_\perp = \frac{\sin(\theta_2-\theta_1)}{\sin(\theta_1+\theta_2)}$$

$$r_\perp = \frac{n\cos(\theta_1) - \sqrt{1-n^2\sin^2(\theta_1)}}{n\cos(\theta_1) + \sqrt{1-n^2\sin^2(\theta_1)}}$$

$$n_1\sin(\theta_1) = n_2\sin(\theta_2)$$

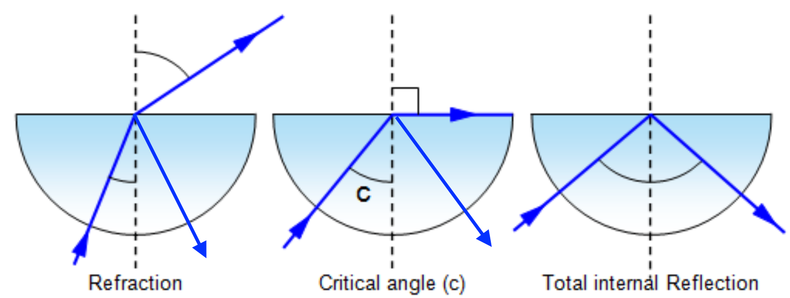
$$\frac{n_1}{n_2} = n$$





When $n^2\sin^2(\theta_1)$
$$\begin{cases} < 1 & |r_\perp| < 1 \\ = 1 & r_\perp = 1 \\ > 1 & r_\perp \text{ is not real!} \end{cases}$$

Due to the general law of continuity, if there is an accurate expression for the laws of reflection just before the limit, it should remain valid afterwards; the challenge is to interpret/guess what analysis says about these imaginary expressions.

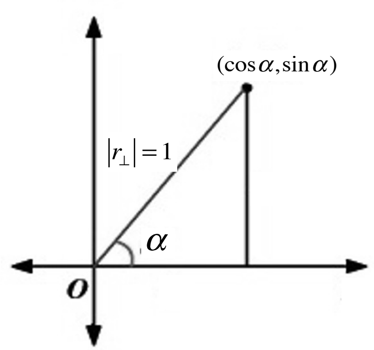


1. Fresnel equations become complex

$$\boxed{r_\perp = \frac{n\cos(\theta_1) - \sqrt{1-n^2\sin^2(\theta_1)}}{n\cos(\theta_1) + \sqrt{1-n^2\sin^2(\theta_1)}}}$$

$$\sqrt{1-n^2\sin^2(\theta_1)} = \sqrt{n^2\sin^2(\theta_1)-1}\cdot\sqrt{-1}$$

$$r_\perp = |r_\perp|[\cos\alpha + i\sin\alpha]$$

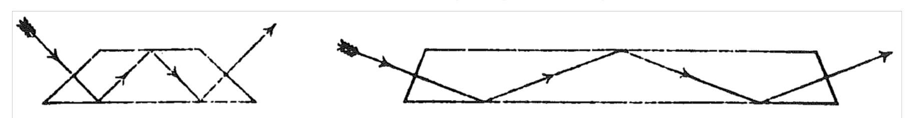


Interpretation:
Phase shift!

It means without a doubt that the periods of vibrations[15] *of the reflected waves, which in the basis of the calculations were supposed coincident at the surface with the ones from the incident waves, no longer coincide […] these periods are retarded or advanced by a certain quantity.*

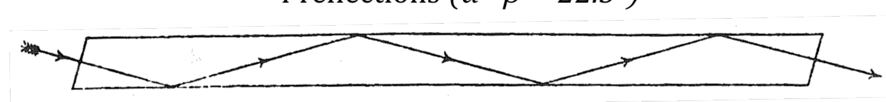
Prisms create circularly polarized light

3 reflections $(\alpha - \beta = 30°)$



$\theta_i = 43°10'$ $\theta_i = 69°12'$

4 reflections $(\alpha - \beta = 22.5°)$



$\theta_i = 74°42'$

Lessons from Case 1

- New physics can be **discovered/predicted** from mathematical considerations (e.g. particle physics). Wigner's "unreasonable effectiveness"

- *One can say that this was the first time in which "nature" was abstracted from "pure" mathematics, that is from a mathematics which had not been previously abstracted from nature itself (Bochner, 1963).*

- Circularly polarized light did not exist in nature*, it was "human-made" thanks to the formalism of complex numbers.

2. Quaternions in Maxwell's Treatise

$$\left.\begin{aligned} 4\pi\mu u &= \frac{dJ}{dx} + \nabla^2 F, \\ 4\pi\mu v &= \frac{dJ}{dy} + \nabla^2 G, \\ 4\pi\mu w &= \frac{dJ}{dz} + \nabla^2 H. \end{aligned}\right\}$$

A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into Scalar quantities, which are completely represented by one numerical quantity, and Vectors, which require three numerical quantities to define them.

The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple co-ordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.

$$\left.\begin{aligned} a &= \frac{dH}{dy} - \frac{dG}{dz}, \\ b &= \frac{dF}{dz} - \frac{dH}{dx}, \\ c &= \frac{dG}{dx} - \frac{dF}{dy}. \end{aligned}\right\}$$

Quaternion Expressions for the Electromagnetic Equations.

618.] In this treatise we have endeavoured to avoid any process demanding from the reader a knowledge of the Calculus of Quaternions. At the same time we have not scrupled to introduce the idea of a vector when it was necessary to do so. When we have had occasion to denote a vector by a symbol, we have used a German letter, the number of different vectors being so great that Hamilton's favourite symbols would have been exhausted at once. Whenever therefore, a German letter is used it denotes a Hamiltonian vector, and indicates not only its magnitude but its direction.

619.] The equations (A) of magnetic induction, of which the first is,

$$a = \frac{dH}{dy} - \frac{dG}{dz},$$

may now be written $\mathfrak{B} = V\nabla\mathfrak{A}$,

where ∇ is the operator

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

and V indicates that the vector part of the result of this operation is to be taken.

Since $\mathfrak{A}$ is subject to the condition $S\nabla\mathfrak{A} = 0$, $\nabla\mathfrak{A}$ is a pure vector, and the symbol V is unnecessary.

What are quaternions? How are they related to vectors?

2. Quaternions in Maxwell's Treatise

Maxwell exhibited his main results in quaternionic form. I went to Prof. Tait's treatise to get information, and to learn how to work them. […] But on proceeding to apply quaternions to the development of electric theory, I found it very inconvenient. Quaternions were in their vectorial aspects antiphysical and unnatural […]. So I dropped out the quaternion altogether, and kept to pure scalars and vectors, using a very simple vectorial algebra in my papers from 1883 onwards.

Heaviside (1893)

Quotation from Crowe (1967)

Lessons from Case 2

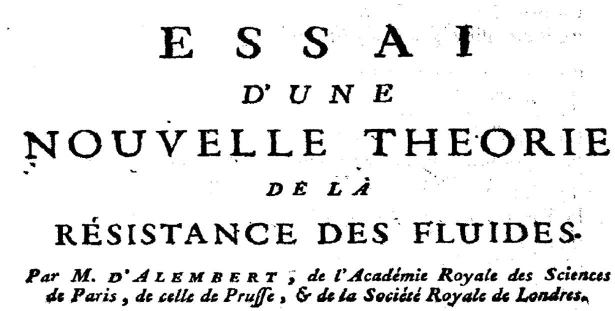
- New mathematics **created** to satisfy the needs of physics. Vectors emerge when Heaviside and Gibbs try to improve the use of quaternions in EM, which resulted in getting rid of them...

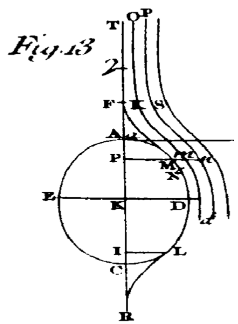
- Scalars and vectors were parts of a single entity (quaternions). Representing physical quantities with quaternions was challenging ("apples and oranges")

- Why do we call the unit vectors *i, j* and *k*? They were originally related with complex numbers!

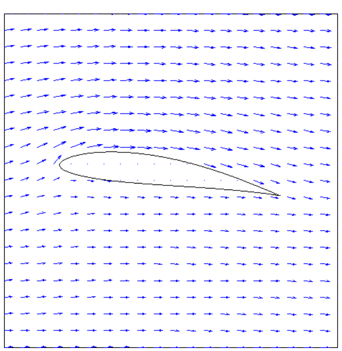
- "Vector algebra war" is a wonderful episode to illustrate different goals/methods/cultures from math and physics.

Complex analysis and hydrodynamics


ESSAI
D'UNE
NOUVELLE THEORIE
DE LA
RÉSISTANCE DES FLUIDES.
Par M. D'ALEMBERT, de l'Académie Royale des Sciences de Paris, de celle de Pruſſe, & de la Société Royale de Londres.



$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$



Incompressible

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Irrotational

$$\nabla \times \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

To solve this system of PDEs, D'Alembert (1749) proposed, for the first time, a complex valued function $f(x+iy) = u(x,y) + iv(x,y)$

Temptation: Should D'Alembert receive the credit for founding complex analysis? Should we rename the CR equations to the D'Alembert equations?

NO!

D'Alembert did **not** conceive a *complex differentiable* function. For him, the imaginary quantities should "destroy themselves"

$$p = \phi\,(x+iy) + \phi\,(x-iy) + i\psi\,(x+iy) - i\psi\,(x-iy)$$

$$q = -i\phi\,(x+iy) + i\phi\,(x-iy) + \psi\,(x+iy) + \psi\,(x-iy)$$

Key take-homes and instructional implications

- The math/physics relation is **complex**; not very useful to talk generally about it; better to look at specific cases and draw situated lessons/conclusions;

- The math/physics interplay is **fruitful**; often one helps the other; but math and physics are **fundamentally different**, and these differences should be made explicit in teaching;

- More could be done to explore the pedagogical potential of the **historical dimension** of the math/physics interplay.