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TWG1: Teaching and learning of analysis and calculus

Designing a Concept Inventory for Real Analysis

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When the effect of teaching innovations that focus on conceptual knowledge is to be investigated, then a test that focuses on this specific form of knowledge is needed. In physics, for example, the so-called Force Concept Inventory is used in this sense to test students' conceptual understanding of classical mechanics. We have developed a concept inventory for the content domain of Real Analysis. In this paper, we report on the test construction, on the content validation by experts, and on the results of a pilot test in Analysis courses at several German universities. We focus on three tasks on the topic of continuity, all of which have been rated as well suited by selected experts.

Keywords: Teaching and learning of analysis and calculus, assessment practices in university mathematics education, Real Analysis, concept inventory, conceptual knowledge.

INTRODUCTION

In this paper, we report on the construction and piloting of a test of conceptual knowledge in Real Analysis. The need for such a test arose following our study on Peer Instruction (Bauer, Biehler, & Lankeit, 2023). In that study, we compared two different variants of Peer Instruction: the "classic" implementation of ConcepTests with voting and peer discussion as described by Crouch and Mazur (2001), versus ConcepTests with voting but without peer discussion, instead with a more detailed explanation by the tutor. Somewhat surprisingly, we found that there was no significant difference between the variants in the results of the final exam. While exam results are a common indicator of academic success, they have limitations as a measurement tool: In addition to the conceptual knowledge that a method such as Peer Instruction aims to foster, written examinations also test other (e.g. computational) skills. Additionally, exam conditions, including time constraints and potential anxiety, can affect students' performance. We believe a test evaluating conceptual knowledge beyond exam results is beneficial and we are therefore developing a test aimed at Real Analysis concepts.

In the present paper we address the following questions: (Q1) On which theoretically-based design principles can the construction of conceptual questions for Real Analysis be based, and what do such questions look like? (Q2) What do experts think about the suitability of these questions and what are the results when the test is used in a pilot run in standard Analysis courses at different universities? In addition to our current purpose, we see it as a long-term goal to provide a test that can be used to investigate teaching innovations aimed at conceptual knowledge. Lecturers in the classes of our pilot tests also appreciated the test as a formative assessment tool when they administered it and then discussed the results with their students.

BACKGROUND

Conceptual Knowledge

The distinction between conceptual and procedural knowledge has received much attention in mathematics education. As Hiebert and Lefevre (1986) point out, the discussion goes back at least to the end of the 19th century, when a focus on understanding (McLellan & Dewey, 1895) or skill development (Thorndike, 1922) was advocated. In the 20th century, the discussion sometimes continued in somewhat different terms and with different emphases, such as semantics vs. syntax (Resnick, 1982) or principles vs. skills (Gellman & Gallistel, 1978). In our project, we follow Hiebert and Lefevre (1986), who define conceptual knowledge as "knowledge that is rich in relationships" and as "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information"(p. 3-4). By definition, a piece of information can be conceptual knowledge only if the holder is aware of its relationship to other pieces of information. In this vein, Anderson and Krathwohl (2001) characterize conceptual knowledge as "interrelationships among the basic elements within a larger structure that enable them to function together" (p. 29).

The Idea of Concept Inventories

In physics, the "Force concept inventory" (FCI) by Hestenes et al. (1992) is a well-known conceptual knowledge test for (part of) classical mechanics, where it probes qualitative preconceptions of students about the concept of force. The FCI has been widely used in physics education. For instance, Crouch and Mazur (2001) prominently employ it in their study on the effectiveness of Peer Instruction: They administered it in their courses over several years at the beginning and end of the term to assess students' conceptual mastery. In mathematics, the idea of the FCI was taken up by Epstein (2007, 2013) through the development of the "Calculus concept inventory" (CCI), which aims to test "conceptual understanding of the most basic principles of differential calculus" (Epstein, 2013, p. 1080). The task content of the CCI is oriented towards calculus courses in the Anglo-Saxon tradition, which differ significantly from introductory courses called "Analysis" at German universities. The latter courses aim at a systematic theory construction based on the epsilon-delta definition of limits and continuity and rigorous proof. For example, theorems such as the Intermediate Value Theorem or the Mean Value Theorem are rigorously proved in these courses on the basis of the completeness axiom of real numbers. In terms of mathematical style, these courses are similar to Real Analysis courses in Anglo-Saxon universities.

Conceptual Knowledge Required in Exercise Tasks

What lecturers require of students in terms of procedural or conceptual knowledge is particularly evident in the tasks that are set as homework. Two recent studies have examined homework assignments in mathematics courses at German universities and classified them according to their design characteristics: Weber and Lindmeier (2020) studied 277 tasks from various mathematics lectures. Wlassak and Schöneburg-Lehnert (2022) focused on assignments to lectures on “Analysis I” courses and studied 530 homework tasks from lectures at different universities. Both studies looked at the numerical proportion of procedural and conceptual or proof-related tasks. Wlassak and Schöneburg-Lehnert (2022) found significant differences between universities. Overall, both studies agree that instructors set about 50 percent conceptual (or proof tasks) and 50 percent procedural (resp. schematic applications or use of theorems for calculation) tasks. All these tasks require extended written answers. In contrast, Bauer (2019) developed short conceptual multiple-choice tasks to be solved (mostly mentally) for Analysis I and II in the tradition of short Peer Instruction tasks focusing on conceptual knowledge. They are suggested as alternative tasks for homework, or Peer-Instruction tasks in the context of lectures and tutorials.

TEST CONSTRUCTION

Aims and Scope of the Test

In the test discussed here, we focus on conceptual knowledge. Much as in physics, we would probe procedural knowledge by a separate test: For instance, Crouch und Mazur (2001) use the FCI to capture conceptual knowledge, while they employ the “Mechanics Baseline Test” (MBT) by Hestenes and Wells (1992) for quantitative problem-solving. We aim to design a test of the conceptual knowledge that introductory courses to Real Analysis in Germany (“Analysis 1”) aim to teach. Since we want to construct the test to fit as many existing courses as possible, we have focused on content widely agreed to be included in such courses. Like Wlassak and Schöneburg-Lehnert (2022), we based this on standard German textbooks and decided on the following contents: (A) completeness of the real numbers, (B) convergence of sequences of real numbers, (C) convergence of series of real numbers, (D) limits and continuity of functions of one real variable, and (E) differentiability of functions of one real variable. The section on limits and continuity, for example, contains as subtopics the sequence criterion and the epsilon-delta criterion for limits and continuity, examples and counterexamples, theorems concerning limits of sums, products, and quotients, as well as the Intermediate Value Theorem and the theorem about maximum and minimum of continuous functions on compact intervals.

The test thus covers essential, but certainly not all, topics of one-dimensional real analysis as taught at universities in Germany and many other countries. We excluded “integration” as it is often covered first in “Analysis 2”, varying by university. Even within the existing topics, there are more subtopics than we can cover with our items because of the naturally limited question pool of such a test.

Design Principles

We constructed the test as a multiple-choice test of 30 minutes in length with the following a priori design principles:

(DP1) The test items relate to definitions and theorems of one-dimensional real analysis. They do not simply test memory for the formulation of definitions or theorems. Instead, they require either their application to a particular situation or to a concrete example, or their connection to other concepts and theorems.

(DP2) The test items are constructed so that they can be solved "in the head" without the need to use symbols and text on a sheet of paper. In addition, (a) they do not require complex mental computations (which would involve procedural knowledge), and (b) they do not require multi-step reasoning (as involved in more complex proof construction).

(DP3) The distractors are chosen based on beliefs about typical students' intuitive misconceptions or misunderstandings, based partly on research and partly on teaching experience.

DP1 is based on the understanding of conceptual knowledge as knowledge that is rich in relationships to other pieces of knowledge. DP2 distinguishes the knowledge to be tested from procedural knowledge, as well as from knowledge needed for proof construction. We are aware that such tests will evoke students' "fast thinking" mode (Kahneman, 2011) and that giving the students more time and paper and pencil would more frequently evoke "slow thinking" modes with (hopefully) more correct solutions. However, we have selected those domain facets where we think giving correct and fast answers is an important component of students' competencies. As far as conceptual aspects and specific misconceptions are concerned, we build on Bauer (2019). However, while the tasks there are constructed with the intention of providing opportunities for discussion (e.g. in the context of peer instruction) and for conceptual engagement during the learning process, the current tasks are intended as quick testing opportunities.

Analysis of Selected Test Items

Regarding the subject of continuity, the test contains five tasks. To give an idea of the type of questions asked, we show three of these tasks (P3, P6, P7 in our terminology) in Figure 1. For reasons of space, we subsequently focus on P3, where we carry out a detailed a-priori analysis and discuss the empirical results below.

We now analyze P3. We describe possible solutions and difficulty-generating elements of the task; in doing so, we "decompress" the task in the sense of Ostermann et al. (2015, p. 54), i.e., we identify the relevant mathematical concepts and their relations that are required in possible solutions of the task. Task P3 relates to the definition of continuity. As for the individual items:

P3(1). The property in Item 1 is sufficient (even equivalent) to infer the required continuity (answer: yes) because it expresses the epsilon-delta criterion for the special case of the point 0. One needs to know the criterion and relate it to the given property, i.e., recognize that the property arises from the criterion by (mentally) replacing both x_0 and $f(x_0)$ with 0.

P3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(0) = 0$. We would like to prove that f is continuous in 0. Can this be concluded from the following statements?

(1) For every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x| < \delta$, the inequality $|f(x)| < \varepsilon$ holds. ☐ *yes* ☐ *no*

(2) For every $\varepsilon > 0$ there is a $n \in \mathbb{N}$ such that for all $x \in \mathbb{R}$ with $|x| < \frac{1}{n}$, the inequality $|f(x)| < \varepsilon$ holds. ☐ *yes* ☐ *no*

(3) We have $\lim\limits_{n\to\infty} f(\frac{1}{n}) = 0$ and $\lim\limits_{n\to\infty} f(-\frac{1}{n}) = 0$. ☐ *yes* ☐ *no*

(4) There is an $\varepsilon > 0$ such that for all $x \in [-\varepsilon, \varepsilon]$ the inequality $|f(x)| \leq x^2$ holds. ☐ *yes* ☐ *no*

P6. Assume that we know about a function $f : [0, 1] \to \mathbb{R}$ that $f(0) = -1$ and $f(1) = 1$ hold. Can we conclude from the following statements that f has a zero?

(1) f is continuous. ☐ *yes* ☐ *no*
(2) f is strictly monotonically increasing. ☐ *yes* ☐ *no*
(3) f is a polynomial function. ☐ *yes* ☐ *no*
(4) f is differentiable. ☐ *yes* ☐ *no*

P7. Are the following statements true for all functions $f : [0, 1] \to \mathbb{R}$?

(1) If f has no maximum, then f is not continuous. ☐ *yes* ☐ *no*
(2) If f is not continuous, then f has no maximum. ☐ *yes* ☐ *no*
(3) If f has no maximum, then f has no minimum. ☐ *yes* ☐ *no*
(4) If f is continuous, then f is bounded. ☐ *yes* ☐ *no*

Figure 1. Three tasks on the topic of continuity

P3(2). The property in statement 2 is sufficient (even equivalent) to infer continuity (answer: yes). The epsilon-delta definition is fulfilled with $\delta = 1/n$. Conceptual understanding of continuity could include that the existence of a neighborhood of x_0 is required for which $f(x)$ lies in a neighborhood of $f(x_0)$. This is more general than the syntactic specification with "ε and δ" and directly yields that a neighborhood can be specified with $\pm 1/n$. One difficulty-generating feature here is that the statement looks very similar to the epsilon-delta definition of continuity but has a prominent discrepancy using $1/n$ instead of δ, where the existence of a large n instead of a "tiny" δ is required. Students who look superficially for an "equivalence" on a syntactic level might find these two expressions too different to be logically equivalent. They might, therefore, not try to think further conceptually that continuity is a consequence of the given property. Also, they might think that requiring the existence of a δ is a stronger condition than the existence of "only" $1/n$, and they might, therefore, think that the condition is necessary but not sufficient.

P3(3). The property in statement 3 is not sufficient to infer continuity (although it is necessary) (answer: no); it refers to the sequence criterion for continuity. A difficulty here is that one must be aware that the sequence criterion requires the convergence of $(f(x_n))$ for *all* sequences (x_n) that tend to zero. Generally, the convergence of special sequences is necessary but not sufficient for continuity. A rigorous proof that it is not sufficient would require a counterexample. In usual lectures, the all-sequences criterion for limits is stressed and contrasted with school-mathematical argumentation, where using "typical" sequences such as $(\pm 1/n)$ is often considered sufficient. This conception might persist among students.

P3(4). The property in statement 4 is sufficient to infer continuity (answer: yes). The difficulty-generating feature is that the formulation is very different from the definition, and students may take this for a quick answer "no". An additional difficulty-generating feature is that ε in this statement has quite a different role than ε in the definition of continuity. If students only look at the statement on a superficial level and search for structural similarities, they might find that it is not enough that "an ε exists" because they know that in the definition, a certain condition must be true for *every* ε. There are various ways to see that the property is sufficient: Arguably, the most conceptual argument uses that $f(x)$ is squeezed between x^2 and $-x^2$, both of which tend to zero; thus $f(x)$ tends to zero, too (Squeeze Theorem). Alternatively, one can argue directly with the epsilon-delta criterion: Since the function $x \mapsto x^2$ is continuous and $|f(x)|$ is bounded by x^2, the same delta works for f as for x^2. Another alternative is to argue with sequences: For a sequence (x_n) that tends to zero, $|f(x_n)|$ is bounded by ${x_n}^2$ for large *n*; since the latter tends to zero, $f(x_n)$ also tends to zero.

As this analysis shows, the items involve the application of the concepts to special situations (special points, concrete sequences, concrete functions) or they refer to other concepts and theorems (in this case the sequence criterion or the Squeeze Theorem) (DP1). Also, it shows that the task can be solved "in the head" without any calculations (DP2). Statement 3 considers a common misconception (DP3).

Content Validation

As Jenßen et al. (2015) point out, content validity is an important quality feature of a test. It serves to support the validity of intended test score interpretations and, to this end, addresses the question of "the extent to which the content of a test or the items of which it is composed actually capture the characteristic of interest" (Hartig, Frey, & Jude, 2008, p. 140). In our project, we conducted expert surveys to ensure this. We selected five instructors at different German universities as experts. All but one of the experts were well familiar with the test items, as they had used the test (as part of a pilot implementation) in their course "Analysis 1" and had discussed the results with their students. The survey was conducted at the item level (cf. Jenßen et al., 2015, p. 14) in the following way: The experts were presented with the tasks, along with a description of the intention pursued by each content area; they then answered for each item the question "How well does this item fit the intention of this domain?" on the four-point Likert scale "very poor / poor / well / very well". In addition, they could comment on each item (optionally) if they rated it as "poor" or "very poor". All five experts answered "very well" for tasks P3 and P7. For P6, four responded with "very well" and one with "well". For each content area (e.g., "Limits and Continuity"), they also answered the question "Do the previous tasks cover the most important facets of this area?" on the four-point Likert scale "no / rather no / rather yes / yes". Four experts answered this question with "yes" and one with "rather yes".

COLLECTION OF STUDENT DATA IN SEVEN COURSES

Data were collected by administering two different tests, each focusing on specific mathematical content areas. The first test focused on real numbers, sequences, and series, while the second test focused on limits of functions, continuity, and differentiability. Data collection took place during the summer term of 2022 (in one "Analysis 1" course) and the winter term of 2022/23 (in six "Analysis 1" courses). A total of 391 participants took the first test across seven courses offered at six different German universities, while 336 participants took the second test across the same courses. Both tests were administered online via "Lime Survey," with students completing the assessments individually within 30 minutes during their respective "Analysis 1" lecture or tutorial group sessions, as determined by their lecturers. The timing of the test administrations varied, with the first test taking place between early December 2022 and mid-January 2023, and the second test taking place between mid-January and the end of February, depending on when the respective topics were completed in each course.

EMPIRICAL VALIDATION – RESULTS

We present the results for the three shown tasks in Table 1. It shows the total percentages of participants who answered the respective items correctly.

Table 1: Percentage of participants (N = 336) with correct answers by item

Task	P3				P6				P7			
Item	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
% Correct	81	44	46	17	76	57	54	68	33	74	74	51

In addition to the total percentages reported in Table 1, we found considerable variation between courses, which however are not significant at the 5% level. The relatively small sample sizes in the courses seem to account for the variability. We consider this to be a weak indication that a strong "course effect" does not exist.

INTERPRETATION OF RESULTS

The data contain many interesting results for the three items shown, but, we will continue to focus on P3, for which we have presented an a priori analysis. We expected P3(1) to be the easiest item (which it was), but we were surprised that only 81% got it right, given its technical and conceptual simplicity. P3(2) and P3(3) are slightly below the "guessing level". These results show the expected higher level of difficulty than that of P3(1). The most positive interpretation is that about 50% of the students ticked the item out of correct understanding, but if we take the possibility of guessing into account, it would be less. In other words, we can identify deficiencies in conceptual understanding that we anticipated in our analysis. Our data show how common this is among our course participants in Analysis I. Comparable data do not yet exist. We expected P3(4) to be the most difficult item, which it was. The percentage of 17% strongly suggests that the wrong answer was not chosen by mere guessing, but some of the reasons for the wrong answer we identified above may be responsible for this poor result.

Another question is, how important is it for students in an Analysis I course to be able to give correct answers under the time constraints we imposed? As described in the content validation section, our lecturers found the items *well* or *very well suited*. However, it is an open question what can be done in a course to achieve better results and whether the course time needed for such support would be justified as compared to the many other objectives of such a course. Subsequent interviews with lecturers may shed more light on this issue. We expect that working on the misconception P3(3) is essential for a deeper understanding of continuity and the concept of limits of functions. P3(2) may require training in more careful reading and conceptual interpretation of formal statements. Solving P3(4) seems to be at a higher conceptual level than basic epsilon-delta arguments and involves qualitative insights that are also relevant in other contexts.

DISCUSSION AND CONCLUSION

In this paper, we have reported on a test designed to capture conceptual knowledge in real analysis. Experts, who validated the test content, found the items to be well or very well suited to the domain in question. As we showed in the previous section for task P3, the results provide highly interesting information about students' conceptual knowledge (or possible lack thereof). The explanatory hypotheses we have formulated suggest a number of further studies focusing on specific content issues. More qualitative research is needed to investigate the thought processes that lead students to wrong or correct answers. The results of our test for this and all the other items suggest directions for possible research. The analyses we conducted for P3 can similarly be applied to the remaining test items. In P6, for example, it is striking that even for the basic item (1) the solution rate is below 80%, which raises the question of further investigations as to whether the students had not yet sufficiently processed the content in question at the time of the test or whether there are more fundamental difficulties here (e.g. in the sense of epistemological obstacles). From a methodological point of view, it is an interesting open question to what extent the test results correlate with the results of final examinations, which usually also include procedural tasks and in which additional factors (such as time pressure) play a role.

NOTES

Author note: The authors played equal roles in the research and publication of this study. Correspondence to this article can be addressed to either of the authors.

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Change of meaning in learning activities about the tangent line and derivatives using DGS

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In this paper we investigate the issue of change of meaning due to semiotic transformation in Calculus teaching activities involving Dynamic Geometry Software. First, we present a phase of a tutorial targeted to students at the transition between secondary school and university, designed by a pre-service teacher in a course of Didactics of Infinitesimal Calculus. In the tutorial, the students could carry out explorations in a DGS structured environment (GeoGebra Applet) to answer questions about tangent lines and on concept of local linearization (micro-straightness). We analysed the transcripts of the interviews, the sheets and the researcher's notes collected during an implementation of the tutorial activity. We present and discuss the case of one student who experienced a change of meaning.

Keywords: Digital and other resources in university mathematics education; Teaching and learning of analysis and calculus; Change of meaning; Dynamic Geometry Software; derivatives.

INTRODUCTION

In this paper we investigate the issue of change of meaning due to semiotic transformation (D'Amore et al., 2012) in Calculus tasks involving Dynamic Geometry Software (DGS; in this case, GeoGebra). The DGS provide very relevant tools to support students' learning processes of conceptualization, that, in mathematics, is strictly related to the semiotic activity. In Calculus teaching, DGS can be relevant mediators between the experience of the subject and the visual representation of mathematical objects; a very interesting case is the one of functions, with particular attention to the interaction between the local-global points of view (Maschietto, 2008).

After designing a research-based tutorial including a GeoGebra Applet, we tested it with students at the transition between secondary school and university (grade 12). We analysed the data collected during an implementation of the tutorial activity. We focused on students' sentences referred to semiotic transformations and to the meaning of the mathematical objects at stake (e.g. curves and straight lines). Moreover, we searched for the students' references to the actions performed during the interaction with the DGS (e.g. to zoom, to compare representations in different sides of the screen, to insert analytical expressions in the algebraic interface).

In this paper, we present and discuss the case of one student who experienced a change of meaning due to a semiotic transformation carried out with a DGS.

THEORETICAL BACKGROUND

D'Amore et al., (2012) proposed an original networked semiotic perspective on learning processes, based on Duval's and Radford's semiotic approaches to mathematics education (Duval, 1995; Radford, 2008) coming to introduce the notion of change of meaning during semiotic transformations, that occurs when "each new representation has a specific meaning of its own not referable to the one of the starting representations, even if the passage from the first to the second ones has been performed in an evident and shared manner" (D'Amore, et al., p. 37). If this change of meaning occurs, the students do not recognize the referential object, thus their learning is interrupted, and they feel confused. In Duval's theory (1995) the semiotic transformations are classified from a structural point of view in two main categories: *treatment*, that consists of transformation within the same semiotic system, and *conversion*, that relates representations in different semiotic registers.

The authors stress that the dichotomy treatment/conversion does not allow to explain the phenomenon of the change of meaning completely. Each passage gives rise to forms or symbols to which a specific meaning is recognised because of the *cultural* processes through which it has been introduced (D'Amore et al., 2012; Radford, 2008) and are not only results of correct codified application of rules within or outside the semiotic system, but students need to re-assign a meaning after the transformation. The main need of the students is to keep control on the meaning connecting the new sign to a generating significant situation that they can refer to, and that was experienced also with their perceptions (*sensuous cognition*, Radford, 2013), becoming part of their ways of reasoning and acting.

RESEARCH PROBLEM AND QUESTIONS

In this paper we focus on the design of activities that could create meaningful contexts for students to develop experiences where to ground their meaning of derivative, and we investigate the possible phenomena of change of meaning occurring in the process of conceptualization of the derivative in DGS environments.

Our research question is:

1- What kind of change of meaning occur due to the use of DGS to perform the semiotic transformation in the case of local linearization of curves in a pre-Calculus activity?

LITERATURE REVIEW

The activity aims to introduce the derivative of a function at a point not only algebraically, but also by considering its visual aspects; therefore, tangent lines have a crucial role in it. Of course, as several studies suggest, this approach could be challenging:

- Students meet tangent lines in different contexts - while studying Euclidean Geometry or Analytic Geometry for example - and develop a concept image which can be an obstacle to them when the adoption of a different point of view, such as the analytical one, is required (Biza et al., 2008).

- The above-mentioned concept image is resistant to the analytical definitions that students encounter (Biza, 2011).
- Students seem to prefer handling derivative algebraically rather than relying on its visual aspects; furthermore, it appears that the graphical interpretation of the definition of derivative is not immediate (Asiala et al. 1997).
- Several episodes of confusion between derivatives and tangent lines' relationship have been observed and reported (Amit & Vinner, 1990).

Yet, the results obtained by Biza et al. (2008) suggest that a reciprocal influence between tangent line and derivative concepts may be the key to success. Furthermore, the choice of working with tangent lines gives the possibility to introduce students to what has been called the "global/local game", typical of Calculus.

DESCRIPTION OF THE TUTORIAL ACTIVITY

The aim of the task design was to develop an activity which could help students in the meaning-construction of the derivative concept. First, we carried out that a deep and careful study of examples and findings from University Mathematics Education research could be a good starting point to achieve this goal. The activity proposed was inspired by Maschietto (2008). In the forthcoming sections, a more detailed description of Observation phase is provided. We omit the details of the other phases for brevity's issues.

Observation phase: Description

Observation phase aims to give students the possibility to encounter and acknowledge Micro-straightness phenomenon (MS), which is defined by Maschietto as "the property of some graphical representations to seem straighter and straighter when zooms around their points are successively performed" (Maschietto, p. 209). It is an exploration activity and involves the employment of a purpose-built GeoGebra Applet: "Functions closely".

The Applet allows users to insert the analytical expression of a function, the coordinates of a point and choose a value for the variable "zoom" from 1 to 20. On the screen, the inserted expressions will appear on the left, the global representation of the function in the centre and the zoomed representation in the neighbourhood of the selected point on the right. The data can be modified any time by the users.

Students are supposed to be divided in small groups (2 or 3 group members). Then, each group is given a list of couples - each composed by a function and a point belonging to its graph (F-P couples) - and is asked to "explore" them using the zoom provided by the Applet. For every couple, the groups are tasked with sketching the zoomed representation for at least three different zoom's values in their notebooks. Finally, the groups are invited to compare the results of their explorations and share their observations with the rest of the class; of course, during the pilot implementation of the tutorial the students could not try this last task, as she worked on her own.

F-P couples are chosen so that some representations will get straighter as zoom's value increases (MS) while others will not. Furthermore, some of the functions have been selected because in the next phases of the activity they could be helpful in developing potential evolutions of students' meanings of tangent line due to potential conflicts between the cultural meaning assigned to different representations. The expected outcome of this phase is the acknowledgement by the students of the MS phenomenon as a property of some – yet not all – functions.

Observation phase: comparison with the literature

Observation phase's development was strongly inspired by Maschietto (2008), Biza (2011) and Biza et al. (2008).

Maschietto's article (2008) - providing an analysis of a teaching experiment involving derivatives, tangent lines and "global/local game" - was crucial to design the tasks, carry out the experiment and foresee how students could deal with the task.

Biza's works (2008; 2011) highlight the presence in some students' concept image of tangent line of features - coming mainly from Euclidean Geometry or Analytic Geometry environments - which tend to be dominant over the definition based on the derivative concept. Her articles deeply influenced our "adjustments" of Maschietto's tasks to our activity's purposes. As far as Observation phase is concerned, they played a crucial role in the choice of functions which could trigger potential conflict factors in students' concept image of tangent line.

Similarities	Differences
- Aim: acknowledgement of Micro-straightness (MS) phenomenon by students. - Tasks.	- Use of a purpose-built GeoGebra Applet. - Global representation always displayed on screen. - Choice of functions and points that could give rise to potential conflict factors in students' concept image of tangent line.

Table 1: Similarities and differences with Maschietto's first session (2008)

DATA COLLECTION AND ANALYSIS

We have at our disposal three different kinds of data – of course after asking students' permission to use them for research purposes.

1. <u>Written materials</u>, consisting of the sheets of paper with the observations and the drawings that students made during the activity.
2. <u>Recorded materials</u>, consisting of the recording of a post-activity conversation between each student and RL.
3. <u>Researcher's notes (by RL)</u> taken during the students' exploration, including:

a. Some students' quotations.
b. Some observations about the time students dedicated to the exploration of certain functions.

The case of Sofia

Sofia (fictional name) is a sixteen-year-old girl who attends the third year of a Chemistry-oriented Upper Secondary School in Italy. She has the highest scores in her class, and she made use of some advanced mathematical concepts – namely, limits, even though she was not given the formal definition - in her school experience. She accepted to take part in our pilot study and try the first phase of our tutorial – Observation - to help us improving the tasks' formulation before an implementation in an actual class.

Sofia had at her disposal a laptop, sheets of paper and a pen. RL has been sitting next to her, without interfering, for all the duration of the Observation phase - in case she had problems with the GeoGebra Applet or with the comprehension of the task.

Data analysis

In the next subsections we will explicit our data analysis and interpretation.

<u>1. Why do we state that Sofia did not acknowledge the micro-straightness phenomenon?</u>

In Sofia's written materials there are a lot of references to aspects closely related to MS, but her post-activity comments and her attitude during the exploration make us think that she did not interpret the phenomenon in the way we would have expected.

During the activity Sofia observed that straight lines sometimes appeared on the laptop's screen after zooming, but she did not do, say or write anything that led us to think that she was really appreciating MS. Unlike the examples provided in literature, she did not seem to consider it as a property: for instance, she did not look surprised when she came across functions that did not behave as the previous ones, such as $f(x) = x^2 - 3|x|\ at\ A = (0,0)$ or $f(x) = |x|\ at\ A = (0,0)$. Another clue in this direction can be found in Sofia's written materials: in the description of some functions, she did not say that they all looked like a straight line but that they resembled "the previous one ["One" here refers to the previous function]" as if "straightness" was not the most important thing to notice at that moment.

Sofia's post-activity comments and written materials are the most important evidence that made us think that she did not acknowledge MS; in particular, the way in which she used the words "straight line" and "curve" played a crucial role in our deduction. In the next part of this section, we will explicit our interpretations of Sofia's post activity comments. Sofia's written materials underwent a similar analysis. We omit it here for brevity's sake.

I: What can you tell me about your exploration? Did you see something new, or did it all go as you imagined?

[17 seconds of silence]

S: [Almost muttering] Well some of them [the curves], well, you could understand they didn't change and [Little pause] well that is because the curve was really wide.

When asked to describe her exploration experience, Sofia did not speak about straight lines, but about curves and effects that zooming had on them.

Straightness does not seem to be the protagonist of her observations: it is not even the first thing that comes to her mind.

I: Really wide. [Little pause]. What do you mean when you say that they didn't change? What did you see when nothing changed?

S: And well [Pause and softly muttering] a piece of the curve [seconds of pause] the point [Pause] I mean on the curve so [Few seconds of pause] if you analyse it in a small part, it does not look like a curve, because it looks like a straight line. [...] [Interrupting, raising her voice] Because it wasn't that curved.

Straight lines appeared later, when Sofia was asked to explain what she meant exactly by "nothing changed".

Straight lines come after curves and zoom operations again in Sofia's speech; furthermore, the way in which she concluded her thought make us think that with the word "straight line" she wanted to highlight "non curviness" rather than "straightness".

I: What about this one? [Pointing the function $f(x)=2x$ at $A= (1,2)$]

S: [Pause] Because it is a straight line.

The second time Sofia used the word "straight line" was to describe what was happening on the screen, while she was dealing with an actual straight line.

The fact that Sofia said "straight line" is not enough to conclude that she acknowledged the micro-straightness phenomenon, as she may have used these words just with a descriptive purpose.

I: Ok. So [Pause] you chose the functions which changed after you used the zoom, right? […] And what was the change?

S: Well [Little pause] you could understand that it was a curve because you [Pause] enlarged them just a little. [Seconds of pause]. I mean [Pause] if you enlarged them too much you couldn't see it was curved.

When asked to clarify for the second time what she meant with "change", Sofia spoke about curves and zoom operations but did not mention straight lines.

Sofia's words make us think that to her the lack of "curviness" was much more significant than "straightness".

I: [Taking Sofia's written materials and reading aloud descriptions of functions 4 and 5].

S: [Interrupting] Because it didn't change. [Little pause]. I mean the image always looked like a straight line while on the other side it looked like [Silence].

The third – and last – time Sofia said "straight line" was to explain what she had written about functions 4 and 5 during the exploration phase of the activity.

Again, we cannot interpret Sofia's words as an acknowledgement of MS. The fact that Sofia wrote: "It seems that the point belongs to a straight line" in the description of function 5 lead us to think that in this case she may want to explain what she had written previously rather than giving importance to "straightness".

I: Ok. Let me ask you one final question. Do you think it is peculiar that some functions look like a straight line if you look at them in a neighbourhood of one of their points?

[40 seconds of pause, Sofia read her work again]

S: Boh [Pause] then it's a property I guess. [Pause, then muttering] Because it is a point [Pause] it is the centre of the curve [Pause] I mean the centre of the point where the curve changes direction.

In this last phase of the discussion, the interviewer asked explicitly to Sofia if she found peculiar that some functions turned into almost-straight lines. Sofia, in her answer, did not mention straight lines but spoke about curves. Sofia's last comment led us thinking that she is "accepting" MS as a fact rather than acknowledging it: she uses words such as "Boh" and "I guess" which expresses doubt and then tries to explain "straightness" using "curviness" - a concept she seems to value a lot.

Our interpretation, based on the data analysis provided above, are that:

1. Sofia accepted - rather than acknowledging - MS phenomenon.
2. Sofia gave more importance to "curviness" - or better "non-curviness" - than to "straightness".

<u>Where do we see "Change of Meaning" in Sofia's behaviour?</u>

In D'Amore et al. (2012) several examples of change of meaning are provided. We report one of them to confront it with some excerpts from Sofia's written materials.

Students, divided in groups, were asked to write in the algebraic register the sum of three consecutive natural numbers. One group answered (n−1) + n + (n+1). Then, a transformation was performed and 3n was obtained. The last expression, however, was interpreted differently by the students, who recognised it as "the triple of a natural number"; furthermore, they stated that the new form could not represent the sum of three consecutive natural numbers, but only the sum of three equal numbers.

$f(x) = x^2 + 3x \; at \; A = (0,0)$	"No, being it a point on a straight line [It is a parabola], it doesn't seem to me that changing the zoom adds some new data".
$f(x) = \sin(x) \;\; at \; A = \left(\frac{\pi}{2}, 1\right)$	"If you look at it without the zoom it seems that the point belongs to a curve, while if you use the zoom it seems that the point belongs to a straight line".
$f(x) = \sqrt{1 - x^2} \; at \; A = (0,1)$	"In order to understand the shape of the curve you need zoom to be less than 9."
$f(x) = \sqrt{1 - x^2} \; at \; A = (1,0)$	"In order to understand that it is a curve and not a straight line, zoom must be less than 8".
$f(x) = \sqrt{x} \; at \; A = (0,0)$	"If you choose a zoom greater than 9 it doesn't seem a curve, but a straight line".
$f(x) = \sqrt{\lvert x \rvert} \; at \; A = (0,0)$	"Until you use a zoom less than 9 you understand that they are two lines".
$f(x) = \sqrt[3]{x} \; at \; A = (0,0)$	"Zoom > 4 it doesn't look like a curve".

Table 2: Excerpts from Sofia's written materials

From these excerpts it emerges that:

1. Zooming too much results in the impossibility to understand the shape of the curves, or, better, if what can be seen on the screen is a curve. We value this observation as another clue towards the great importance that Sofia gives to "curviness" and we interpret it as a confirmation that the transformation changes the meaning (from a curve to a straight line).
2. Some zoom's values have been identified by Sofia as "boundaries" for the "Loss of Curviness" phenomenon. We interpret these "boundary zoom's values" as signals of "Change of Meaning": before a certain zoom's value she can be sure she is looking at a curve, once it is passed "curviness" is lost and the object is no more recognized as the same, thus she cannot go on in its reasoning but a new one should begin, about a new object (a straight line).

We regard Sofia's situation as similar to the one proposed above: Sofia, like the group of students, gave a different meaning to a representation after it underwent a treatment (zoom with a value higher than the boundary value). We classify zoom as "treatment" - from a structural semiotic point of view - because:

- It was an explicit transformation - since Sofia was told how zoom worked in the Applet - which does not involve a change of semiotic register.
- Sofia did not give any signs to have moved from the graphic register throughout all the activity: for example, she spoke about curves – and not about functions – all the time.

DISCUSSION

To conclude this section, we continue the analogy between the two episodes by quoting an interesting observation from D'Amore et al. (2012)'s article:

> From an, so to speak, "external" point of view, we can trace back to seeing the different algebraic writings as equally significant since they are obtainable through semiotic treatment, but from inside this picture is almost impossible, bound as it is to the culture constructed by the individual in time. (ibid., p. 38)

The same could be said for Sofia's experience: from outside the graphical register, it seems easy to re-build the connection between the two sides of the screen and, therefore, between the two representations, but Sofia could not make it: to her the two were "culturally" separated, since losing "curviness" (not only) to her means losing the "curve" itself.

"Change of Meaning" could explain Sofia's missed acknowledgment of MS at a point of a curve. As a matter of fact, it could be that she did not value MS - even though she perceived some of the curved lines getting straighter – because it was impossible from her perspective to make a synthesis between the two sides of the screen. Indeed, at a certain point, they represented two different objects according to her: curves on the left, not – curved lines on the right. Hence, the incompatibility of the global and local representations, due to "Change of Meaning", could have broken the relationship between the two points of view, crucial to value MS as significant.

Sofia during the activity said several times that what was happening was "similar to the previous one". Her words seem to suit the previous observation: it looks like that to her the connection between different functions' right images (local representations, all resembling straight lines) was stronger than the one between left and right images (global and local representations respectively) of the same function.

CONCLUSION

In our pilot study, we identified an interesting case of change of meaning due to a semiotic transformation carried out with a DGS in semiotic transformations in the case of local linearization. In particular, we observed it in the case of treatment within the dynamic graphical register available in GeoGebra. Sofia's experience is particularly interesting because the outcomes were different from literature-reported ones (Maschietto, 2008) and unexpected to us: she did not seem to acknowledge MS phenomenon, and the cause is the change of meaning.

Our first contribution to existing literature concerns the change of meaning due to treatments in DGS; indeed, D'Amore et al. (2012) stressed that treatment is more rarely considered source of difficulties for students, while, as they showed in case of algebraic treatment, the process of attribution of a new meaning to representations is necessary also in this case, and sometimes even more difficult than conversion.

In this case, the new cultural meaning assigned by the students cause a change of mathematical object (from curve to straight line) that is incompatible with her personal perception (‘a curve cannot be straight’). Her choice of limiting the zoom’s values could be interpreted as a “limited” acceptance of zooming as a legitimated strategy to learn something new about curves, since it leads to betray the “essence” of a curve.

This issue is specific of DGS and open to a new research question, that we will investigate in the future. Indeed, in DGS, the semiotic transformation of the representations cannot be completely decided by the students; the transformation tools are given as “black boxes” thus the students’ attribution of a new cultural meaning is limited by a sort of “fixed behaviour of the representation”. This is not the case of paper and pencil environments. The use of DGS could thus lead to new phenomena of change of meaning like the one we observed in Calculus teaching.

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The complexity of mixed methods research – case study of a project on students' meanings for the fundamental concepts of calculus

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We report on a project whose aim it is to investigate advanced high school students' meanings for derivative / rate of change and integral / accumulation in intra- and extra-mathematical contexts. The project is ongoing with, so far, 725 questionnaires and 207 interviews. We relate a selection of initial results and raise issues pertaining to mixed methods research in large-scale mathematics education projects.

Keywords: Teaching and learning specific topics at university, teaching and learning of analysis and calculus, accumulation, meaning, mixed methods research.

BACKGROUND AND AIM

This paper arose from a project on students' meanings for the fundamental concepts of calculus. Students' personal meanings for mathematical concepts have recently attracted increased attention. In particular, Thompson (2016) has distinguished what someone knows about, say, the derivative, from what they mean by the derivative: The meaning of a student's understanding is linked to the space of implications (ideas, associations, explanations, solutions, …) the understanding mobilizes for the student.

Meanings for the fundamental concepts of elementary calculus that students acquire in high school may have a crucial influence on their tertiary studies. We take the fundamental concepts of calculus to be the derivative / rate of change of a quantity, the (definite) integral / accumulating quantity, and the fundamental theorem. In doing so, we stress quantitative thinking in extra-mathematical contexts. This may motivate students, support solving everyday problems, foster interdisciplinary connections, and showcase the broad applicability of calculus. The primary catalyst for our project has been the wide dissatisfaction with students' knowledge in calculus (e.g., Kouropatov & Dreyfus, 2013; Thompson & Harel, 2021). Our project thus also has a didactical motivation, namely, to pave the way for the development of improved materials and practices for teaching and learning calculus.

In this paper, we consider the question how to investigate students' meanings on a large scale, taking the notion of integral / accumulation as a case study. In particular, we ask how students' meanings depend on the context in which a situation is presented.

Several large-scale research efforts on calculus have been undertaken in the past. For example, Epstein (2013) has designed a Calculus Concept Inventory; Greefrath et al. (2016) have based their research on Basic Mental Models; we have compared their approach to ours (Dreyfus et al., 2022) and found them to be substantially different.

While a qualitative approach is the natural way to investigate delicate nuances in high school students' meanings, quantitative methods allow for larger samples. We have therefore decided on a mixed methods approach. Standard approaches to mixed methods research (e.g., Creswell & Plano Clark, 2007) propose two options: (A) start with a qualitative investigation, discover a phenomenon, and then build a questionnaire to investigate the scale of the phenomenon; (B) start with a quantitative study, and then interview subjects to gain deeper insights into the results. It has been claimed that mixed methods research in mathematics education allows for a comprehensive understanding of complex phenomena (Kelle & Buchholtz, 2015). On the other hand, even when the rationale for and manner of integrating qualitative and quantitative components is made explicit (Choudhary & Jesiek, 2016), formulating mixed methods research questions and linking them to mixed methods data analysis is notoriously difficult (Onwuegbuzie & Leech, 2006).

DESCRIPTION AND METHODOLOGY OF THE PROJECT

Exploring the nature of students' personal meanings requires fine-grained research with qualitative methods that allow for iterative adjustments of the research process and instruments. On the other hand, investigating the meanings of students with different backgrounds, different teachers, in different schools and even school systems requires data on a large scale, as do later didactical suggestions, if they are to have more than local validity. Quantitative methods may be expected to reveal statistical connections; such connections may in turn be explained by subsequent qualitative study. Hence, a mixed methods approach has the potential to reveal students' meanings for the fundamental concepts of calculus at scale.

From prior research we knew some meanings of high school students for derivative and integral that could be expected. We also knew we wanted to use situations with and without extra-mathematical context. But we did not know which situations and which formulations would elicit meanings from students. Pilot interviews with 98 high

The drawing shows a loaf of bread with a slice shown x cm from the left end of the bread. Which of the following graphs could represent the volume V of the bread to the left of the slice as a function of the distance x from the left end to the slice?

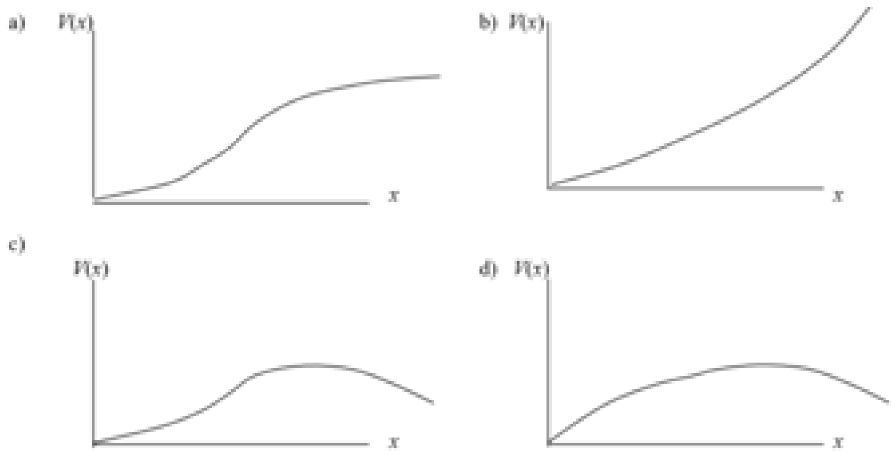


Figure 1: Bread loaf item (P. W. Thompson, personal communication, October 16, 2020)

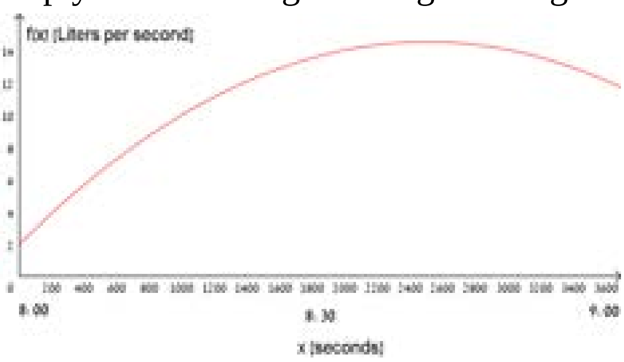
school calculus students and teachers allowed us to try a multitude of situations - mathematical ones, kinematic ones, and everyday ones, with either time or distance as independent variable – as well as a wide variety of formulations, including graphical and algebraic ones. Figure 1 presents one example requiring accumulative thinking.

During the pilot phase, we realized that we were unlikely to learn about specific meanings students might or might not have by asking them to solve problems in a questionnaire. Thus, we decided to systematically investigate which meanings students identify with, and in which contexts or situations they identify with these meanings: We proposed statements of hypothetical students that each express a distinct meaning, and asked respondents in what measure they identified with each such statement. We chose meanings discovered by prior theoretical and empirical research (e.g., Sealey, 2014; Zandieh, 2000) as well as meanings that arose in the pilot interviews.

In separate questionnaires, we related to five fundamental concepts: Constant Rate of Change, Instantaneous Rate of Change, Accumulation Calculation, Accumulation Function, and Fundamental Theorem of Calculus. In this paper, we focus on Accumulation Calculation, that is the meanings students associate with what they are doing when they carry out the calculation of an integral or accumulation. The situation we used for Accumulation Calculation is presented in Figure 2.

Monday morning at 8:00 the pool was empty. Workers began filling it. The given function represents the flow of water into the pool during the first hour (3600 seconds), from 8:00 to 9:00. The flow of water is measured in litres per second.

$$f(x) = -\frac{1}{500{,}000}x^2 + \frac{1}{100}x + 2$$



Alona said that using this data, it is possible to estimate the amount of water that accumulated in the pool from 8:00 to 9:00. The students discussed the meaning of her statement to them.

Figure 2: The Accumulation Calculation situation

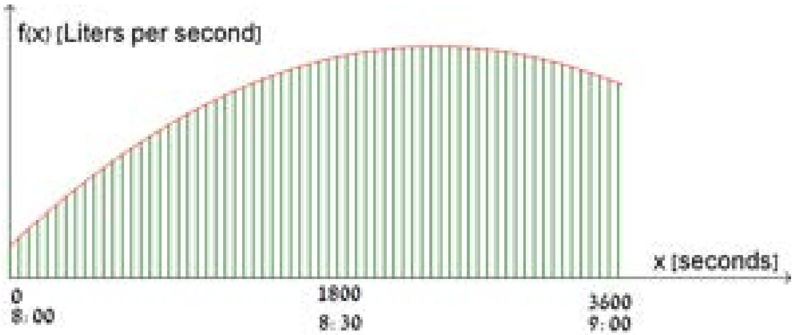
In Figure 2, Alona presents a claim about an everyday situation concerning an accumulating quantity of water. The pilot interviews showed that students' meanings may be different if Alona presented her question about the integral. In a parallel questionnaire, we therefore replaced Alona's claim in Figure 2 by "Alona asked what the meaning of the integral $\int_0^{3600} f(x)dx$ is in this case". We refer to these two questionnaires as the quantity setting (Q) and the integral setting (I).

We adapted the two Accumulation Calculation questionnaires to four different contexts, making as few changes in formulation as possible. Two contexts were extra-mathematical: filling a pool (as in Figure 2), and motion on a straight line; the other

Anna: For me, if we calculate the values of the primitive function of the function f(x) at the hours 8:00 and at 9:00, the difference between them will give amount of water that accumulated.

Ariel: For me, the area between the graph of the function and the x-axis in the relevant interval gives the amount of water that accumulated in the pool between the hours at the hours 8:00 and at 9:00.

Lina: For me, if we draw vertical lines between the function f(x) and the x-axis, at each point, and we sum them, we will get the amount of water that accumulated.



Sapir: For me, we take small time intervals on the x-axis and in each interval, we determine a corresponding constant flow. If we multiply the constant flow value by the length of the time interval, and we sum all the products, we will get approximately the amount of water that accumulated. For example, we can take intervals of length a on the x axis and calculate: $f(0) \cdot a + f(a) \cdot a + f(2a) \cdot a + f(3a) \cdot a + \cdots$

Dino: For me, the definite integral from 0 to 3,600 on the given function will give the amount of water that accumulated.

Vadim: For me, if we take the values of the given function at every point and we sum them, we will get the amount of water that accumulated.

Ron: For me, if we draw rectangles whose width is a short interval on the x-axis, and whose height is the height of the given function at a corresponding point, the sum of their areas will give approximately the amount of water that accumulated.

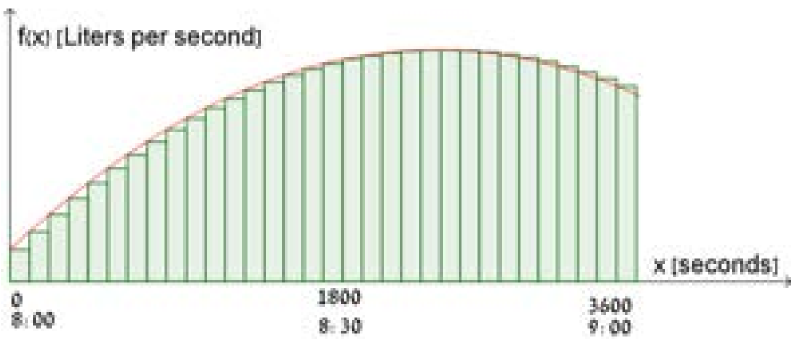


Figure 3: The hypothetical students' answers

two contexts were intra-mathematical: area and formal-mathematical. In total, we thus had eight Accumulation Calculation questionnaires: 2 settings × 4 contexts. [1]

The questionnaire presented hypothetical students' answers to Alona's question. The seven answers to the situation in Figure 2 are presented in Figure 3. The names of the hypothetical students were chosen to reflect the intended meanings: Anna – Antiderivative, Ariel – Area, Lina – Lines, Sapir - Sum of Products, Dino - Definite Integral, Vadim – Values, and Ron - Rectangles. The hypothetical students' formulations in Figure 3 were modelled on how actual students of the same general population as the respondents to the questionnaires expressed themselves (see Dreyfus et al., 2022, for an example).

We kept the differences between the eight versions of the Accumulation Calculation questionnaire as small as possible. We made three adaptations:

- Context: for example, for the area context, we replaced "the amount of water that accumulated" by "the accumulated area".
- Setting: In the integral setting, we replaced Dino by Ahmed: "For me, the integral of the function according to *x* gives an accumulation. In this case, water accumulates, and therefore we get the amount of water in the pool at 9:00."
- In the area context and quantity setting, we omitted Ariel's statement because it was part of what Alona stated.

Students were asked to react to each statement by marking answers to the two Likert scales presented in Table 1. These two scales were presented immediately after each statement, with X replaced by the name of the hypothetical student who made the statement.

<table>
<tr><td>To what degree is X's statement correct, in your opinion?</td><td>1
Not at all</td><td>2
Not very</td><td colspan="2">3
Idon't know</td><td>4
Fairly</td><td>5
Very</td></tr>
<tr><td>How close is X's statement to your way of thinking?</td><td>1
Not at all</td><td colspan="2">2
Not very</td><td colspan="2">3
Fairly</td><td>4
Very</td></tr>
</table>

Table 1: Scales used in the questionnaire

When piloting the questionnaires with the second scale only, we found that many students made their choice based on their judgement of correctness. We then inserted the first scale to ensure that students answered the second scale based on their way of thinking. This decision was successful as shown in Table 2. In Table 2, we included all statements in all 8 questionnaires, except the ones where the respondent chose "3 - I don't know" in the first scale. We also combined the negative choices (row and column "1 or 2"), and the positive ones (row "3 or 4" and column "4 or 5"). Approximately 26% of all responses in Table 2 were "off-diagonal", the vast majority of them (545 out of 599) being "correct but not my way"; off-diagonal choices were more frequent for Lina, Sapir, Vadim, Ron, and Ahmed than for Anna, Ariel, and Dino, presumably because the latter three statements made simple identifications between integral, area and antiderivative.

Correct / My way	1 or 2	4 or 5	No answer	Total
1 or 2	558	545	2	1105
3 or 4	54	1051	1	1106
No answer	1	13	68	82
Total	613	1609	71	2293

Table 2: Diagonal versus off-diagonal answers

The 8 Accumulation Calculation questionnaires were administered together with questionnaires on the other four fundamental concepts, 28 questionnaires altogether. They were administered to 725 students learning mathematics at the advanced level in grade 11 or in grade 12, after they studied integration of polynomial and trigonometric functions. The questionnaires were distributed randomly in each class during mathematics lessons. Each student answered two questionnaires in about 30 minutes. Time was not limited. 400 students answered Accumulation Calculation questionnaires.

SOME RESULTS

As mentioned in connection with Table 2, there are many fewer off-diagonal elements for Anna, Ariel, and Dino than for the other statements. This raises the question whether respondents identified more with these three statements than with the others? Table 3 answers this question for the pool context.

Statement	1 or 2	3 or 4	No answer	Total
Anna	28	66	4	98
Ariel	12	84	2	98
Lina	69	27	2	98
Sapir	87	8	3	98
Vadim	71	25	2	98
Ron	56	40	2	98
Dino (Q only)	7	41	0	48
Ahmed (I only)	29	18	3	50

Table 3: My way in Accumulation Calculation, context Pool (P)

Table 3 shows that many respondents to the questionnaire identify with Anna, Ariel, and Dino (75±9%); far fewer respondents (24±16%) identify with the other statements. This may be related to the classroom instruction they got. Typically, teachers might, in an introductory lesson, relate integral to area and show the area as a sum of rectangles (Ron), and in the following lessons mainly treat the computation of areas (Ariel) by definite integrals (Dino) using antiderivatives (Anna).

Differences between the other five statements are small. Respondents seem to identify with the answers of Lina and Vadim as much as with those of Ron, Sapir, and Ahmed.

We interpret this as showing that the respondents' reasons for identification are simplicity and familiarity rather than depth of understanding. In post-questionnaire interviews, many referred to the inefficiency of the computations needed by Lina, Sapir, Vadim, and Ron but only few were disturbed by Vadim's addition of values or Lina's addition of lines. With few exceptions, the picture is similar in the other contexts.

In pilot interviews, we noticed that students seemed to react differently in different contexts. This raises, for each statement, the question how many students identify with each statement in the different contexts and settings. In Table 4, we present respondents' identification with Anna's statement, in the different contexts.

Context / My way	Pool	Motion	Area	Formal
1 or 2	28	55	29	55
3 or 4	66	42	68	50
No answer	4	6	3	1
Total	98	103	100	106

Table 4: Identification with Anna's statement in the different contexts

Table 4 shows that close to 70% of the respondents identify with Anna's statement in the Pool and Area contexts, but only between 40% and 50% in the Motion and Formal contexts. We do not yet have an explanation for these results; in fact, given that students spend much time in class using antiderivatives to compute integrals in a formal context, it appears surprising that identification with Anna in the formal context is so low. Post-questionnaire interviews will be needed to understand these quantitative results.

Parallel questions could be asked for the other 7 statements (other than Anna). Many other questions could be asked about different frequency distributions, for example how respondents' answers depend on setting. Decisions what to examine are not easy.

We now turn to issues that can be investigated by means of contingency tables such as the one in Table 5, which compares Lina and Vadim. Based on Table 3, one might expect the distributions for Lina and Vadim to be rather similar since their statements both consider the integral or accumulated quantity as a sum of numbers rather than a sum of products, Lina graphically and Vadim numerically.

Vadim / Lina	1 or 2	3 or 4	No answer	Total
1 or 2	224	54	1	279
3 or 4	60	47	3	110
No answer	3	0	8	11
Total	287	101	12	400

Table 5: Integrals as graphical (Lina) versus numerical (Vadim) sums of numbers

Table 5 shows that most respondents (56%) do not think like Lina nor like Vadim. This is to be expected (see above, Table 3). More interesting is that among the remaining 44%, most identify with one of them but not with both. Respondents seem not to link between Lina's and Vadim's ways of thinking; here 'link' only refers to identification with ways of thinking, not to a mathematical connection. An only slightly different situation pertains with respect to the ways of thinking of Sapir and Ron.

Many additional comparison questions between statements could be asked using similar contingency tables. For Accumulation Calculation we have 7 meanings and hence 21 contingency tables, and separately for context and setting, we have 8×21=168. Comparisons between contexts and settings yield similar numbers of additional contingency tables, each of which may raise theoretically or didactically interesting issues. The question arises how to select which of these issues to examine.

The questionnaires were not intended to and cannot answer deeper qualitative questions about students' meanings, but they do raise such questions, for example:

- Given that so many more students identify with Ariel than with Ron (Table 3), what in their meaning for the area connects to the amount of water or distance?
- What causes many students (almost 30%) to either identify with Lina's way of thinking but not with Vadim's, or identify with Vadim's way of thinking but not with Lina's (Table 5)?
- The frequency, with which the respondents identify with a hypothetical student, depends on context. Can this dependence be explained by the meanings the respondents hold?
- Same question about dependence on setting instead of context.

We began carrying out post-questionnaire interviews, 109 so far, most of them before analysing the questionnaire results. One aim of these interviews was reverse validation, namely whether interviewees' meanings for the statements are indeed the ones we intended. While we cannot yet fully support this, we have no indications to the contrary.

These interviews have yielded results reported elsewhere. For example, Noah-Sella et al. (2023) interviewed 21 students who also study physics at the advanced level. Many of them brought up physics without being prompted. Several of them related mathematics to formulas, algebraic manipulations, and rote procedures as opposed to physics which they said emphasized understanding. This understanding is often related to graphical thinking: Students reason co-variationally with graphs and use them to solve problems. Such qualitative phenomena may lead us back to the quantitative data, to find out how frequent they are.

DISCUSSION

Our research has led to preliminary conclusions about students' meanings for the fundamental concepts of calculus, and more are expected when we will analyse the quantitative data more thoroughly and link it to qualitative data more systematically.

However, our research has also raised issues about doing mixed methods research on students' meanings in ways that maximize insights for large samples.

In terms of Options A and B, we have adopted both, but neither in the clean way described in the background section. We started with a catalogue of meanings and used the pilot interviews to refine that catalogue, as well as to decide on methods and formulations likely to evoke meanings. Similarly, we did not yet carry out a systematic qualitative investigation to explain the results produced by the quantitative analysis, but the post-questionnaire interviews already yielded qualitative results that ask for further quantitative investigation. We conclude that in complex cognitive-epistemological research, modifications of options A and B may lead to cycles of qualitative and quantitative stages, the design of each being determined by results of the previous one.

Our choices of which quantitative questions to ask, and which not to ask, have been informed by interviews and theoretical reflections but largely based on intuition. There is a need for a systematic yet manageable approach to the selection of educationally relevant questions. In terms of the introduction, although we have made the rationale for integrating qualitative and quantitative components in our project explicit (Choudhary & Jesiek, 2016), we have not yet succeeded in formulating a coherent collection of mixed methods research questions and linking them to mixed methods data analysis (Onwuegbuzie & Leech, 2006).

We expect that further methodological considerations will enable us to make the research educationally productive, and lead to results that inform the design of calculus instruction at the high school level so that it achieves two aims: Calculus as part of human culture that interrelates mathematics and the real world; calculus with meanings for the fundamental concepts that usefully prepare the students for their tertiary studies.

NOTES

1. In the formal context / quantity setting, the function $f(x)$ was introduced as the rate of change of a function $g(x)$ with $g(0) = 0$, and Alona claimed that using the given data, it is possible to estimate the value of the function $g(x)$ that accumulates from $x = 0$ to $x = 3600$.

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The study of functional equations to highlight the role of order in proof and proving at the interface between algebra and analysis

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Abstract: Functional equations are mathematical objects that are defined within the framework of algebra of functions (i.e. equation involving only the four arithmetical operations) for which the establishment of solutions most often requires recourses to methods of proof and proving specific to analysis. In this paper, we focus on the first Cauchy functional equation $f(x + y) = f(x) + f(y)$ *to highlight the role of order in such process, and to argue that the study of such functional equations is an effective means of simultaneously developing proof and proving skills and an understanding of the concepts involved when working with ordered sets of numbers, at university.*

Keywords: Teaching and learning of logic, reasoning, and proof; Teaching and learning of specific topics in university mathematics; Epistemology and didactics; Order in Mathematics; Cauchy functional equations.

INTRODUCTION

Functional equations are mathematical objects that are defined within the framework of algebra of functions (i.e. equation involving only the four arithmetical operations: addition, subtraction, multiplication, and division) for which the establishment of solutions most often requires recourses to methods of proof and proving specific to analysis. In this respect, working on these equations is relevant to address issues of interactions between algebra and analysis. In the cases that we will examine in this paper, we will highlight the possible role of order in this process. This presentation falls in a wider project consisting in making more visible the role of order in the French curriculum. Indeed, order plays a role in many areas of mathematics, as Sinaceur points out:

> A priori or by nature, the notion of order in mathematics is intrinsically neither geometric, although it is easily represented by the relation "to be situated between…", nor algebraic, although it is expressed by the relation of inequality, nor analytic, although it is implied in the notions of limit and convergence. It thus appears in contemporary mathematics as a transversal notion, present on many of the paths that link one discipline to another. (Sinaceur, 1992, p.115) [1]

In the French educational system, order is nearly never studied for itself neither at secondary level nor in early university courses. Moreover, in analysis, its role is often hidden by the recourse to limits for defining objects (such as Integral) or for proof and proving theorems (such as the Intermediate value theorem).

The main goal of this paper is to develop an epistemological analysis to address the following research question: "In which respect is the first Cauchy functional equation $f(x+y) = f(x) + f(y)$ a good candidate for shedding light on the possible role of order at the interface between algebra and analysis?". Such analysis is a first step for developing a didactical engineering (Artigue, 1991). In a preliminary section, we briefly remind the definition of irrational numbers by Dedekind who, according to Sinaceur (1992) "reduced continuity to order". In a second section we present some epistemological issues concerning functional equations in Cauchy (1821) by Jean Dhombres. In a third section, we focus on discontinuous solutions of the first Cauchy functional equation $f(x+y) = f(x) + f(y)$. In a fourth section, we present three proofs that solutions of this Cauchy equation in the class of monotonic functions are linear functions to enhance the relevance of this functional equation to put on the scene with undergraduates the role of order at the interface between algebra and analysis. Finally, we discuss didactic implications and present briefly a forthcoming experiment.

REDUCTION OF CONTINUITY TO ORDER: DEDEKIND'S CREATION OF IRRATIONAL NUMBERS

The title of this section is borrowed from Sinaceur (1992) who claims in a section entitled "Reduction of continuity to order" that:

> It was undoubtedly Dedekind who, by wishing to provide "a purely arithmetical and perfectly rigorous foundation for the principles of infinitesimal analysis", highlighted the structure of the ordered set of $\mathbb{R}$. For him, this means finding a true definition of the nature of continuity. (ibid, p.110) [3].

We briefly remind here that Dedekind (1872) defines a cut in the set of rational numbers as a pair (A, B) such that $A \cup B = \mathbb{Q}, A \cap B = \emptyset$, and $\forall\, x \in A, \forall\, y \in B, x \leq y$. After having shown that there are infinitively many cuts that are not operated by a rational number (which he names its incompleteness), he defines completeness as the property that every cut of a given ordered set be operated by an element of the set. Consequently, to complete the set of rational numbers, for every cut not operated by a rational, he creates a new number, an *irrational* one. He then proves that the new set is a complete (in the sense above) ordered set. Based on this construction, the definition of a least upper bound (a supremum) comes: an upper bound for a given subset M of an ordered set E is an element that is greater than or equal to any element of M. A least upper bound for M, if it exists, is the smaller among the upper bounds of M. Given a cut (A, B) in the sense of Dedekind, the unique element operating the cut, if it exists, is the supremum of the subset A, and the infimum of the subset B. In a complete ordered set E, every bounded above (resp. below) subset of E admits a supremum (resp. infimum) that is unique. Due to this close relation between cuts and supremum (resp. infimum), it is not seldom that they are used concomitantly in a proof: creating a cut, assuming the existence of the supremum (resp. the infimum), showing that this element is a candidate to have the desired property, and proving it with order consideration. An example can be found below in this paper for the second and third proofs that “if a solution of the Cauchy equation $f(x+y) = f(x) + f(y)$ is monotonic, then it is a linear function”.

Before moving to the next session, we would like to remind that the property of density (in-itself) of an ordered set is an important issue when considering the elaboration of the theory of real numbers: between the discrete set of integers, and the continuous set of real numbers, there is, among others, the dense (in-itself) incomplete set of rational numbers; the dichotomy discrete-continuous does not capture the mathematical fact that there are dense ordered sets that are not continuous (Durand-Guerrier, 2016). Note that the property “To be dense (in-itself)” for an ordered set is different of the relation “To be dense in…” between a subset of an ordered set and this set.

THE ROLE OF FUNCTIONAL EQUATIONS IN CAUCHY’S ALGEBRAIC ANALYSIS

Jean Dhombres, a French historian of mathematics, studied functional equations as a mathematician at the beginning of his career and published with J. Aczel a treatise (Aczel & Dhombres, 1989) which deals with modern theory of functional equations in several variables and their applications to mathematics, information theory, and the natural and social sciences. In a paper published in 1992, he examines the role of the four fundamental functional equations studied by Cauchy in the first part of his *Course of Analysis of the Ecole royale Polytechnique* (Cauchy, 1821). In a chapter of his course entitled “*Determination of a continuous function of a single variable verifying certain conditions*” [4], Cauchy treats simultaneously the four functional equations conserving or exchanging addition and multiplication:

(A) $\Phi(x+y) = \Phi(x) + \Phi(y)$ (B) $\Phi(x+y) = \Phi(x)\Phi(y)$

(C) $\Phi(xy) = \Phi(x) + \Phi(y)$ (D) $\Phi(xy) = \Phi(x)\Phi(y)$

The first one (A) is the Cauchy equation that we will study in the next sections. In the class of continuous functions considered by Cauchy, the solutions of (A) are the linear functions; those of (B) are the exponential functions; those of (C) are the logarithmic functions composed with the absolute value, and those of (D) are the power function with arbitrary real exponent composed with the absolute value. The resolution by Cauchy of the first functional equation (A) is made in two times: first algebraic manipulation leading to the form of the solutions defined on the set of rational numbers $\mathbb{Q}$; second using the fact that $\mathbb{Q}$ is dense in the set of real numbers $\mathbb{R}$ and the continuity of the searched functions, he proves that the only solutions defined and continuous on $\mathbb{R}$ of equation (A) are the linear functions.[5] Although the functional equation had already been studied before Cauchy, Dhombres (1992, p.28) underlines the novelty and the fecundity of this method that Cauchy then successfully applied to equations (B), (C) and (D), and allow him to solve completely in the class of continuous functions the functional equation: $\Phi(x+y) + \Phi(x-y) = 2\Phi(x)\Phi(y)$, using the density in $\mathbb{R}$ of the set of dyadic numbers. Considering this, Dhombres claims that relying on the set of rational numbers for solving the first four equations was motivated by the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, as is the set of dyadic numbers with the standard order. This allowed him to fully justify, in the case of continuous functions, that the form established for rational numbers holds for real numbers, that was previously, and even later often taken for granted by mathematicians. Dhombres, at the beginning of the paper, wondered why Cauchy paid attention to the four functional equations (A), (B), (C) and (D). In the conclusion, he considers that for Cauchy, they were only a transitory step, not goal in themselves (ibid, p. 48). He also pointed the relevance of solving these equations in a delimitated class, here the class of continuous functions. This choice provides the regular solutions that we are used dealing with at the secondary-tertiary transition, with proofs that are accessible at this level. Considering here the class of continuous functions, Cauchy embeds the solutions in the domain of analysis. It seems that Cauchy did not search solutions in class of functions else than the continuous ones. This will be done later by G. Hamel in a paper published in 1905, that we present in the next section.

DISCONTINUOUS SOLUTION OF THE CAUCHY FUNCTIONAL EQUATION: $\boldsymbol{f(x+y) = f(x) + f(y)}$

In a paper published in 1905, Hamel considered discontinuous solutions of the Cauchy functional equation $f(x+y) = f(x) + f(y)$ (A). It is known since Cauchy that looking for continuous solutions, the solutions are the linear functions. In addition, it is easy to prove that if the solutions are searched among functions defined on the sets of rational numbers, then the solutions are linear functions in form $f(x) = Kx$ without any additional hypothesis on the functions. The question raised by Hamel is: "And what happens if we don't assume that the solutions defined on the set of real numbers are necessarily continuous functions?". In his paper of 1905, Hamel proves the existence of discontinuous functions solutions of the Cauchy equation (A); he did this by introducing a basis for the real numbers (named today Hamel Basis) that in modern terms would be expressed as: "the set $\mathbb{R}$ of real numbers is a linear space over the field $\mathbb{Q}$ of rational numbers" (Aczel & Dhombres, 1989, p.19). Moreover, Hamel establishes that such functions are totally discontinuous:

> Each of these discontinuous solutions of the functional equation is totally discontinuous; in any neighbourhood of any point of the (x, y)-plane there are points of the "curve" $y = f(x)$ [6]. (Hamel, 1905, pp.461- 462)

A consequence of this theorem is that when considering a graphical representation on a real interval of a discontinuous solution of (A) (that is not in the form $g(x) = Kx$), given a point of the plan with a rational abscissa α, and an ordinate different of $K\alpha$ there will be points of the graphical representation in every neighbourhood of this point. Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, on the graphical representation, there will not be only one point that will appear on the vertical line corresponding to the point of rational coordinates $\big(\alpha, g(\alpha)\big)$ that lies on the line with equation $y = Kx$.[7] More precisely the graph of such a totally discontinuous solution is dense in $\mathbb{R} \times \mathbb{R}$. In Durand-Guerrier et al. (2019), we analyse a similar phenomenon in the case of the functional equation for exponential, which we show relevant for a discussion with undergraduates on the $\mathbb{Q}$-incompleteness versus the $\mathbb{R}$-completeness, and related issues with graphical representations. Relying on this experience, we hypothesise that the Cauchy functional equation (A) would be a good candidate for designing an activity at the secondary-tertiary transition and in teacher training program aiming at shedding light on the crucial role of completeness/incompleteness of the standard order on the numbers sets at the interface between algebra and analysis.

In the next section, we focus on proofs that every monotonic function with domain of definition $\mathbb{R}$ and solution of the Cauchy functional equation (A) is a linear function.

THREE PROOFS THAT EVERY MONOTONIC FUNCTION SOLUTION OF THE CAUCHY FUNCTIONAL EQUATION (A) IS A LINEAR FUNCTION.

Hewitt and Zuckerman (1969, p.121) underline that a consequence of the theorem above established by Hamel is that: "If f satisfies (A) [8] and is continuous at some point, or is bounded above or below on some interval, then $f(x)$ has the form kx."

It is also the case if f is monotonic (Aczel & Dhombres, 1989, p.15).

Theorem: *if a function defined on* $\mathbb{R}$ *satisfies equation* (A) *and is monotonic on* $\mathbb{R}$*, then there exists a real* k *such that* $\forall x \in \mathbb{R}\ f(x) = kx$.

We provide below three proofs of the theorem above shedding light on the role of order in the study of the Cauchy functional equation (A). The first proof relies on the fact that every real number is the limit of a pair of adjacent rational sequences; the second and the third ones on the definition of the set of real numbers by Dedekind's cut method. The proofs are done in the class of *increasing functions from* $\mathbb{R}$ *to* $\mathbb{R}$; in the three proofs, f denotes a function of this class.

Proof 1, with adjacent rational sequences

Given a real number α, u and v two adjacent rational sequences converging to α, with u an increasing sequence and v a decreasing sequence with $u \leq v$, we have:

$$\forall n \in \mathbb{N},\ u_n \in \mathbb{Q},\ \wedge\ v_n \in \mathbb{Q}\ \wedge\ u_n \leq \alpha \leq v_n \text{ and } \lim_{\infty} u = \lim_{\infty} v = \alpha$$

A f is increasing on $\mathbb{R}$, $\forall n \in \mathbb{N}\ f(u_n) \leq f(\alpha) \leq f(v_n) (*)$

As $\forall n \in \mathbb{N}\ u_n \in \mathbb{Q}, v_n \in \mathbb{Q}$, then $(\forall n \in \mathbb{N}\ f(u_n) = u_n f(1)) \wedge (f(v_n) = v_n f(1))$

Then we have: $\forall n \in \mathbb{N}\ u_n f(1) \leq f(\alpha) \leq v_n f(1)$ (**)

As u_n and v_n converge to α, and $\forall n \in \mathbb{N}\ u_n \leq \alpha \leq v_n$, we have:

$$\alpha f(1) \leq f(\alpha) \leq \alpha f(1) \text{ (***)}$$

Finally, we conclude that $f(\alpha) = \alpha f(1)$, from which follows:

$\forall x \in \mathbb{R}\ f(x) = x f(1)$, i.e. f is linear.

Proofs 2.1 & 2.2, using the Dedekind' s cuts.

Given a real number α, there is a cut (A_1, A_2) of the set of the rational numbers for which α is the only real number operating this cut; i.e. α is the supremum of A_1 and the infimum of A_2.By the definition of A_1 , A_2 et α, $\forall x \in A_1\ \forall y \in A_2\ x \leq\ \alpha \leq y$ (*)

Proof 2.1.

Given $b \in A_1$ and $c \in A_2$ we have $b \leq\ \alpha \leq c$ (from *); then, as f is increasing:

$$f(b) \leq f\ (\alpha) \leq f(c)(**).$$

As $A1$ and $A2$ are subsets of the set of rational numbers, we have:

$$f(b) =\ bf(1) \wedge f(c)\ =\ cf(1)$$

Then, by substitution in (**), we have $bf(1) \leq f(\alpha) \leq cf(1)$(***)

1st case: $f(1)\ =\ 0$; then $f(\alpha) = 0$ hence $\forall x \in \mathbb{R}, f(x) = 0$.

2nd case: $f(1) >\ 0$ [9]; by dividing by $f(1)$ in *** we have $b \leq \frac{f(\alpha)}{f(1)} \leq c$ ****

We deduced that $\forall x \in A_1\ \forall y \in A_2\ x \leq\ \frac{f(\alpha)}{f(1)} \leq y$ *****

This proves that $\frac{f(\alpha)}{f(1)}$ is operating the cut (A_1, A_2).

Because there is a unique real number operating a cut, we conclude that:

$\frac{f(\alpha)}{f(1)} = \alpha$, and finally, $f(\alpha) = \alpha f(1)$, from which follows: $\forall x \in \mathbb{R}\ f(x) = xf(1)$, i.e. f is linear.

Proof 2.2

We first prove that $(f(A_1), f(A_2))$ is a cut of $f(\mathbb{Q})$ operated by $f(\alpha)$.

Let us consider $e \in \mathbb{R}^{+*}$.

As (A_1, A_2) is a cut of $\mathbb{Q}$, there exist $x \in A_1$ and $y \in A_2$, such that $o \leq y - x \leq \frac{e}{f(1)}$

Let us consider c and d two such elements.

From $0 \leq c - b \leq \frac{e}{f(1)}$, and $f(1)\ > 0$,we get: $0 \leq cf(1) - bf(1) \leq e$;

as $b \in A_1$, $f(b) = bf(1)$; as $c \in A_{2,}$ $f(c) = cf(1)$; then we have: $0 \leq f(c) - f(b) \leq e$. It follows that:

$$\forall \varepsilon \in \mathbb{R}^{+*} \exists w \in f(A_1)\ \exists z \in f(A_2), o \leq z - w \leq \varepsilon$$

This proves that: *($f(A_1), f(A_2)$)* is a cut of $f(\mathbb{Q})$ (*)

As α is operating the cut (A_1, A_2), and f is an increasing function, we have:

$$\forall x \in A_1 \forall y \in A_2\ f(x) \leq f(\alpha) \leq f(y)$$

By definition of $f(A_1)$ and $f(A_2)$, we have: $\forall w \in f(A_1)\ \forall z \in f(A_2\ )\ w \leq f(\alpha) \leq z$

This proves that $f(\alpha)$ operates the cut $(f(A_1), f(A_2))$(**)

From (*) and (**) we conclude that $(f(A_1), f(A_2))$ is a cut of $f(\mathbb{Q})$ operated by $f(\alpha)$.

We now prove that $\alpha f(1)$ is also operating the cut.

As A_1 and A_2 are subsets of the set of rational numbers, we have:

$$\forall x \in A_1 \ \forall y \in A_2 \ f(x) = xf(1) \wedge f(y) = yf(1)$$

Given $d \in f(A_1)$, and $b \in A_1$ such that $d = f(b)$ we have $d = bf(1)$, and given $h \in f(A_2)$ and $g \in A_2$ such that $h = f(g)$ we have $h = gf(1)$.

As $f(1) > 0$, and $b \leq \alpha \leq g$, we have $bf(1) \leq \alpha f(1) \leq gf(1)$.

From which follows: $d \leq \alpha f(1) \leq h$ and finally:

$$\forall z \in f(A_1) \ \forall w \in f(A_2) \ \ z \leq \alpha f(1) \leq w.$$

This last assertion means that $\alpha f(1)$ is operating the cut $(f(A_1), f(A_2))$ (***)

Thanks to the uniqueness of the real number operating the cut, we conclude that:

$f(\alpha) = \alpha f(1)$, from which follows: $\forall x \in \mathbb{R}\ f(x) = xf(1)$, i.e. f is linear.

In these three proofs that *any increasing function solution of the functional equation* (A) *is a linear function*, the role of order is highlighted. In the second and third proofs, we refer only to properties related to order, without involving limits of sequences This is an illustration of the claim by Sinaceur that Dedekind reduced the continuity [of the set of real numbers] to order.

DIDACTIC IMPLICATION

From the above, there are two main points of interest from our didactic perspective.

The first concerns the important and surprising result that there are totally discontinuous functions among the solutions of the functional equation (A). In university courses, when such functions are introduced, it is common for the professor to give examples that, for the students, seems to be constructed *for this purpose*, except for the Dirichlet function, the indicator function of $\mathbb{Q}$ in $\mathbb{R}$, whose usefulness can be easily demonstrated. Such a presentation does not highlight the rationale for considering totally discontinuous functions, which might appear as pathological monsters, that should be relegated, as suggested by Lakatos (1976). However, our experience with the case of the Cauchy functional equation (B) whose continuous solutions are exponential functions, shows that this provides a rich opportunity to highlight the role of completeness/incompleteness, and allow graphical proofs to be questioned, justifying Bolzano and Dedekind's concerns that geometry-based proofs are not appropriate when moving on to analysis (Durand-Guerrier, 2022a). Starting with equation (A), instead of equation (B) could allow the emphasis to be placed on the topological properties, because the algebraic calculations are easier.

The second is that when solving the equation in the class of monotonic functions, the solutions are necessarily linear. In the French syllabus, this kind of results are seldom taken into consideration. This, together with the usual practice of working mostly in the set of real numbers, leaves in the shadow the role of order and the topological properties relevant to ordered sets (completeness/incompleteness; connectivity/non-connectivity; compactness/non-compactness, etc.). This is likely reinforced by the usual practice in first-year university courses of giving privilege to the following characterization of the Supremum:

$$(\forall x \in F, x \leq M) \ \wedge (\forall \varepsilon > 0 \ \exists x \in F, M - x < \varepsilon)$$

and the corresponding sequential characterisation, which favour a point of view linked with limits. Although these characterisations are useful in many cases, for some proofs it may be more efficient to use the definition of Supremum (resp. Infimum) as the minimum (resp. maximum), if any, of the upper bounds (resp. lower bounds). In Durand-Guerrier (2016) we report the case of Master students in a teacher training program in France working on a fixed-point theorem for an increasing function, who initially thought the continuity of the function in the interval $[0,1]$ of domain $\mathbb{R}$ was a necessary condition. Once they realised that this was not the case, they looked for a proof using the sequential characterisation; none of them search for a proof consisting in considering the supremum of a well-chosen subset as a candidate for a fixed point and proving that this is the case. This proof is efficient and holds as soon as we are in a complete lattice (Tarski, 1955). This is not to say that proofs using the sequential characterisation should be replaced by proof using the definition; rather, we consider that multiple proofs activities should be proposed and discussed with students at the secondary-tertiary transition and in teacher training programs as an efficient means of simultaneously increasing skills in proof and proving, as well as understanding of concepts. (Durand-Guerrier, 2022b). This is particularly important in the case of order which, as mentioned above, is a transversal notion at the interface of several areas of mathematics (e.g. Algebra, Geometry, Analysis, Combinatorics, etc.).

A FORTHCOMING EXPERIMENT WITH CAUCHY EQUATION (A)

The second author of this paper has for years proposed activities based on functional equations, including the Cauchy equation (A). Naturalistic observations support the conjecture of their relevance to address some of the issues developed in the previous sections regarding order. The next step is to design an experiment around the Cauchy functional equation (A) to test our conjecture. The population we will consider for this experiment will be made of small groups of volunteer students following a teacher training program in different contexts (third year university, master's degree, preparation to the French *Agrégation*), leaving for other experiments the suitability for the secondary-tertiary transition. This choice is based on the hypotheses that working on the Klein's second transition for these students moving from university to secondary education shed light on the transition from secondary to tertiary education (Winsløw & Grønbæk, 2014). The experiment is planned in the spring fall 2024. We will follow the methodology of didactical engineering (Gonzales-Martin & al. 2014), with an initial open question as:

The goal is to solve the functional equation $f(x+y) = f(x) + f(x)$ under various hypothesis on the domain and the property of the function. You are asked to formulate your hypotheses and to prove the assertions done under these hypotheses.

Our didactic organisation will comprise two sessions. The first will consist of a period of individual research followed first by a discussion in small groups and then by a group discussion. We will collect the questions and the answers produced by students, both written and oral during this session. The second will be collaborative work in small groups, starting with a few questions that did not emerge during the first session to carry out specific work on the concepts of continuity, completeness, and monotony. We will also conduct interviews with students having participated at the two sessions.

CONCLUSION

In this paper we provide motivations for studying functional equations as a means of highlighting the role of order in proof and proving at the interface between algebra and analysis. We show that even the simplest functional equation has unexpected solutions in the set of real numbers as soon as we look for solutions without assuming continuity of the functions, whereas the solutions in the set of integers or of rational numbers are exactly what we expect, i.e. linear functions. We then give three proofs, one using sequences, the two others using Dedekind's cuts, that solutions in the class of monotonic functions are linear. We consider that, from a didactic perspective, this highlights the relevance of introducing multiple proofs activities at university as a means of simultaneously developing proof and proving skills and an understanding of concepts involved. In the case of ordered sets, we consider that this could contribute to a better appropriation by undergraduate students of the general topological concepts that they will encounter later, and which are known to be difficult. A forthcoming experiment aims at testing our hypotheses will be designed in the spring fall 2024.

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1.Our translation from French.
2.Our translation from French.
3.Our translation from French.
4. We will discuss below what happens if we do not impose to the function to be continuous.
5.Our translation in English from German: Jede dieser unstetigen Lösungen der Functional gleichung ist total unstetig; in jeder beliebigen Nähe eines jeden Punktes der (x, f) −Ebene liegen Punkte de "Kurve" $f = f(x)$.
6.Since the restriction of the function on the set of rational numbers is in all cases of the form $g(x) = Kx$, the point of coordinates (α, g(α)) with α rational are on the straight line with equation y = Kx, whatever the solution continuous or discontinuous.
7.In the original text, the author refers to this equation by (1). For being homogeneous along the text, we changed (1) in (A) everywhere.
8. For f an increasing function solution of the Cauchy functional equation (A), $f(1) \geq 0$.

A small intervention: students' self-work sessions in calculus lectures

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This study describes the effects of a small intervention, performed within the framework of participatory action research, in first semester calculus at an engineering college. The project is a collaboration between a pure mathematician and a mathematics education researcher, to study the effects of students' self-work (SW) on students' achievements, self-efficacy, learning habits and classroom discussion. In all classes, students were given quizzes with peer instruction; in some, they were also given 10' SW sessions. Both changes can be easily incorporated in large, coordinated courses. Multiple forms of data were collected: quiz results, questionnaires, exam questions, and reflections. The findings show that in groups with SW students' engagement, self-efficacy and achievements on the final exam improved, whereas failure rate lowered.

Keywords: Teaching and learning of analysis and calculus, Teachers' and students' practices at university level, Novel approaches to teaching.

BACKGROUND

For over a decade, mathematics educators have advocated departing from lecturer-centred pedagogies and to "…reform collegiate mathematics teaching in a way that aligns with… more student-centred approaches" (Vroom et al, 2022, p. 2). Alternative approaches, particularly active learning, have been shown to be beneficial (e.g., Crouch and Mazur, 2001; Freeman *et al*., 2014). Yet, the traditional lecture remains dominant in undergraduate mathematics courses (Melhuish et al., 2022; Vroom et al., 2022). The reluctance to drop traditional lecturing was probed by a few scholars (Dawkins & Weber, 2023; Pritchard, 2010; Vroom et al., 2022) who claim that mathematicians believe that lecturing has been satisfactory in many aspects and has many merits (e.g., immediate students' feedback, introducing students to disciplinary thinking, slowing down the pace of "doing mathematics"). However, the lecturers express a sincere desire for an engaging and active classroom environment (Woods & Weber, 2020).

Most instructors need to consider institutional and curricula constraints, contradictory departmental contexts, the physical barriers of the classroom, or even concerns about student evaluations (Vroom et al., 2022). Those teaching a coordinated course also suffer under the "tyranny of content" (Kensington-Miller et al., 2013), and feel time pressure to cover the same material as other instructors. Finally, adopting a new pedagogy requires a considerable investment of time and effort, occasionally requiring a skill set that research mathematicians do not possess or wish to promote, especially with no guarantee for improved learning outcomes. Goodchild (2023) suggests "[mathematicians] want substantive and practical suggestions to address the issues they experience with students' learning. They want empirical evidence for the effectiveness of interventions…". Accordingly, Dawkins and Weber (2023) suggest alternative formats for classroom innovations, e.g., designing small pedagogical modifications to the lecture format, that consider mathematicians' beliefs on lecturing and are easier to apply. They recommend studying the effects of such modifications on cognitive and affective aspects of students' learning.

Although there is some scepticism regarding radical reforms (such as IBL or flipped classroom), active learning is widely acknowledged as beneficial, and most lecturers believe it increases students' engagement. Freeman et al. (2014) defines active learning as anything that "engages students in the process of learning through activities and/or discussion in class… It emphasises higher-order thinking and often involves group work" (p. 8413-8414). The meta-analysis performed by Freeman et al. (2014) firmly supports the claim that active learning practices in STEM education leads to substantial increase in examination grades and reduces the failure rate.

Students' engagement is also related to affective aspects such as motivation and self-efficacy, defined as an individual's belief in their ability to reach goals (Bandura, 1977). Bandura writes that the best way to build self-efficacy is to engage in experiences that build mastery of a concept. Students with low self-efficacy can be motivated to try harder, but this requires extrinsic motivation. Ponton et al. (2001) write about the importance of self-efficacy in engineering education and suggest providing students with more "mastery experiences" to increase it. Goodchild (2023) agrees that "students need to be given problems in which they are likely to experience success and a sense of personal achievement and growth when the problem is solved" (p. 91).

The study described here concerns a modification in a first semester calculus course. Both authors are calculus lecturers, who perceived their students as relatively passive learners and wanted to increase students' engagement and to improve students' learning habits, particularly their preparation for the lesson. The paper describes the effect of using Self-Work in some groups, reports initial findings and their implications.

RATIONALE AND RESEARCH QUESTIONS

This study was conducted in a first semester calculus at an engineering college in Israel. About 800 students are split into 16 lecture groups. The syllabus is fixed and the final exam is common, therefore, rigid time and content constraints apply. The authors taught 5 (of 16) lecture groups and were also the designers and performers of the study. This context naturally lends itself to 'participatory action research' (PAR) framework, as it addresses "close-to-practice research involving teachers and researchers working together to address problems in practice" (Wright, 2021, p. 160), performing actions and reflecting on it. PAR recognizes the advantage of combined work of academic researchers and teacher researchers, who are well acquainted with the classroom environment; it aims to generate practical knowledge, accounting for teachers' perspectives, the challenges they face and the opportunities they encounter during their work. PAR is carried out within teachers' own classrooms and involves critical reflections over a long period of time. Thus, Wright (2021) claims, findings may be more relevant and applicable to other classroom situations.

The authors hypothesised that if students arrived better prepared for the lessons, (e.g., by reviewing the previous lesson and practising exercises) it would enable them to participate more during the lesson and improve their self-efficacy. The authors designed the pedagogical modification aimed to have a substantial impact on student experience, while leaving course content intact, and changing the use of class time only a little. The authors added online quizzes with peer instruction (following Crouch and Mazur, 2001) to all classes (see below Table 1). In some classes they added self-work (SW) sessions: a weekly habit of solving an exercise alone, without discussion, for 10 minutes. Hence, the research questions examined in this study are:

1. Did the peer instruction activity affect students' achievements in the course?
2. In what ways does adding self-work affect: (a) students' engagement during the lesson; (b) students' communication through written work; (c) students' self-efficacy concerning mathematical content; and (d) students' problem-solving abilities?

The instructors taught in their usual style, the only difference between the groups was the weekly SW sessions. This study compares differences between the groups and not between students' pre/post course status. The SW questions and implementation were adapted to the course and studied as they were initiated (fitting the PAR framework).

METHOD

Table 1 presents the 5 groups taught by the lecturers. Students have lecture four hours per week, and a two-hour recitation section with TAs, who also give homework.

Group	Industrial	Comp. Sci.	Bio-Medical	Electrical	Software	Total of 252 (258) students
~N students	25 (26)	52 (57)	59 (54)	58 (66)	58 (55)	
Self -Work	With	With	Without	Without	With	With 135(138) Without 117 (120)

Table 1 Breakdown of lecture groups, with N at the start and (end) of the semester

Active learning was incorporated in the lectures in two ways. During the semester all groups were given four capstone problems that students solved through peer instruction, using an online quiz app called MathMatize: (a) students were asked to solve the problem on their own and submit their answers anonymously; (b) volunteers were asked to explain their answers, and (c) students were asked to solve the same problem again after peer discussion, followed by a class summary. After each quiz, the students were asked to evaluate their technical and conceptual understanding of the relevant mathematical material, their ability to solve the problems by themselves and after group discussion. At the end of the semester, a final review session was given completely through peer instruction, where students worked for two hours and solved 5 problems (see Online Resource 1 - MathMatize questions).

Three of the five groups practiced weekly *self-work* (SW) sessions. Two groups (one for each lecturer) were control groups (see Table 1). The lecturers wrote questions on the board, and the students were asked to work on them by themselves for 10 minutes. Some problems were of a technical nature (e.g., computing a limit) and some were more conceptual (e.g., deciding if a claim was true). Some questions were based on previously taught content (thus encouraging students to go over their notes before the lesson) and some problems were designed to scaffold learning of new material, thus making students' ability to solve them relevant for their work and understanding of the current lesson. Often the SW was just to do another standard example that the lecturer would normally have done anyway (see Online resource 2). At the end of the semester, a survey was given with the final MathMatize review. Both instructors wrote a weekly reflection to keep track of their impressions over the semester.

The findings are based on students' answers in the online quizzes and questionnaires, students' grades in the final exam and analysis of answers to questions on the final exam. The findings are supported by evidence from lecturers' weekly reflections and selected students' quotations from conversations and free responses on the final survey.

Analysis of exam questions

Two final exam questions were chosen for analysis, one concerning the intermediate value theorem and the other relates to inverse functions. A rubric was developed to evaluate the quality of student logic and argumentation and the quality of student written communication. Both aspects were graded on a scale 1-4 (1 - below standard, 4 - exceeds standard). The rubric was used to grade questions experimentally; then it was discussed and rewritten until both authors scored responses the same way. Then, each author graded one question for all students who answered it, in order to preserve grading uniformity. Online Resource 3 presents the two exam questions, the rubric, and translated examples of student answers together with explanations. To avoid bias, each author graded a question that she did not grade in the actual exam, and the exam questions were graded blinded to student identity or lecture group.

Analysis of lecturer reflections

Both lecturers/authors wrote weekly reflections during the course. The SW questions were documented together with their impact, students' behaviour during the SW sessions, the lecturer's perception of classroom dynamics and quality of discussion. Notable student comments were also recorded, as well as conclusions for future lessons. At the end of the semester each instructor summarised her own reflection (see Online Resource 4) and recurring themes in both summaries were analysed.

To counteract the potential bias in lecturers' reflections, the reflections were summarised independently, and then similar themes were extracted. In addition, other forms of data were collected from students. However, the author/lecturer duality has also an advantage since both authors were fully invested in the process, could adapt the pedagogical change, and troubleshoot challenges to suit the needs of their course. In fact, this is considered an advantage according to the PAR framework (Wright, 2021).

FINDINGS

Peer instruction

Table 2 shows the results of the four online quizzes (sequences, continuity, derivatives, integrals) and the final review. The first four columns show the results of groups with/without self-work (SW) in Rounds 1-2, where in round 1 students solved the problem alone and in round 2 with peers. The final two columns show the percentage of improvement between the two rounds. The groups with self-work seemed to be better at answering the questions on average in Round 1 or Round 2. These groups also had a greater average amount of improvement between the two rounds. It seems that the groups with self-work benefited more from the peer instruction activity. This trend is less noticeable at the beginning of the semester, where the classroom norms and students' habits are not yet set. However, later in the semester, the students in the self-work groups improved faster. This is supported by the rest of the data below.

Percentage of correct answers	**Round 1**		**Round 2**		**Improvement**	
	With SW	Without SW	With SW	Without SW	With SW	Without SW
Sequences	36.7	30.3	41.3	35.6	4.5	5.4
Continuity	40	29.3	75.1	60	35	30.7
Derivatives	64.3	65.5	85.7	88	21.3	22.5
Integrals	2.6	0	32.8	21	30.3	21
Final-1	29.2	45.8	73.1	46.2	43.9	0.5
Final-2	26.6	45.5	72.7	54.8	46.2	9.4

Table 2 Summary of results on MathMatize questions

On each quiz, students were asked if they felt able to answer the problem on their own, and if class discussion helped. Table 3 presents the percentage of positive answers.

Percentage of positive answers	**Do Alone**		**Discussion helped**	
	With SW	Without SW	With SW	Without SW
Sequences	32.9	15.1	59.4	58.1
Continuity	29.5	7.1	73.2	35.7
Derivatives	49.3	46.3	57.7	44.4
Integrals	21.7	14.3	64	66.7

Table 3 Student experiences during peer instruction

The groups with SW felt that they are better at solving problems on their own than the groups without SW, which reflects higher self-efficacy. The SW groups also reported that the peer discussion helped them more. Thus, in general, students in the SW groups felt that the activity was more beneficial to their learning.

Final exam achievements

Table 4 shows the failure rate and average grade on the final exam for all groups. Both groups earned a similar average, slightly above the average of the groups that had other lecturers. However, the failure rate for students with SW is lower than without SW and lower than the other lecture groups.

	With SW	Without	Course Total	Other groups
Final Avg	59.8	59.8	58.2	57.5
% Failure	41.6	45.5	45.4	46.2
N	127	111	790	552

Table 4 Achievement in final exam

Finally, Figure 4 shows the grade breakdown for the entire course.

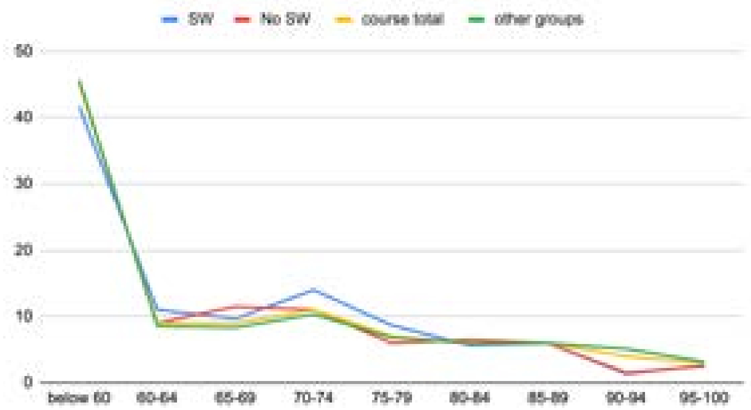

Figure 4 Grade Breakdown in the Course

The x-axis shows a grade range, and the y-axis shows the percentage that received that grade. The students without SW (red) follow the course averages (yellow) very closely, and those of the other course groups (green). The students with SW (blue) have a lower failure rate, and more students earning a lower passing grade, particularly in the grade range 70-80. The students with only peer learning (red), differ from the rest of the course (green) only at the range 60-65, where more students with peer learning barely passed. This corroborates results of Freeman (2014), that active learning helps students on the verge of passing.

A Closer Look at Exam Questions

Student's answers to two questions from the final exam were analysed; two aspects were evaluated using the assessment rubric (Online Resource 2): quality of logic and argumentation (L) and quality of written communication (W). Table 5 presents the average grades. The students with SW scored better on the inverse problem, and the opposite on the IVT problem. Since SW seemed to help on one of the problems and the opposite on the other, no conclusions can be drawn. However, the overall exam grade was improved by SW (Figure 4). Further analysis of exam problems is needed.

	Inverse function			**IVT**		
	N answered	L	W	N answered	L	W
With SW (N=127)	82 (64.6%)	0.8	0.8	56 (44.1%)	1.8	1.7
Without SW (N=111)	74 (66.7%)	0.6	0.7	53 (47.7%)	2.2	2

Table 5 Average grades in exam (L=Logic & argumentation, W=Written comm.)

Lecturer reflections

Both lecturers reported that initially, they struggled to get students to work during SW sessions, and that it took several weeks to develop the classroom culture. At the beginning of the semester, both instructors considered giving SW up because they weren't sure if it was worth the use of valuable class time (the sessions lasted longer than the planned 10 min.), but over the course of the semester the students accepted the SW as part of the lesson and tried seriously to solve the given questions. By the end, both lecturers were convinced of the value of SW. The class culture also shifted. In the fifth week of the semester, for example, one of the lecturers wrote an exercise and just before starting to solve it students asked whether they should start by themselves. This would have never come up in a classroom without self-work.

The lecturers themselves went through a process of action research and felt that if they used SW again, they would be better at choosing the questions, and motivating students to try SW. This is because they would be able to communicate their confidence in SW to students and incorporate SW more naturally into the lessons. The lecturers identified the type of questions that best fit their goals: questions that are good concept review, a computation similar to ones just presented in the lecture, or examples that preview something that will be covered in the lesson. Both authors believe that if chosen well, SW questions can actually *save* time. Both lecturers found that class discussions were more focused after using SW, and the students asked deeper questions. More students came to talk to the lecturers after active learning activities, during the breaks for example, and students seemed to have a better grasp of the material. However, there were also students who just sat there and refused to try and waited for the "real" lesson to start. Finally, both lecturers felt that it gave students feedback regarding their understanding of the material and gave them insight into what the class was struggling with.

DISCUSSION AND CONCLUSIONS

First semester calculus courses are usually characterised by the need to teach a large mass of mathematical content in a relatively short time. This is especially true in coordinated courses. As a result, time constraints are a major consideration in choosing how to teach (pedagogy) and what (content). In this study, two active learning activities were incorporated in lectures: all five groups used peer instruction with online quizzes and three of the groups also had weekly self-work sessions. The findings demonstrate that students thought both activities had positive effects on their learning (Figure 1), and that the students' achievements in the final exam were the same or above the total course average, with lower failure rates (Table 4 and Figure 4).

The positive effects of using peer instruction in STEM courses has been well documented. Indeed, we support the results of Ponton et al (2001), that providing students with the mastery experience through online quizzes contributed to higher self-efficacy overall, a better learning environment and learning experience. The failure rate and higher grades on the final exam that the students achieved support the literature about the correlation between higher achievements and self-efficacy, and corroborates Freeman et al (2014) that active learning lowers the failure rate and increases achievement. A future research direction is to study if the effects depend on the initial self-efficacy of students, i.e. there are different effects on students that enter the course with high self-efficacy and students who enter the course with low self-efficacy.

Regarding the effect of adding self-work (SW) sessions to the lessons, both lecturers wrote in their weekly reflections that at some point, they debated whether to discontinue the self-work sessions because of their duration and because the peer instruction includes a phase of self-work (although the MathMatize quizzes were given 4 times during the semester and the SW sessions were given weekly). However, the findings demonstrate that continuing with the self-work sessions throughout the semester had several important positive aspects. The self-work groups benefited more from the peer instruction activity (Table 2), also they seemed to have more students earning a lower passing grade instead of failing (Figure 4).

Finally, both lecturers indicated in their reflections that as the semester progressed, they felt a shift in classroom culture. Interestingly, the improvement in groups with SW is noticeable when looking at the trend along the semester's timeline (Table 2). This can be attributed to the time it takes to establish classroom norms. In the lecturers' reflections, both lecturers wrote that during the first weeks of the course it was difficult to get students to work alone and that many students simply waited patiently for the "real lesson" to begin. Over time, it seemed students became accustomed to SW and needed less prompting from the lecturers. The lecturers themselves learned how to use SW more efficiently, in a way that encouraged student participation and demonstrated the relevance of the SW. The lecturer's weekly reflections reported SW helped focus the class and gave the students an outlet to check how well they were following the course, which made the lecture overall more productive.

In their free response on the final survey students wrote that they felt that the SW problems helped them figure out how well they understood the material, and to realise what their strong or weak points were. In addition, both lecturers wrote in their reflections that during the break time students asked the lecturers whether their solutions were correct and well written. Thus, SW influenced students' engagement, improved their self-efficacy (Table 3) and influenced students' work at home (Figure 1). Students with SW scored slightly higher on the final exam (Table 5), and more achieved a lower passing grade as opposed to a high failing grade without SW (Figure 4). Thus, self-work promoted change in classroom norms and students' learning.

More research into student writing and exam questions is certainly needed to determine if SW affects student writing capabilities. This implementation of SW did not provide feedback for written (home) work throughout the semester. Nevertheless, exam solutions were examined to check whether SW sessions had some effect on students' written communication in the final exam. Grading exam solutions written under pressure may not provide the same complete insight as homework where multiple drafts can be written to practice communication. Indeed, the findings were inconclusive and more research into exam scores and student writing is needed.

This paper describes a small pedagogical change that can have a big impact on student experience, students' learning habits and classroom norms. This type of change can be easily incorporated even in coordinated courses with a common syllabus and a large lecture, without requiring instructors to make big changes to their lecture style. Instructors can apply it in their own class regardless of what other instructors are doing. It does not require a massive time and energy investment in the creation of learning materials, and it does not require special means. It is accessible to a wide range of instructors and students and its effects should be studied further.

Finally, both authors were also the lecturers for the five groups, fulfilling the role of architect, engineer, and construction worker, metaphorically speaking, through the entire process (PAR). Although measures were taken to make the research as objective as possible (see above) accounts of the authors' teaching reflections are biassed to some degree. On the other hand, both authors were fully acquainted with the instructional context (e.g., syllabus, student population), thus were able to constantly consider and adapt the study's design according to the *real* learning environment within which they operate. Weber (2012) stated that "mathematicians are unlikely to implement teaching suggestions if these suggestions are at variance with their pedagogical goals and beliefs or " (p. 464). Goodchild (2023) writes "if mathematics teaching and learning in higher education is to change, it is up to mathematics teachers to be the change agents" (p. 74). Continuing this, Dawkins and Weber (2023) propose a model for improving advanced mathematics instruction, that relies on increased mathematicians' involvement in designing easily adopted pedagogies that could be simply up-scaled. This work provides initial evidence for the efficiency of this approach.

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Poniendo la lupa en la Acción: ¿Qué tan sencilla o compleja puede ser?

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De acuerdo con la teoría APOE, cuando un individuo siente la necesidad de ver un Proceso como una estructura estática a la que puede aplicar Acciones y, las aplica, se dice que ha encapsulado ese Proceso en un Objeto. No obstante, la realización de una Acción puede hacerse a través de distintos caminos cognitivos. Este estudio presenta un diseño, obtenido a partir de un análisis teórico de la Acción 'suma gráfica de funciones'. Se plantea que, aunque se trate de la misma Acción, la naturaleza distinta de las funciones que se suman incide en la forma de actuar y en la complejidad de la Acción en cada caso. Conocer los requerimientos cognitivos en cada camino es crucial para determinar la complejidad de una Acción. Este estudio proporciona una contribución teórica en relación con la comprensión de la estructura Acción en APOE.

Palabras clave: Función, Representación gráfica, APOE, Estructura Acción.

APOE: UNA TEORÍA QUE EVOLUCIONA

> *"este libro no puede, ni debe, ser considerado como la 'última palabra' sobre la teoría APOE"* (Arnon et al., 2014, p. 4)

Los primeros cimientos de la teoría APOE se construyeron en la década de 1980, cuando Ed Dubinsky (1935 – 2022), su fundador, se encontró con la noción de *abstracción reflexiva* de Piaget. Han transcurrido cuatro décadas desde que Dubinsky y sus colaboradores comenzaron a reflexionar sobre la construcción del conocimiento en matemáticas avanzadas. Aunque esta teoría se ha consolidado con el tiempo, sigue evolucionando hasta el día de hoy permitiendo explorar sus constructos a un mayor grado de especificidad y profundidad. A continuación, presentamos una breve descripción de la teoría APOE y algunos temas recientes de investigación en los que se ubica este estudio.

La teoría APOE en la construcción de conocimiento matemático

La Teoría APOE es una teoría constructivista que, en términos de un modelo cognitivo, permite describir aquello que puede construirse en la mente de un individuo mientras aprende algún concepto matemático. En la teoría APOE, las construcciones involucradas se consideran etapas que, a su vez, se refieren a las estructuras mentales: ***A**cción*, ***P**roceso*, ***O**bjeto* y ***E**squema*. Para el tránsito de una estructura a otra se emplean mecanismos mentales como interiorización, encapsulación, des-encapsulación, reversión, coordinación y tematización. A continuación, se describe cada una de las estructuras mentales y su interacción con algunos de los mecanismos (ver Figura 1).

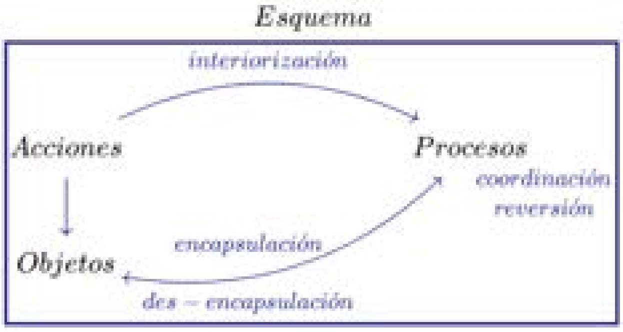

Figura 1: Estructuras y mecanismos mentales para la construcción de conocimiento matemático (adaptada de Arnon et al., 2014, p. 18).

En la Figura 1 se observa que la construcción de la comprensión de un concepto matemático inicia con Acciones sobre Objetos que el individuo previamente ha construido. Por ejemplo, para un individuo que ha construido el concepto del conjunto de los números reales como Objeto, podemos considerar la Acción de tomar un elemento de un conjunto y transformarlo de alguna forma para asignarle un único elemento de un segundo conjunto. Lo anterior es una manera de iniciar el camino hacia la construcción del concepto función. Una Acción es externa, cada paso de la transformación se hace de manera explícita y sin omitir alguno. Aunque la Acción es la estructura más básica, es indispensable para el desarrollo de las demás estructuras.

Un Proceso es una Acción *interiorizada*, es decir, una estructura mental que realiza la misma transformación que la Acción, pero ahora ejecutada totalmente en la mente del individuo; de manera consciente, reflexiva y controlada. Lo anterior surge como producto de la repetición de la Acción y de su interiorización a través de la reflexión sobre ella. Continuando con el ejemplo de función, cuando un individuo repite esa Acción en diferentes conjuntos, y reflexiona sobre la Acción como una transformación dinámica, comienza la interiorización de ésta para "ver la función como un tipo de transformación que empareja elementos de un conjunto, llamado dominio, con elementos de un segundo conjunto, llamado recorrido" (Arnon et al., 2014, p. 30). Otra manera de construir Procesos es a través de los mecanismos *reversión* y *coordinación*.

Una vez que el individuo ha construido un Proceso, el cual tiene una naturaleza dinámica, puede sentir la necesidad de verlo como una estructura estática para aplicarle Acciones (o Procesos). Lo anterior constituye el comienzo de la *encapsulación* del Proceso en un Objeto mental. Acciones como formar un conjunto de funciones, operarlas o establecer sus propiedades, motivan la encapsulación del Proceso función en el Objeto cognitivo función. Por otra parte, realizar una Acción o un Proceso sobre un Objeto, en algunos casos puede requerir de la *des-encapsulación* del Objeto a su Proceso para examinar sus propiedades (Asiala et al., 1996). Finalmente, la interacción entre las estructuras y los mecanismos mentales y, en general, "todo su conocimiento conectado (explícita o implícitamente) a ese concepto" (Arnon et al., 2014, p. 110) conduce a Esquemas que tienen una naturaleza dinámica y coherente. Por ejemplo, para el caso de función, ante una situación matemática, si el individuo identifica si ésta corresponde, o no, a una situación funcional, es un indicio de que su Esquema de función es coherente.

Desde su inicio, la teoría APOE se ha mantenido como un cuerpo teórico dinámico y en constante evolución. Especialmente las *descomposiciones genéticas*, que consisten en modelos cognitivos que describen un camino viable por el cual un individuo podría construir su conocimiento matemático, han evolucionado significativamente a medida que ha progresado la investigación. Este avance ha generado otro tipo de reflexión profunda dentro de la teoría. Por ejemplo, Oktaç et al. (2021), han examinado con detalle descubrimientos recientes acerca de los puntos de transición, también conocidos como niveles intermedios entre concepciones. Asimismo, tanto Dubinsky et al. (2013) como Villabona et al. (2022) han abordado una nueva estructura potencial denominada *Totalidad*. En particular, Villabona (2020) ha abierto una reflexión acerca del tipo de concepción que podría tener un individuo que realiza algunas Acciones y otras no, sugiriendo la existencia de Acciones más complejas en comparación con otras.

Este estudio tiene como objetivo determinar y clasificar los tipos de Acción que se pueden realizar en el contexto de funciones. Para ello, es esencial considerar tres aspectos fundamentales. En primer lugar, la función es un Objeto matemático central en las matemáticas y posibilita la aplicación de distintos tipos de Acciones con variados grados de complejidad. En segundo lugar, es crucial explorar Acciones en un contexto poco convencional para los estudiantes, como lo son las representaciones gráficas, dado que en general la enseñanza suele priorizar lo algebraico. Una de estas Acciones es la suma gráfica de funciones, donde la naturaleza de las funciones que se suman puede complejizar la Acción como se ejemplificará más adelante. En tercer lugar, es necesario examinar la manera en que los individuos aplican estas Acciones. Como señala Mamolo (2014), más allá de la capacidad de actuar sobre un Objeto, resulta crucial comprender cómo se lleva a cabo dicha Acción sobre ese Objeto así como la naturaleza de la actividad matemática y estrategias de solución que emergen de los estudiantes (Proulx, 2015). Por lo anterior, elegimos el concepto de función que, debido a su importancia en la enseñanza de las matemáticas, ha sido ampliamente estudiado en la investigación en Matemática Educativa, como se expone a continuación.

EL CONCEPTO FUNCIÓN Y SU REPRESENTACIÓN GRÁFICA

Desde una perspectiva cognitiva, el concepto función se ha considerado como fundamental y complejo. Varios estudios han identificado concepciones erróneas y obstáculos en su proceso de aprendizaje (Leinhardt et al., 1990). Asimismo, existen estudios que se han enfocado en la instrucción y en el diseño de actividades para abordar estas dificultades (Paoletti y Moore, 2018). Otros enfoques han explorado la función desde una perspectiva epistemológica (Sierpinska, 1992), mientras que algunos han analizado su transición entre diferentes niveles educativos (Artigue, 2008).

Desde la teoría APOE, los estudios sobre la comprensión de este concepto han tenido una naturaleza tanto teórica (Dubinsky, 1991) como empírica (Dubinsky y Wilson, 2013). Breidenbach et al. (1992) señalan que para que un estudiante manifieste una comprensión del concepto de función, este debería mostrar evidencias de una concepción Proceso, sin embargo, muchos universitarios apenas alcanzan una concepción Acción. Otros estudios declaran que, como requisito previo para la comprensión de otros dominios, un individuo debería tener un Esquema de función (Martínez-Planell y Trigueros, 2019) ya sea que se requiera a la función como Objeto o como Proceso.

Aunque el concepto de función ha sido ampliamente estudiado desde la matemática educativa, se ha buscado analizar su comprensión en términos de la teoría APOE, ya que nuevos cuestionamientos nos están llevando a otro tipo de estudios con el fin de mirar algunos fenómenos desde distintas perspectivas. Es notable e interesante observar cómo estos enfoques novedosos pueden integrarse y explicarse dentro de los mismos constructos que conforman la teoría APOE, evidenciando su versatilidad y aplicabilidad. Este estudio hace parte de esos nuevos planteamientos, pues, aunque la estructura Acción es fundamental para iniciar la construcción de conocimiento, no hemos encontrado estudios que pongan su foco de atención en esta estructura y, en particular, en una representación gráfica.

Por otro lado, una dificultad persistente a lo largo de la historia ha sido la necesidad de asociar una expresión algebraica a la representación gráfica. Como señala Sierpinska (1992), las primeras definiciones del concepto de función se centraban en una expresión algebraica, lo que llevó a matemáticos reconocidos a considerar que, si una gráfica no podía ser representada algebraicamente, no correspondía a una función. Asimismo, Leinhardt et al. (1990) evidenciaron que, a pesar de conocer la definición de función, algunos estudiantes no logran determinar si una gráfica es representativa de una función. En nuestro estudio, consideramos la representación gráfica interesante por varias razones. En primer lugar, como ya se ha mencionado, la representación gráfica ha sido menos favorecida que la algebraica cuando se trata de enseñar funciones. De ahí que las situaciones en un ambiente gráfico resulten inusuales para los estudiantes. En segundo lugar, las situaciones en un ambiente gráfico permiten evidenciar mayor conciencia en los estudiantes sobre las propiedades y las operaciones entre funciones, a diferencia de si se realizaran solo de manera algebraica mediante algoritmos que podrían estar memorizados.

ELEMENTOS METODOLÓGICOS E INSTRUMENTO DE INVESTIGACIÓN

La teoría APOE propone un ciclo metodológico de investigación que comprende las siguientes componentes: Análisis teórico, Diseño e Implementación de la Enseñanza y Recolección y Análisis de Datos. Lo que se muestra en este documento es el producto de un análisis teórico, en particular se presenta el diseño y análisis de una situación que involucra la Acción ‘suma gráfica de funciones’. Nuestro análisis teórico toma en consideración estudios previos de la enseñanza y el aprendizaje del concepto, apoyados o no en APOE; estudios epistemológicos del concepto; abordaje de la noción de función en algunos libros de texto y nuestra experiencia en su enseñanza y aprendizaje. A partir del análisis teórico, hemos diseñado un instrumento con un enfoque en la suma gráfica de funciones. Como enfatiza Oktaç (2019), el diseño (de problemas, situaciones, cuestionarios, entrevistas) es fundamental en la teoría APOE, pues a través de ellos obtenemos información sobre los mecanismos y las estructuras mentales que desarrolla un individuo mientras construye su comprensión de un concepto matemático o le da solución a una situación matemática.

La situación que hemos diseñado y que a continuación se presenta permite, con la misma Acción suma gráfica de dos funciones, analizar cómo esta se hace más compleja cuando una de las funciones varía un poco. Es decir, dentro de la misma Acción, estamos observando que la naturaleza de la función, que está determinada por las propiedades asociadas al Proceso de función, incide al momento de llevar a cabo la Acción.

Explorando la complejidad en la suma gráfica de funciones

Una de las situaciones (Situación 1) que se propone es la que se muestra en la Figura 2 que contiene, a su vez, dos situaciones: suma entre funciones constantes (a la izquierda del eje vertical) y suma entre una función constante y una lineal no constante (a la derecha del eje vertical). La manera en que está diseñada esta situación resulta del análisis teórico, en donde se ha pensado que, aunque se trate de la misma Acción, alguien que vea esta situación como dos situaciones independientes, podría actuar de una manera, cuando las funciones tienen la misma naturaleza (como en el caso de las constantes) y de otra en caso contrario.

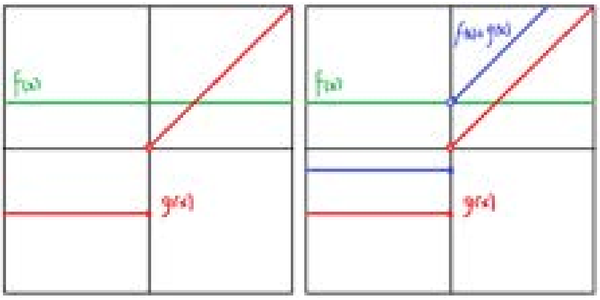

Figura 2: Situación 1 y su solución.

Este estudio infiere que la complejidad para llevar a cabo una Acción puede ser determinada en dos sentidos. Por un lado, está influenciada por la cantidad de elementos necesarios del Esquema asociado al Objeto y la forma en que estos interactúan. Por ejemplo, una Acción puede requerir la des-encapsulación de los Objetos, ya sea para examinar las propiedades y características asociadas a sus Procesos, o para la posterior coordinación de estos. Por el otro lado, está influenciada por la interacción del individuo con el Objeto a través de su Esquema asociado y su nivel de desarrollo. Esta interacción define un camino, y la concepción que el individuo tenga sobre las funciones dadas juega un papel crucial al momento de realizar la Acción. De esta manera, la Acción puede volverse tan compleja para el individuo como esa interacción.

Para ejemplificar lo anterior, a continuación, se describen algunos caminos, derivados de nuestro análisis teórico, que podrían seguirse al momento de abordar la situación presentada en la Figura 2. Entendemos por camino cognitivo una descripción que incluye los elementos cognitivos en términos de estructuras y mecanismos para llevar a cabo la Acción. Estos caminos pueden diferir en cada individuo de acuerdo con su Esquema de función.

DOS POSIBLES CAMINOS PARA LA SUMA GRÁFICA DE FUNCIONES

Con el fin de profundizar en la comprensión de la descripción de los caminos que acá se presentan, a continuación, se procede a esclarecer parte de la terminología utilizada y a plantear algunas distinciones.

Terminología y distinciones

Concepción Proceso de función: Una concepción Proceso de función implica la comprensión de función como "un tipo de transformación que empareja elementos de un conjunto, llamado dominio, con elementos de un segundo conjunto, llamado recorrido" (Arnon et al., 2014, p. 30). La expresión "tipo de *transformación*" infiere que la concepción Proceso de función es independiente de la representación, es decir, es general. Sin embargo, las Acciones que indican la transformación en cada representación cambian y, en consecuencia, también los Procesos. Encontrar el valor de una función en una representación algebraica difiere de una representación gráfica, no obstante, es posible establecer equivalencias de esta transformación, como se explica a continuación para el caso de la representación gráfica.

Concepción Proceso de función en una representación gráfica: Arnon et al. (2014) señalan que una de las dificultades en los estudiantes para transitar de una representación a otra, es que hay una carencia del "significado cognitivo del concepto (planteado por la descomposición genética)" (p. 181). Para el concepto función, el significado cognitivo se asocia a la idea general de transformación; al respecto, Dubinsky (1991) señala que "puede ser posible que el sujeto coordine el Proceso de función y su gráfico" (p. 115). Lo anterior sucede, cuando el estudiante entiende que, para un valor en el eje horizontal, la altura del gráfico corresponde al valor de la función. Así, la coordinación se da a través de la equivalencia entre el valor de f para un valor particular, con los elementos visuales que proporciona la gráfica para ese valor (signo y distancia al eje horizontal). Si bien es posible pensar que la comprensión del concepto función implica una concepción Proceso y que este hecho se puede interpretar en diferentes representaciones, las propiedades asociadas a dicho Proceso pueden ser diferentes dependiendo del tipo de función como se ejemplifica a continuación.

Función constante y función lineal: Para examinar las propiedades asociadas al Proceso de función cuando ésta es constante y está representada gráficamente, un individuo puede pensar y reflexionar sobre cómo es que se está dando la transformación y determinar que, para cualquier valor sobre el eje horizontal, la distancia a la gráfica, y el signo, son siempre los mismos. En términos de variación, la altura al gráfico no varía, independientemente si se varía el valor en el eje horizontal. De manera similar para el caso de la función lineal no constante, por cada unidad que se varíe en el eje horizontal, la variación en la altura es siempre la misma.

Con lo anterior, en este documento expresiones como “concepción Proceso de función constante”, no significa que existen diferentes concepciones Proceso de función, sino que las propiedades asociadas al Proceso difieren de acuerdo con el tipo de función. También cuando se haga alusión a un tipo de concepción, nos estamos refiriendo, en particular, a concepciones asociadas a un dominio gráfico. A continuación, se describen dos caminos, junto con sus requerimientos cognitivos, que pueden pensarse en ambos casos (ver Figura 2), si las dos funciones son constantes o, si una es constante y la otra lineal. Una síntesis de estos caminos se muestra en la Figura 3.

Camino 1: Infiriendo propiedades de la función resultante a partir de las propiedades de las funciones dadas

En este camino el individuo reconoce que las funciones dadas son rectas y que esta propiedad la heredará la función resultante. Además también reconoce, de qué tipo será la función resultante, es decir, constante (recta horizontal) o lineal (recta inclinada). Lo anterior hace referencia a la *forma* de la función, lo cual requiere de propiedades del Objeto y, por tanto, de una concepción Objeto de función. Una vez que el individuo ha reconocido la forma que tendrá la función suma a partir de las propiedades de las funciones dadas, el siguiente paso tiene que ver con su *ubicación*. Para que el individuo acierte en la ubicación de la función resultante, éste debe reflexionar sobre las propiedades asociadas al Proceso de función, tanto en f como en g, ya sea para determinar un punto (para el caso donde ambas son constantes) o dos puntos (en cualquier caso) por donde pasará la recta. Esto último requiere de la des-encapsulación de cada una de las funciones a su Proceso para llevar a cabo una Acción sobre un valor x (o dos cuando se requieran dos puntos) en el eje horizontal. Para sumarlas gráficamente en ese punto (o puntos), el individuo deberá tener en cuenta elementos visuales informados por las mismas gráficas de f y g. De esta manera, la des-encapsulación se hace con la intención de realizar la siguiente Acción: Examinar propiedades y características del Proceso función en términos de distancias y signos para un valor (o valores) en el dominio de la función suma. Esta Acción inmediatamente se generaliza sobre todo el dominio de la función resultante para trazar una recta, ya sea horizontal o inclinada. La generalización es posible gracias a que, de antemano, el individuo sabe qué forma tendrá la función resultante.

En síntesis, el camino anteriormente descrito requiere, en primer lugar, de una concepción Objeto de función para determinar la forma de la función resultante y; en segundo lugar, para su ubicación, se requiere de la des-encapsulación de cada uno de los Objetos f y g a su Proceso para comparar distancias y signos en un valor (o valores) sobre el eje horizontal; esto último es una Acción que inmediatamente se generaliza para trazar la función resultante.

Camino 2: Como una traslación

Este camino se enfatiza en que una de las funciones, por ejemplo f, al ser constante trasladará verticalmente a la otra, g. Es decir, en este camino el individuo identifica inmediatamente el efecto gráfico que tiene sumarle una función constante, ya sea que ésta esté por encima o por debajo del eje horizontal, a cualquier función. Lo anterior requiere de una concepción Objeto de función para aceptar que es posible sumarlas y, para reconocer que sumarle una constante a otra función no cambiará la *forma* de la no constante. Sin embargo, analizar de qué manera trasladará f a g, es decir, la *ubicación* de la resultante, requerirá del mecanismo de des-encapsulación para examinar las propiedades y características, pero solo de f pues, en este camino, el estatus de g es de Objeto, es decir, "algo" que se va a tomar y a mover según lo indique f. Así, la des-encapsulación de f se hace con el fin de reconocer su signo y su distancia al eje horizontal; en primer lugar, para saber en qué sentido desplazar a g, hacia arriba si f es positiva o hacia abajo si f es negativa y; en segundo lugar, para determinar cuánto la trasladará, lo cual se abstrae de su distancia al eje horizontal.

Como se observa, en este camino, al igual que en el camino 1, dos aspectos son importantes: Reconocer la forma de la función resultante y, su ubicación. Para lo primero, se requiere de una concepción Objeto de las funciones dadas y; para lo segundo, de la des-encapsulación de f a su Proceso. También en todo momento g conserva su estatus de Objeto pues, es sobre la totalidad del Proceso de g que el Proceso de f está actuando.

Si se comparan los dos caminos anteriormente descritos para la suma gráfica entre dos funciones, el camino 2 muestra el camino con menos requerimientos cognitivos pues, al igual que en el camino 1, se requiere de una concepción Objeto de función, pero, a diferencia, solo se requiere de la des-encapsulación de una de las funciones dadas, como se resume en la Figura 3.

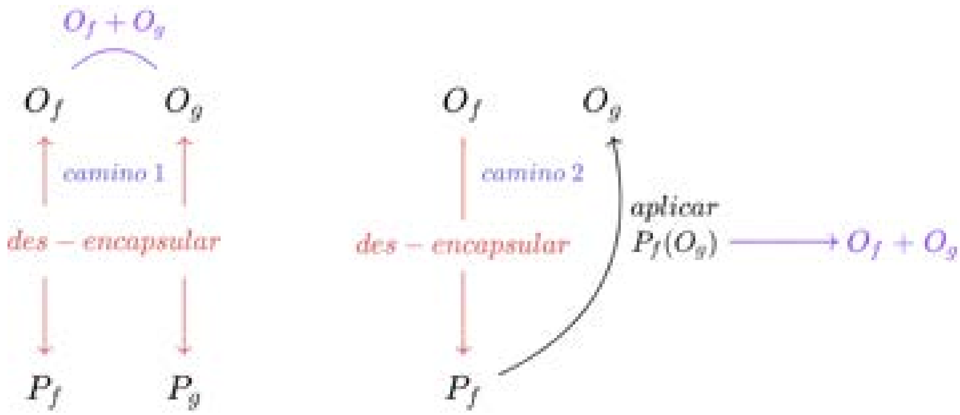

Figura 3: Caminos cognitivos para la suma gráfica de las funciones f y g.

IMPLICACIONES TEÓRICAS Y DIDÁCTICAS

Si bien APOE es una teoría consolidada, todavía está en constante evolución. Investigadores que se enmarcan en esta teoría continúan revisándola, y estudios recientes muestran la necesidad de incorporar nuevos constructos o profundizar en los existentes. Aunque la estructura Acción ha estado presente desde los inicios de la teoría APOE, y es la base para el desarrollo de otras estructuras, estudios como Mamolo (2014) y Villabona (2020) han planteado una serie de cuestionamientos nuevos sobre la forma de actuar y las Acciones. Este estudio pretende aportar en esa dirección.

Analizar qué significa que se pueda realizar una Acción y otra no y; explicar en términos cognitivos desde la teoría APOE, por qué algunas Acciones, o formas de actuar, son más complejas que otras, permitirá esclarecer sobre lo que significa tener una concepción Objeto de un concepto o, explicar qué tipo de concepción tiene un individuo cuando puede realizar algunas Acciones y otras no.

Lo que se ha mostrado en este documento hace parte de un análisis teórico. Este análisis se relaciona con el planteamiento de que, la Acción puede hacerse tan compleja dependiendo de la interacción que el individuo tenga con el Objeto a través de los elementos de su Esquema. Para explorar tal interacción, cabe resaltar la importancia del diseño de las situaciones matemáticas como la que se mostró en este documento. Esta situación, además de ser inusual, permite la posibilidad de tomar diferentes caminos cognitivos, contrario a si se presentara en una representación algebraica, por ejemplo. Respecto a qué llevaría a un estudiante a seguir el camino 1 o el 2, o tal vez algún otro, planteamos que se relaciona con el nivel de desarrollo de su Esquema de función. Un estudiante con un Esquema coherente de función, puede estar inclinado a tomar el camino con menos demanda cognitiva, el 2. Independientemente de cuál camino tome, situaciones como la que se mostró, y el diseño de entrevistas didácticas, nos darán información sobre la manera en que los elementos del Esquema están interactuando y cómo esta interacción, supeditada por la forma de actuar del individuo sobre el Objeto, determina la complejidad de llevar a cabo una Acción.

Debido a la relación intrínseca que hay entre la teoría APOE y la enseñanza-aprendizaje, profundizar teóricamente en la construcción de un concepto naturalmente tiene implicaciones didácticas. La forma en que un individuo puede actuar sobre diversos objetos matemáticos influye en la interiorización de esas Acciones. El conocimiento de las diversas Acciones que se pueden aplicar sobre un mismo Objeto ayuda en el diseño de actividades para motivar su interiorización y por ende la construcción de una concepción Proceso. En particular, el ámbito gráfico proporciona un espacio adecuado donde se pueden introducir nuevas acciones que son novedosas para los estudiantes.

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How many differentiable functions are there? – A reflection on functional thinking and the Baire category theorem as a component of mathematical horizon content knowledge

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The Baire category theorem implies that the set of continuous functions that are differentiable in at least one point is meager in the space of all continuous functions. In this sense, the "typical" continuous function is nowhere differentiable. Drawing on the notion of "horizon content knowledge", we discuss this observation in the context of the didactic principle of "functional thinking", which is important for the teaching of functions in German schools. We examine potential components of this horizon content knowledge by illustrating the application of the Baire category theorem to prove that "typical" continuous functions are nowhere differentiable with a sequence of tasks that may be implemented in a first-year course on real analysis.

Keywords: Teaching and learning of specific topics in university mathematics, Teaching and learning of analysis and calculus, Baire category theorem, functional thinking, horizon content knowledge.

INTRODUCTION

This is a discussion paper and epistemological in nature. We investigate a case of mathematical *horizon content knowledge* (Loewenberg Ball & Bass, 2008; Jakobsen et al., 2013) for prospective mathematics teachers (PMTs), which may be included in their education at university in the context of real analysis. Based on the observation that PMTs learn that a "typical" real number is irrational because $\mathbb{R} \setminus \mathbb{Q}$ is uncountable while $\mathbb{Q}$ is countable, we raise the question of how to address a similar phenomenon for the case of functions. In this context, an application of the *Baire category theorem* shows that a "typical" continuous function is nowhere differentiable, contrasting significantly with the sets of functions usually dealt with in school. Here, the reason is that the set of functions that are differentiable in at least one point is meager in the set of all continuous functions. In both cases, numbers and functions, it is the mathematical discourse at the university level that clarifies the scope of the definitions or characterizations of the mathematical objects given in school, but in only one case is it usually made explicit in PMT education.

In this paper, we therefore raise the question how prospective mathematics teachers' horizon content knowledge about typical functions can be enhanced by answering which continuous functions are typical from a mathematical point of view. One of the challenges here is that the Baire category theorem is usually presented in the context of abstract or functional analysis, and is therefore not necessarily part of the curricula of prospective mathematics teachers at university. Thus, we outline a potential inclusion of this application of the Baire category in a course on real analysis to strengthen PMTs' horizon content knowledge about functions. In doing so, we aim to initiate a discussion about the inclusion of this theorem into PMT education by outlining its relevance in terms of the didactic paradigm "functional thinking" (Krüger, 2019), which is a vital component in the didactical discussion about teaching functions at school. In this regard, our investigations are embedded into the discussion about "[m]aking university mathematics matter for secondary teacher preparation" (Wasserman et al., 2023). We have a German context in mind, but since the issues raised here are predominantly epistemological, we believe they translate to other contexts as well.

PROBLEMATIQUE

Determining what a "typical" instance of a particular class of mathematical objects looks like requires not only specifying how "typicality" is characterized, but also checking the scope of a definition and determining what zoo of objects actually falls under the definition. Pupils and prospective mathematics teachers experience this need for instance in the context of real numbers. They learn that the set of irrational numbers is an uncountable subset of $\mathbb{R}$ and that the set of rational numbers is countable. From this point of view, elements from $\mathbb{R} \setminus \mathbb{Q}$ may be described as "typical". A similar situation arises in the context of functions: Here, the set of continuous real-valued functions on an interval is a metric space (with the supremum metric), and the notion of *comeager set* can be used to characterize "typicality" in metric spaces. In this sense, it can be proven with the Baire category theorem that a nowhere differentiable function is a "typical" continuous function. [1]

Based on the premise that prospective teachers should know about "typical" real numbers, we argue that they should not only be aware of the consequences of the definition of real numbers for the real numbers themselves, but that the consequences of the general definition of real-valued functions should also be part of PMTs' mathematical horizon content knowledge.

Mathematical horizon content knowledge

Different facets of (prospective) teacher knowledge have been conceptualized during the last decades, probably originating with Shulman's (1986) seminal distinction of content, pedagogical, and pedagogical content knowledge. Loewenberg Ball and Bass (2009) have added a facet to this discussion called *horizon content knowledge* in their framework of *mathematical knowledge for teaching*. It is "an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated", and it shall contain amongst others "a sense of the mathematical environment" of what is currently taught or "[m]ajor disciplinary ideas and structures" (Loewenberg Ball & Bass, 2009, p. 6). Horizon content knowledge is useful to grasp mathematically what pupils say, to "anticipat[e] and mak[e] connections", and to "catch[] mathematical distortions or possible precursors to later mathematical confusion or misrepresentation"

(Loewenberg Ball & Bass, p. 6). Jakobsen et al. (2013, p. 3128) argued further that horizon content knowledge is “an orientation to and familiarity with the discipline […] that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory”. In this sense, horizon content knowledge is not directly part of school relevant *specialized content knowledge* (Loewenberg Ball & Bass, 2009), which would be “immediately about the content being taught” (Jakobsen et al., 2013, p. 3128).

Admittedly, it is somewhat vague what constitutes mathematical horizon content knowledge for a school subject. In the case of functions for the upper secondary level, we would like to indicate what it may include.

Teaching real numbers in school and at university

Our investigation is embedded into the teaching and learning of real numbers and functions in (upper-level secondary) school and connects to the intricacies related to the set of real numbers (e.g., Barquero & Winsløw, 2022; Durand-Guerrier, 2016; Wasserman et al., 2022). In Germany, real numbers are often introduced in grade 10 or 11 (out of usually 13) as an extension of the set of rational numbers to formally enable the solution to equations like $x^2 = 2$, which are unsolvable over the rational numbers. In this context, prospective teachers (and pupils) discuss proofs by contradiction that there is no rational number whose square is equal to 2. While such epistemological issues are covered in school mathematics, students also work with π in geometry, too, without worrying much about the existence of irrational numbers.

Passing to a graphical representation of the graph of the function $x \mapsto x^2 - 2$, which looks like a gapless curved line, the question of the existence of a root is almost blurred, as it is “obvious” that one must exist. Indeed, this is a consequence of the intermediate value theorem, which in turn fails over subsets of $\mathbb{Q}$ instead of $\mathbb{R}$. One may thus argue that to make this theorem “provable” a characterization of the set of real numbers is necessary. In upper secondary schools, pupils work extensively with real-valued functions defined on subsets of $\mathbb{R}$ and with irrational numbers such as e. In this respect, it seems rather uncontroversial that PMT are taught an axiomatic approach to $\mathbb{R}$ in their courses on real analysis at university. Similarly, real functions are defined, and thus the question of what a “typical” real function looks like is within the reach of pupils. As we have mentioned above, the scope of the concepts used and defined in school is discussed in PMT education in the context of numbers but not yet for the case of functions.

Functional thinking in mathematics teaching at school

In the last two or three years of upper-level secondary schools, real functions (i.e., those from intervals to $\mathbb{R}$) are intensively discussed, in particular in the context of differentiability, integration, and mathematical modeling, and a general definition of functions may be addressed, too. In this context, “elementary” functions (polynomial, rational, exponential, and trigonometric functions, their inverses, and finite combinations) comprise the main class of functions considered in school. This is, of course, for a good reason and fits into the didactic concept of *functional thinking*, which is an important guiding idea in current German mathematics education. It is based on ideas by Felix Klein (see Krüger, 2021) and has been intensively discussed for at least forty years (e.g., Greefrath et al., 2016; Roth & Lichti, 2021; Vollrath, 1989). It is characterized as "a way of thinking, which is typical for handling functions" (Vollrath, 1989, p. 6; own transl.) and emphasizes the use of representations as well as the shifts between them (e.g., symbolic form of a function term, graphs, and tables). Drawing graphs of functions is one particular vital point. Functional thinking also aims for teachers to support their pupils develop appropriate conceptions by considering three so-called *basic ideas* ("Grundvorstellungen") of functions (vom Hofe & Blum, 2016). For instance, Roth and Lichti (2021, p. 4, own transl.) argue that

> […] one can only deal with a mathematical concept, such as that of a function, using suitable representations. Even the development of basic ideas about functions themselves can only succeed by means of their representations and their interconnection, that is, the change between such representations.

Here, the first basic idea *function as a mapping* emphasizes that each point in a domain is assigned a unique point in a codomain, the second *function as covariation* underlines that, given a functional relationship $y = f(x)$, changes in x specify how y changes, and the third *function as an object* stresses that a function is an object itself and operations may be performed on it (e.g., deriving it) (Greefrath et al., 2016).

While functional thinking sensitizes teachers for a large variety of ways to deal with functions in school, its discussion in mathematics education literature does almost not emphasize the aspect of functions we consider here, namely the question what a "typical" function is from a mathematical point of view and that most functions appearing in school are far away from this (e.g., Barzel et al., 2021; but see Tietze et al., 2000). We would like to emphasize at this point that we do not see this as a shortcoming, since the functions from school are already very rich and useful in terms of functional thinking. For example, elementary functions are very fruitful for modeling and are rich enough to address structural properties (e.g., linearity, functional, or differential equations) as well. However, understanding the typicality of functions may indeed belong to the basic idea of *function as an object* in school and university.

Nevertheless, a few instances of "pathological" functions (which do not fall into the classes described above) are in fact encountered in school, such as piecewise defined functions with "jumps" or "kinks", cumulative distribution functions like $x \mapsto \int_{-\infty}^{x} \frac{\exp(-t^2)}{\sqrt{\pi}} \mathrm{d}t$ without an "elementary" term, or the Dirichlet function $x \mapsto 1$ if x is rational and $x \mapsto 0$ if x is irrational as an example for a non-integrable function. In the context of real and functional analysis in the 19th and 20th century, the very functions initially deemed pathologies (in a sense going beyond those listed above for the context of school) have actually led to significant theoretical foundations and developments. Tietze et al. (2000, p. 186, own transl., emph. orig.) describe this situation as follows:

So-called 'pathological functions' force a foundation of the fundamentals, and consequently, we finally obtain the *set-theoretical concept of function*. As is usually the case with new findings, this concept of function is initially only partially or almost not at all accepted outside the areas of mathematical basic research. The breakthrough of the set-theoretical concept of function came only with *Bourbaki*, and even this was not the last stage in the development and exactification process of an evolving mathematics. […]

Therefore, including the questions of which and "how many" functions are ignored when only dealing with elementary functions, as well as which are "typical" and in what sense, is relevant in mathematics teacher education not only from a scientific point of view, but also in terms of functional thinking. Furthermore, it is not difficult to imagine that pupils may actually ask whether there are "other" functions besides those they encounter in class (a similar issue plays a role in the contextualization of examples and theorems in real analysis courses at university, see Discussion).

In view of the pathological functions in both school and university mathematical discourses as well as the foundations of real analysis in the sense of the axiomatization of real numbers and their consequences, applications of the Baire category theorem seem to us to be relevant to belong to PMTs' horizon content knowledge about functions as it is embedded in the teaching contexts that (future) teachers are confronted with.

THE BAIRE CATEGORY THEOREM

The following definitions are standard and either included in a two-semester course on real analysis in Germany (Analysis I and II) or are immediate generalizations of the notions of distance $|x-y|$ for $x, y \in \mathbb{R}$ as well as open and closed subsets of $\mathbb{R}$. Hence, the following definitions are accessible to students enrolled in such a course. Let (X, d) denote a metric space.

- (X, d) is called *complete* if every Cauchy sequence is convergent in it. For instance, $\mathbb{R}$ with the standard metric is complete. The *open ball of radius* $r > 0$ *around* $x \in X$ is $B(x, r) \coloneqq \{y \in X : d(x, y) < r\}$ and the corresponding *closed ball* is $\bar{B}(x, r) \coloneqq \{y \in X : d(x, y) \leq r\}$.
- Let $C[a; b]$ denote the set of continuous functions $[a; b] \to \mathbb{R}$ and $d_s(f, g) \coloneqq \|f - g\|_\infty \coloneqq \sup_{x \in [a;b]} |f(x) - g(x)|$ the *supremum metric*. Then, $(C[a; b], d_s)$ is complete. In this metric space, the ball $B(f, r) = \{g \in C[a; b] : \|f - g\|_\infty < r\}$ contains all functions whose graph lie in the r-strip around the graph of f.

The following definitions characterize "smallness" of subsets and "typical" elements of a metric space. Let $A \subseteq X$.

- The *closure* $\bar{A}$ of a subset A of a metric space (X, d) is the union of A and all its accumulation points in X (i.e., $\bar{A} = \bigcap_{C \supseteq A,\ C \text{ closed in } X} C$). The interior A° is the set of all points $a \in A$ for which there is an open ball centered at a and completely contained in A (i.e., $A^\circ = \bigcup_{O \subseteq A,\ O \text{ open in } X} O$).

- A is *dense (in X)* if $\bar{A} = X$; A is *nowhere dense* if $(\bar{A})^\circ = \emptyset$ (i.e., $(\bar{A})^c$ is dense); A is *meager* if it is the countable union of nowhere dense sets; and A is *comeager* if its complement is meager. Elements of comeager sets are called *typical*.

The intuition is that a nowhere dense set may be considered as very thin, since its points do not accumulate, and meager sets are merely countable unions of nowhere dense sets. For instance, $\mathbb{Q}$ is a meager subset of $\mathbb{R}$ since it is the countable union of singletons. In the following, we derive an analog statement for $(C[a;b], d_s)$.

The *Baire category theorem* is as follows (Bridges, 1998, p. 297):

> Let (X, d) be a complete metric space and D_k, $k \in \mathbb{N}$, a countable collection of open dense subsets of X. Then, $D \coloneqq \bigcap_{k\in\mathbb{N}} D_k$ is dense. By passing to complements, this implies that X cannot be the countable union of closed nowhere dense sets.

We also recall the *Stone-Weierstraß theorem* (Bridges, 1998, p. 216):

> The set $P[a;b]$ of polynomial functions is a dense subset of $(C[a;b], d_s)$. That is, for each $f \in C[a;b]$ and $\varepsilon > 0$, there is a polynomial $p \in P[a;b]$ such that $\|f - p\|_\infty < \varepsilon$.

In words, this means that the ε-strip around the graph of f is thick enough such that the graph of p fits in there.

These theorems, and thus the application to the comeagerness of the set of nowhere differentiable functions, for which we construct a sequence of tasks below, can thus be presented in a German course on Analysis I and II. Depending on the course, these theorems may be proven or not. The point we try to make is that these theorems and the following application are in reach of such a course, even if the proofs are omitted.

Proving the Baire category theorem in the context of a course on real analysis

The Baire category theorem may be proven by mimicking the construction of nested intervals, which has likely appeared in a lecture on real analysis (e.g., for a proof of the intermediate value theorem): It must be shown that for each $x_0 \in X$ and $\varepsilon_0 > 0$ the set $D \cap \bar{B}(x_0, \varepsilon_0)$ is non-empty. Since D_1 is open and dense, there is a ball of radius $1 > \varepsilon_1 > 0$ centered around some $x_1 \in X$ such that $\bar{B}(x_1, \varepsilon_1) \subseteq D_1 \cap B(x_0, \varepsilon_0)$. By openness and density of the D_k, $k \in \mathbb{N}$, we may proceed inductively and find $x_k \in X$ and $0 < \varepsilon_k < 1/k$ such that $\bar{B}(x_k, \varepsilon_k) \subseteq D_k \cap B(x_{k-1}, \varepsilon_{k-1})$ for $k \geq 1$. Then, $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, thus convergent to some x the complex space X, which satisfies $x \in \bar{B}(x_0, \varepsilon_0)$ and $x \in D_k$ for $k \geq 1$. (Bridges, 1998, pp. 279–280)

Sequence of tasks about continuous nowhere differentiable functions

In the following, we illustrate a sequence of tasks based on the Baire category theorem that might be used in a course on real analysis to prove the following theorem:

> The set $D \coloneqq \{f \in C[a;b] : f \text{ is differentiable in at least one point}\}$ is meager in $(C[a;b], d_s)$. In particular, there are continuous nowhere differentiable functions on $[a;b]$.

The existence of a continuous nowhere differentiable function can of course be illustrated with the *Weierstraß functions* of the form $x \mapsto \sum_{k=0}^{\infty} a^k \cos(b^k x)$ for $0 <$

$a < 1$ and an odd integer $b > a^{-1}(1 + 3\pi/2)$ and other examples, too (Bridges, 1998, p. 54). Applying the Baire category theorem, however, leads to the much stronger result and answers the initial question we posed about how many differentiable functions are in the space of continuous functions in terms of "meagerness". From a didactic point of view, this theorem also raises the question of how to adequately represent a "typical" continuous function, given that it is impossible to draw an accurate picture of its graph.

We now follow Heuser (1992, pp. 261–262) for the construction of the tasks. For convenience, let $[a; b] = [0; 1]$ (the general case is similar). For $n \in \mathbb{N}$ let

$$F_n = \left\{ f \in C[0; 2] : \exists x_0 = x_0(f) \in [0; 1]: \sup_{0<h<1} h^{-1} |f(x_0 + h) - f(x_0)| \leq n \right\}$$

denote the set of functions whose difference quotients at one point are bounded by n.

i) Show that F_n is closed in $C[0; 2]$ for every $n \in \mathbb{N}$.

Now, assume that every function in $C[0; 2]$ was differentiable in at least one point in $[0; 1]$. The goal of the following tasks is to derive a contradiction.

ii) Under the given assumption, show that $C[0; 2] = \bigcup_{n \in \mathbb{N}} F_n$.
iii) Use the Baire category theorem to show that there is an $m \in \mathbb{N}$ and a closed ball $\bar{B}$ of positive radius in $(C[0; 2], d_s)$ such that $\bar{B} \subseteq F_m$.
iv) Use the Stone-Weierstraß theorem to show that there is a polynomial $p \in P[0; 2]$ and an $r > 0$ such that $\bar{B}(p, r) \subseteq F_m$ for the m from (iii) and conclude that F_m contains all functions f such that $|f(x) - p(x)| \leq r$ for all $x \in [0; 2]$.
v) Show that there is a sawtooth function f, whose ascending steps have a slope $> m$ and whose falling steps have a slope $< -m$, such that $f \in C[0; 2] \setminus F_m$ for the m from iii).
vi) Conclude that there is a function in $C[0; 2]$, which is not differentiable at any point in $[0; 1]$.

We sketch some parts of the proof: For i), let $(f_k)_{k \in \mathbb{N}} \subseteq F_n$ denote a convergent sequence with limit f in $(C[0; 2], d_s)$. This means that $f_k \to f$ uniformly on $[0; 2]$. For every k there is an $x_k \in [0; 1]$ such that $\sup_{0<h<1} h^{-1} |f_k(x_k + h) - f_k(x_k)| \leq n$. By compactness of $[0; 1]$, $(x_k)_{k \in \mathbb{N}}$ has a convergent subsequence converging to an $x_0 \in [0; 1]$; we may thus assume that $(x_k)_{k \in \mathbb{N}}$ is this subsequence. Let $\varepsilon > 0$ and $h \in (0; 1)$ and define $k_1 < k_2 < k_3$ in $\mathbb{N}$ such that $|f(t_0 + h) - f(x_k + h)| \leq \varepsilon h/4$ for $k > k_1$, $|f(x) - f_k(x)| \leq \varepsilon h/4$ for $k > k_2$ and $t \in [0; 2]$, and $|f(x_k) - f(x_0)| \leq \varepsilon h/4$ for $k > k_3$. Then, $|f(x_0 + h) - f(x_0)| \leq |f(x_0 + h) - f(x_k + h)| + |f(x_k + h) - f_k(x_k + h)| + |f_k(x_k + h) - f_k(x_k)| + |f_k(x_k) - f(x_k)| + |f(x_k) - f(x_0)|$, which implies that $|f(x_0 + h) - f(x_0)|/h \leq \varepsilon/4 + \varepsilon/4 + n + \varepsilon/4 + \varepsilon/4 = n + \varepsilon$ for $k > k_3$. Thus, $f \in F_n$ and i) is proven. ii) follows from the assumption and the definition of the sets F_n, and iii) follows from the second part of the Baire category theorem, because $(C[0; 2], d_s)$ is complete and thus one of the F_n cannot be nowhere dense.

DISCUSSION

The aim of this article was to stimulate a discussion about the mathematical horizon content knowledge of (prospective) mathematics teachers about functions, while at the same time contextualizing the challenges in school and the didactic principle of functional thinking. Starting from the premise that it is fairly uncontroversial that PMTs learn what can count as a "typical" real number and in what sense, we argued that PMTs can also learn what a "typical" continuous real function is and in what sense.

On the one hand, we have presented a sequence of tasks that use the Baire category theorem to show that the set of continuous functions, which are differentiable in at least one point, is meager in the space of continuous functions with the supremum metric. This sequence can in principle be used in a fairly elementary way in a course on real analysis, even in exercises, even though the Baire category theorem traditionally appears in abstract or functional analysis. On the other hand, we have justified the treatment of the theorem in the context of mathematics teacher education by locating it in PMTs' horizon content knowledge. A crucial point here is that, although the didactic concept of functional thinking suggests vivid ideas about continuous or differentiable functions, which are helpful for the elementary functions dealt with in school mathematics, these ideas are problematic with regard to the general concept of real function that is indeed introduced or introducible in school, too. Of course, this is not to argue that the Baire category theorem should be taught in school, but to argue that PMT should have insights into the breadth and limits of fundamental concepts such as number and function, which are achievable within their subject-specific resources. For instance, the elementary functions, which are *typical in school* in terms of their frequency of use, are not *typical in real analysis* in terms of their generality.

We will now underpin the relevance of the Baire category theorem with further examples that can either be understood in principle with school knowledge, even if not proven, or contextualize elements from real analysis at university level (see e.g., Jones, 1997/8). The derivative of a differentiable function does not have to be continuous, of course. But "how discontinuous" can it be? The Baire category theorem can be applied to show that the set of discontinuities is meager and thus the set of points of continuity is dense. Additionally, real analysis courses may cover the fact that if there is an $n \in \mathbb{N}$ with $f^{(n)}(x) = 0$ for all $x \in \mathbb{R}$, then f is a polynomial. More is true: If f is infinitely differentiable and for every $x \in \mathbb{R}$ there is a derivative order $n = n_x$ with $f^{(n)}(x) = 0$, then f is a polynomial (Deiser et al., 2016, pp. 224–225).

The Baire category theorem can also be used to contextualize the *Thomae function*, which is, to our knowledge, traditionally used to prove the existence of a function that is discontinuous at each rational, continuous at each irrational number, and nowhere differentiable: This function assigns 1 to 0, $1/q$ to every rational number p/q with, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and $\gcd(p, q) = 1$, and 0 to every irrational number. Students may now wonder whether there is a function $\mathbb{R} \to \mathbb{R}$ that is continuous at all rational and discontinuous at all irrational points. Using the Baire category theorem, one can prove that this is not the case (Abbott, 2015, ch. 4). This observation places the Thomae function in the bigger picture of (dis)continuous functions and answers the obvious follow-up question.

As a last example, we mention a theorem by Morgenstern (Jones, 1997/8, p. 367): Contrary to a belief from the 19th century, the *typical* C^{∞}-function (i.e., infinitely differentiable) is *not analytic at any point* (i.e., locally representable as a power series at any point). This underlines a drastic difference between *real differentiable functions* and *complex differentiable functions*, which are in turn always analytic (Lang, 1999).

NOTES

1. What may count as "typical" is clearly not unambiguously defined in mathematics. Our examples illustrate that one can deal with this idea in terms of "(un)countability" (set theoretic) and "(co)meagerness" (topological). We note that there is also a measure theoretic version in terms of "zero/full measure" as well. The relationship between the topological and measure theoretic notions is however quite delicate. For instance, there are meager sets with full measure and comeager sets with zero measure (Oxtoby, 1980).

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Structuralist praxeologies in the perspective of Klein: the case of connectedness in analysis

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This article is a continuation of our previous work on structuralist aspects of analysis at university in the framework of the Anthropological Theory of the Didactic. We now examine how the teaching of abstract structures (in particular metric spaces and topology) may contribute to Klein's perspective of "Elementary Mathematics from a higher standpoint". Precisely, on the basis of a real and abstract analysis textbook used in the transition between Bachelor and Master degree programs in mathematics and in the light of the notion of structuralist praxeology and its dialectics, we discuss whether we can defend teaching the notion of connectedness to teacher students as a means to link real and abstract analyses in the spirit of Klein.

Keywords: Teaching and learning of analysis and calculus; Transition to, across and from university mathematics; Curricular and institutional issues concerning the teaching of mathematics at university level; Structuralist praxeologies; Anthropological Theory of the Didactic

INTRODUCTION

At the beginning of the last century, Felix Klein developed material for university lectures for teacher students in the form of three volumes *Elementary Mathematics from a higher standpoint.* Still today, his seminal ideas and methodological orientations continue to inspire mathematics education research (Weigand, McCallum, Menghini, Neubrand and Schubring, 2019). In particular, Klein posed the issue of the relation between school mathematics and academic mathematics, in other words, how a future teacher can be introduced to further advances in mathematics so that this knowledge is useful for his or her role as a secondary school teacher. A main general principle in this endeavour is to underline the mutual connections between problems in the various sub-disciplines of mathematics, offering a synthetic and holistic view on mathematics, and to emphasize relations with problems posed at school. This also concerns their mutual motivation and in particular addressing the rationale of school mathematics content beyond references that are currently emphasised as being immediately relevant to everyday life.

At Klein's time, modern mathematics had not yet taken off, but Klein was aware that a process of conceptual rewriting of mathematics was taking place in the natural historical development of the field, and that the fruits of this process should make it possible to modernise and invigorate mathematics teaching at all levels:

> The normal process of development […] of a science is the following: higher and more complicated parts become gradually more elementary, due to the increase in the capacity to understand the concepts and to the simplification of their exposition. It constitutes the task of the school to verify, in view of the requirements of general education, whether the

> introduction of elementarised concepts into the syllabus is necessary or not. (Klein and Schimmack 1907, p. 90)

This discourse resonates with the didactical benefits of mathematical structuralism later put in the fore by Bourbaki in the Manifesto *The Architecture of Mathematics*. Concepts and structures are key ingredients of an integrated perspective on mathematics, and generality may clarify and simplify the exposition. We do not claim that Klein was structuralist in his pedagogy; on the contrary, he paid great attention to balance logic and intuition. Our perspective, in a context where mathematical structuralism has massively impacted university mathematics curricula, is to examine the extent to which the teaching and learning of abstract mathematical structures at university can contribute to the implementation of Klein's vision or depart from it.

We focus in this paper on the case of the relationship between real analysis, as it is taught at the secondary-tertiary transition, and abstract analysis (metric spaces, topology, Banach spaces, Hilbert spaces,…). In previous work (Hausberger and Hochmuth, 2023), we gave historical anchor points of the emergence of such a realm of structures that generalised real analysis and applied a model, initially developed in the framework of the Anthropological Theory of the Didactic (ATD) for abstract algebra, to account for the transition from real to abstract analysis throughout the Bachelor and Master degree programmes. We used this model to study excerpts of a textbook (Bridges, 1998), used at the Bachelor-Master transition and chosen for its didactic project: to make visible how the concepts and theorems of abstract analysis enlighten real analysis, which is first recapitulated in view of its generalisation. The main point was to detect the continuities and ruptures that might be observed in the shift towards abstraction that accompanied the rise of mathematical structures in analysis, or in our ATD terms the development of structuralist praxeologies. Moreover, the model puts in the fore a dialectic of objects and structures (see theoretical framework), in other words a dialectic of the particular and the general or of the concrete and the abstract that characterise structuralist thinking. The vitality of this dialectic was observed in relation to the issue of motivating abstract concepts.

This paper is a continuation of the previous work, but with a slightly different focus: we now consider relationships with school mathematics and teacher training, in the spirit of Klein. As a case study, we analyse from the viewpoint of structuralist praxeologies the tasks assigned by Bridges in his textbook around the topological notion of connectedness in metric spaces and in relation to real analysis. For example, Bridges generalises the intermediate value theorem (IVT) which is used as a main motivation for connectedness. Our main research question is the following: on the basis of this textbook and in the light of our analysis tools, can we defend teaching the notion of connectedness to teacher students as linking real analysis and abstract analysis in the spirit of Klein? We begin by presenting our theoretical framework, and then go on to analyse selected extracts from the textbook using these tools. Finally, we draw conclusions in relation to the problem posed by Klein, and conclude by outlining a few extensions we envisage to this research.

THEORETICAL FRAMEWORK

According to ATD (Chevallard & Bosch, 2020), every human activity consists in the coordination of a praxis and a logos, hence the key notion of a praxeology, represented by a quadruple [T/τ/θ/Θ]. Its practical-technical block (or know-how) consists of a type of tasks T together with a corresponding technique τ (useful to carry out the tasks $t \in T$). The technological-theoretical block (or know-why) comprises the technology θ, a discourse on the technique, and the theory Θ, the ultimate level of justification. We continue by presenting the tools developed specifically in ATD to analyse praxeologies based on mathematical structuralism. The reader may wish to consult our previous paper (Hausberger and Hochmuth, 2023) for connections with other works offering a praxeological modelling of mathematical practices in the analysis track at university.

The starting point is the consideration of mathematical structuralism as a method, which consists of reasoning in terms of classes of objects, relations between these classes and stability properties for operations on structures (Hausberger, 2018). The general view of structures thus allows particular properties of objects to be demonstrated by making them appear as consequences of more general facts (theorems about structures). Dually, generalisations are put to the test of objects, hence a *dialectical relationship between objects and structures*. In praxeological terms, the structuralist method consists in the passage from a praxeology **P** = [T/?/?/$\Theta_{\text{particular}}$] where it is unclear which technique to apply, to a structuralist praxeology $\mathbf{P_s}$ = [T^g/τ/θ/$\Theta_{\text{structure}}$] where, modulo generalisation of the type of tasks (T^g), the theory of a given type of structure guides the mathematician in solving the problem. It is important to point out that this transition (called type 1) leads to a structuralist praxeology whose rationale is related to its ability to solve concrete problems (related to **P**) by a gain in technology permitted by the insight of structures.

Moreover, Hausberger (2018) distinguishes two structuralist levels of praxeologies: at level 1, structures act as a vocabulary and appear mainly through definitions (e.g. a task of type T "show that a given function between given metric spaces is bounded" is solved by checking that the definition of boundedness is satisfied); at level 2, the technique mobilises general results about structures (e.g. any continuous function from a compact space into a metric space is bounded). In the process of developing the level 2 contextualised (since the function and metric spaces are given) structuralist praxeology, an abstract task is assigned (prove the theorem) to establish its technology.

The latter task is of type T^g "show that any function between metric spaces that fulfils given conceptual properties is bounded". At this stage, it remains more or less an isolated task. But in the teaching and learning of structures, the stage is reached when praxeologies based on such abstract types of tasks (that concern abstractly defined classes of objects, e.g. generic functions between generic metric spaces) are developed. In this context, the key structuralist insight (for the preceding example) that compactness is preserved by continuous mappings comes in the fore together with other connections between the various concepts that are involved (continuity, closedness, boundedness, compactness). This is called the type 2 transition, situated between the Bachelor and Master degree programs. It is important to note that the new purely abstract praxis *Π'* shall be anchored on reasoning with concepts that take their origin in the logos of previously developed structuralist praxeologies denoted $\mathbf{P}_s$. This connection between the two types of transition is vital for a sound (properly motivated) development of abstract analysis.

One shall note that generic objects such as generic real functions already appear in early analysis courses in the context of abstract tasks of what we called pre-structuralist praxeologies (Laukert et al., 2023), but the properties of functions and their domain/co-domain ($\mathbf{R}$ or $\mathbf{R}^n$) that play a role are not fully elucidated in terms of structures (topological concepts, metric spaces and functional analysis), hence the terminology pre-structuralist. In fact, analysis mixes different kinds of structures and some results in real analysis that closely intertwine different structuralist aspects may be hard to extend to natural structuralist statements or may lead to different general statements that capture some aspects of the initial problem while abstracting other aspects. Real analysis certainly constitutes a body of praxeologies that cannot be reduced to its structuralist dimensions developed and revealed through the abstract analysis.

To conclude this theoretical framework, let us emphasize key features of structuralist praxeologies that relate to the perspective of Klein. Our main research hypothesis is that smooth transitions of type 1 (and 2, to some extent) with a vitality of the dialectic of objects and structures allows to meet the vision of Klein since: i) structuralist concepts become tools to solve concrete problems ii) structures unify different branches of mathematics (e.g. geometry and analysis, by bringing geometrical insights into analysis through topology) iii) the conceptual perspective brings a new foundation to real analysis in terms of more general principles, which increases the understanding of the reasons why theorems hold true iv) the type 1 transition connects university to school mathematics. On the opposite, discontinuities in the type 1 transition hinders the realisation of Klein's perspective.

ANALYSIS OF THE TEXTBOOK EXCERPTS ON CONNECTEDNESS

The subchapter starts with a definition of connectedness: a metric space is called connected if it is not the union of two disjoint nonempty open subsets (p. 158). Since the family of open subsets gives the topology of the metric space, connectedness is a topological property. Connectedness formalizes the idea that a metric space, or a subspace of it, "cannot be split into smaller, separated parts". Since closed subsets are precisely the complementary sets of the open subsets, connectedness can analogously characterized by the non-existence of two disjoint nonempty closed subsets whose union gives the whole space (3.4.1 (ii)). The point of view that connectedness is a topological notion is further strengthened by presenting a characterization using continuous functions (i.e. functions whose preimage of an open set is an open set): a metric space is connected if and only if there is no continuous mapping from the space onto $\{0,1\}$, the typical disconnected subset (Exercise 3.4.2.2). It is also a first step in the direction of a generalisation of the intermediate value theorem (IVT) as it connects the concepts of connectedness and continuity of mappings.

Immediately afterwards the following question is posed: what are precisely the connected sets in **R**? This question indicates the type of task T_1 "determine connected subsets of a given metric space". The proof for **R** (Proposition 3.4.3) uses the intermediate value property (IVP) of an interval I, which we see as a pre-structuralist characterisation: I contains the segment $[c,d]$ for any two points $c < d$ in I. Intervals have been defined comprehensively as subsets of the form $|a,b|$ (open, closed or semi-open) and characterized by this property (Proposition 1.3.3) in the chapter on real analysis. The supremum axiom of **R** is used, but not enlightened in structuralist terms (concept of completeness). The embryo of a praxeology that is being developed remains of structuralist level 1, since it uses mainly the definition of connectedness (and that of open/closed subsets).

Every result on connectedness, contextualised to **R**, can thus be reduced to the case of intervals. This leads to question what the structure-oriented extension of connectedness in metric spaces may bring to real analysis. The answer remains unclear at this stage. The IVP involves the partial order in **R** while connectedness applies to topological spaces in general, but the benefit of generality remains to be seen.

The following Exercise 3.4.4 establishes the type of tasks T_2 "prove that a subset of a metric space is connected". In the assigned tasks, subsets are given abstractly by union/intersection of nonempty closed subspaces. The aim is obviously to prove a structuralist theorem to feed a level 2 contextualised structuralist praxeology based on the same type of tasks, but no example of such application is given. The next Proposition 3.4.5, posing the stability of connectedness under taking adherence points (S to $\overline{S}$), as well as Proposition 3.4.6 and exercises 3.4.7 1-2 fit with T_2: again, the considered subsets are defined abstractly by other properties (e.g. $\overline{S} \cap T \neq \varnothing$ is non empty and S,T connected). So far, the course focuses on structuralist stability properties of connectedness. With regard to **R**, however, the results are without specific gain: e.g., intervals obviously remain intervals by adding adherence points.

New notions are introduced and considered through exercises 3.4.7.3-6: chain connectedness, connected component, total disconnectedness (the connected component of a point is the singleton), local connectedness. There is again a task of type T_2, resulting in the theorem that a compact, chain connected set is connected. Moreover, further properties of connected components are established, but without any contextualisation, except the following: the notions of connected components and local connectedness are applied in Exercise 3.4.7.7 to give a conceptual proof of the description of open subsets of **R** as unions of open intervals. This characterisation has already been proved in Proposition 1.3.6 of the real analysis chapter. Here, too, we cannot see any particular gain from the abstract treatment in metric spaces (except aethetics to the eye of a structuralist mathematician, which is not a didactical criteria).

Then two tasks of type T_3 “prove that a subset of a metric space is totally disconnected” are assigned. In the case of **R**, any countable subset like **Q** turns out to be disconnected (first task), but also the set of irrational numbers (second task), which are known to be uncountable. One may note that the type of tasks T_2 may have been assigned, instead of T_3, to both the rational and irrational numbers (viewed as metric spaces for the distance inherited from **R**), in order to introduce the notion of totally disconnectedness from a bottom-up perspective.

The stability of connectedness under continuous mappings is expressed by Proposition 3.4.8. That the range of a continuous mapping from a connected metric space is connected is an immediate consequence of the topological definitions of connectedness and continuity. The consequence for continuous mappings from a connected metric space to **R** is then addressed as the generalised IVT (figure 1) and is an important consequence of this structuralist principle: theorem 3.4.9 thus elucidates the structure of the domain of the function so that the IVP on the range holds, but the codomain remains **R** without further structuralist insights. Also, the role played by completeness of **R** remains somehow implicit in this result, as it was in the observation that intervals are the connected subsets in **R**. Finally, this new theorem is not applied to more general contexts than that of the IVT itself (real numbers), where as a contextualised level 2 structuralist praxeology based on T_4 “prove that the range of a mapping has the IVP” could have been developed to assign a practical rationale to connectedness and the generalised IVT altogether.

A very important consequence of Proposition (3.4.8) is the following generalised *Intermediate Value Theorem.*

(3.4.9) Theorem. *Let f be a continuous mapping of a connected metric space X into $\mathbf{R}$, and a, b points of $f(X)$ such that $a < b$. Then for each $y \in (a, b)$ there exists $x \in X$ such that $f(x) = y$.*

Proof. By Propositions (3.4.8) and (3.4.3), $f(X)$ is an interval. The result follows immediately. □

Figure 1: the generalised intermediate value theorem

Then a couple of exercises that draw general consequences from the continuity of the distance in metric spaces are posed (e.g. 3.4.10.1 about unbounded connected metric spaces), without hinting at the scope of these results and without further enlightening the real analysis context.

Proposition 3.4.11 connects to the notion of compactness and uniform convergence: whenever X is connected, the uniform continuity of every continuous real valued function on X is equivalent to the compactness of X. We see this result as emblematic of the transition of type 2. Nevertheless, the uniform continuity of functions defined on compact, i.e. bounded and closed, intervals of **R**, is a well-known result in real analysis. That bounded and closed are not only sufficient but also necessary is usually justified by counter-examples considering continuous functions on unbounded as well as on bounded and open intervals. The proposition enlightens the role of the concept of compactness in a more general context considering real-valued functions on connected metric spaces. It in particular shows in which sense compactness is necessary for uniform continuity to hold and establishes the real analysis result on a more general ground. Dually, the real analysis background serves as an anchor point for the rise towards abstraction, hence serves the transition. Unfortunately, these connections remain implicit in the textbook, they are not discussed.

In the end of the chapter, the notion of path-connectedness is considered in metric spaces. It generalises the idea that a subset does not consist of isolated separate parts in the sense that two points can be joined by a path lying wholly in the subset. Proposition 3.4.12 establishes that path-connectedness implies connectedness. Path-connectedness is a notion which is considered in multivariable real analysis contexts to also generalise the concept of interval in **R**. Another, perhaps more straightforward possibility, would be convexity, where the choice of paths is restricted to straight lines. Convexity is stronger than path-connectedness. That convexity is not discussed as alternative can be interpreted as a symptom for the focus of the author on topological perspectives in his presentation and a lack of meta-discourse on the raison d'être of generalised notions in a view to Real Analysis needs. Regarding $\mathbf{R}^n$, Proposition 3.4.13 establishes that connected open subsets of $\mathbf{R}^n$ are path-connected, i.e. connectedness and path-connectedness are equivalent for open sets in $\mathbf{R}^n$. The obvious question about the case of closed connected subsets of $\mathbf{R}^n$ is not raised (a negative answer). Thus, with a view on a real analysis context, the difference between connectedness and path-connectedness remains weakly clarified.

Altogether, the praxeology based on the type of tasks T_5 "Prove that a subset of a metric space is path-connected (or not)" is only weakly developed. Obviously, neither T_2 nor T_5 are relevant in **R**, since the connected subsets in **R** are the intervals and intervals are trivially path-connected. Are the types of tasks T_2 and T_5 relevant when contextualised to $\mathbf{R}^n$ i.e. in the multivariable real analysis context? Exercises 3.4.16.1 contains the only example of subsets in $\mathbf{R}^n$ whose path-connectedness is questioned. The set $B=\{(x,y)\in \mathrm{R}^2 : 0<x\leq 1, y=\sin\frac{\pi}{x}\}$, i.e. the graph of a function defined on a semi-open interval, and the set $A=\{(0,y)\in \mathrm{R}^2 : -1\leq x\leq 1\}$, i.e. the set of adherence points of the graph, are considered. Then $A\cup B$ is not path-connected. We may infer that, in the structuralist perspective of the author, this task of type T_5 is assigned in order to present a counter-example concerning path-connectedness of closed connected subsets and stability of path-connectedness under taking adherence points.

At this point, we asked ourselves why the following relationship to real analysis, which also appears relevant for school mathematics, was not established: at school, the continuity of a function is typically explained by the fact that the graph of the function can be drawn “without lifting the pen”. This descriptive property can be formally interpreted as the path-connectedness of the graph. This is supported by the observation/theorem that a real-valued function defined on $\mathbf{R}$ is continuous if and only if its graph is path-connected in $\mathbf{R}^2$.

Finally exercises 3.4.16 2-3 question the stability of path-connectedness under union and intersection, which holds true under some extra conditions. Again, these are structuralist aspects whose practical usefulness is not underlined through contextualised level 2 structuralist praxeologies.

DISCUSSION

To summarise, we observed that the definition of the concepts and their embedding in the theory of metric spaces, as well as resulting new concept-based proofs, appear to be poorly motivated, despite the relationship with the intermediate value theorem (IVT). A contributing factor is that the scope of structuralist praxeologies on the topic of connectedness is scarcely or not developed at all and the types of tasks remain fairly limited in number and diversity. In particular, level 2 contextualised structuralist praxeologies are missing. As a result, a raison d’être cannot come to life. In addition, the role played by other properties of real numbers (such as completeness or the ordered field structure) is not discussed in relation to the IVT. Only connectedness is elucidated. This is another reason why the results in the chapter under consideration tend to remain isolated. Large parts focus on theoretical development (the type 2 transition), which for the reasons mentioned has little explicit connections to the type 1 transition. These observations, formulated in praxeological terms with a view to structuralist aspects and their transitions, have consequences on Klein’s project that we will now underline.

The starting point of the contribution was the question of the educational benefits to teach student teachers connectedness in metric spaces. Criteria for an answer emerged from connections between Klein’s project and the point of view of structuralist praxeologies, mainly the idea of smooth structuralist transitions as developed in the theoretical framework. From our analysis of a typical textbook, our position is divided. We begin by the drawbacks, which relate to inadequate didactic means to achieve the type 1 transition on the topic of connectedness.

a) Basically, the concept of connectedness remains weakly motivated by considerations on the real numbers. Starting from intervals and their properties, other possibilities to generalise their features could have been reflected in the transition to $\mathbf{R}^n$, such as convexity or path-connectedness. To decide between these different alternatives, the question of the stability of properties under a continuous mapping could have been asked, in the structuralist spirit. In particular, this structuralist behaviour is the key point to tackle the following issue: which alternative leads to a generalisation of the intermediate value theorem?

b) The textbook primarily presents conceptual proofs whose potential to tackle interesting questions in the perspective of Klein's project is still doubtful. The general concepts appear to be motivated by the goal of generalisation itself, but are hardly anchored in real analysis issues (other than the IVT). For example: Exercise 3.4.7.7 (determination of the open sets of the real line) mobilises a conceptual proof of a connection already established in real analysis, which may be interesting for pure mathematicians and may be satisfying in terms of aesthetic value, but the added practical value for student teachers remains at least unclear and is actually not worked out.
c) With regard to the concept of connectedness, the text to a certain extent constitutes something that remains stuck between two worlds: on the one hand, a (relatively) concrete world of real analysis based on real numbers as school objects and already considered in terms of axioms, and on the other hand a (relatively) abstract world of metric spaces. A dialectic between concrete and abstract (in our framework, objects and structures) is thus not brought to life.

Nevertheless, the text provides starting points for a depiction of connectedness that could contribute to Klein's agenda. What are such starting points?

a) Connectedness in **Q** could have been investigated: are the connected sets there also the intervals? Unlike the textbook, the definition of a totally disconnected subset would have arisen as a concluding step of this investigation and not as a starting point.
b) Similarly, path-connectedness may have been related to the process of formalisation of the intuitive idea that the curve of a continuous function can be drawn without lifting the pen: continuous real functions are characterised by the path-connectedness of their graph. Exercise 3.4.16.1 of the textbook further elaborates on this idea to construct a counter-example to structuralist assertions without making explicit this connection with the intuitive notion of graph from school mathematics.
c) The question we formulated in point a) of the drawbacks could have been investigated to motivate the notion of connectedness. The idea that a missing point in the interior of an interval decomposes it into two disjoint closed subsets may serve as an argument to introduce the definition, among other arguments. On a meta level, the following issue needs to be addressed to implement didactical aspects of this investigation: what degree of generalisation beyond metric spaces would be adequate to foster the development of helpful structuralist insights among teacher students? In particular, to what extent should general topology be developed?

Although the textbook attempted to elaborate abstract analysis from real analysis in a bottom-up perspective, the overarching viewpoint of structures is dominating the presentation. In our opinion, fulfilling Klein's project would require a different kind of textbook that more successfully implements a dialectical point of view between real and abstract analysis, in other words between objects and structures.

GENERAL CONCLUSION AND OUTLOOK

The notion of structuralist praxeology in ATD, with its structuralist levels, its two transitions and the objects-structures dialectic, has made it possible to study Klein's didactic problematic by virtue of the continuities and ruptures that it highlights in the passage from real analysis to abstract analysis. Both the research on structuralist praxeologies and that on Klein's pedagogical programme are inscribed in a questioning of transitions (Klein's double discontinuity for the former and transitions within university for the latter).

The use of the notion of structuralist praxeology is not limited to the analysis of what already exists, but also makes it possible to engage in didactic design with the aim of developing structuralist praxeologies useful to future teachers. In this endeavour, particular attention should be paid to the objects-structures dialectic and the process of questioning objects (e.g. continuous real functions and their properties) in the light of structures (Hausberger, 2019) may be envisaged in the didactic perspective of Questioning the World (Chevallard & Bosch, 2020) in ATD. This is one of the main avenues for the development of this work, as our analysis has shown punctual weaknesses of a textbook while at the same time identifying avenues for the implementation of Klein's ideas, around the notion of connectedness as an emblematic case.

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Engineering students' deed-oriented learning opportunities in a mathematical discourse about integrals

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Explorative and ritualistic routines, and the interplay between them have been the main foci of commognitive research on routine use, with less focus on the routine of deeds. The study presented here concerns engineering students' routines of deeds while using an animation tool for visualizing the mathematical object 'integral' as an area in a motion-context. We study the students' learning opportunities in four task situations. Inspired by the work by Nachlieli and Tabach (2019), we modified their methodological tool for analysing ritual-enabling, exploration-requiring opportunities to learn, and added a third component: deed-oriented opportunities to learn. In our study, we observed opportunities for deed-oriented routines to enable more explorative routines.

Keywords: Teaching and learning of mathematics in other disciplines, Teaching and learning of analysis and calculus, Exploration-requiring learning opportunities, Deed-oriented learning opportunities.

INTRODUCTION

Engineering students are introduced to a variety of tasks during their studies. Some of the tasks are more authentic than others. Studies show that there is a gap between educational practices and professional practices where the tasks during studies are more mathematical inclined and the mathematics used in workplaces are more applied to tangible objects. There is a concern that the engineering studies are not addressing the specific needs of engineering students for preparing them for their professions (González-Martín et al., 2022).

The mathematical object 'Integral' is one of the important objects engineering students learn about during their studies. In a frequently-used textbook for mathematics in engineering education, *Calculus: A complete course* by Adams and Essex (2022), 'Integral' has received considerable attention. The book introduces the object as realizations of areas as limits of sums, then introduce the definite integral and its properties, which leads up to the fundamental theorem of calculus. Then a basic area problem (find the area of region R) is presented with following examples and tasks for the students to solve. Thereafter some integral techniques are presented: integration by parts, integrals of rational functions, inverse substitution, other methods for evaluating integrals, improper integrals, the trapezoid and midpoint rules, Simpson's rule and other aspects of approximate integration. Lastly, applications of integration and techniques are introduced (e.g., how to calculate volumes of solids) and corresponding tasks provided before other mathematical topics are focused on in the rest of the book.

The tasks offered to the students are mostly mathematical, with less attention to a real-world context.

Our study contributes to the research on engineering students' learning opportunities in mathematics. We view learning opportunities as students' opportunities "to build or strengthen connections among related mathematical ideas—and to consider these ideas in relation to how [other] students think about the ideas" (Silver et al., 2007, p. 261). In this paper, we investigate engineering students' mathematical discourse while they are using an animation tool designed for educational purposes in engineering education. The tool animates two cars' motion and visualizes their velocity-time curves (the tool is presented more in detail later in the context of study).

THEORETICAL BACKGROUND AND RESEARCH QUESTION

We take a commognitive perspective on learning. Within commognition, a mathematical discourse is determined by the participants' word use (oral or written use of mathematical keywords), visual mediators operated upon (such as graphs, diagrams, symbols etc), routines (established patterns in how to solve a task which are repeated in similar situations) and narratives (stories about mathematical objects that can be endorsed). Visual mediators are any visual realization of an object of the discourse. The object may be a primary object existing outside the discourse and artifacts created for communication purposes. A narrative is utterances, spoken and written, framed as a story about mathematical objects operated upon which can be endorsed by the mathematical community (e.g., theorems and axioms). The discursive process of convincing that a narrative can be endorsed is the substantiation of a narrative (Sfard, 2008).

Routines: Rituals, explorations and deeds

Routines are repetitive patterns we turn to in certain situations: from saying 'hello' to the cashier when buying items in a store to how you regularly proceed in solving a familiar mathematical task. When we meet a familiar situation, we are most likely to behave in a way we have learned by others, leading us to act in similar ways. A routine consists of three parts: initiation, procedure and closure. The routine is being initiated by a task to complete, where you conduct a procedure and decide under which conditions the procedure is completed (Sfard, 2008).

There are three types of routines: explorations, rituals and deeds (Sfard, 2008). An explorative routine is characterized by explorative participation, focusing on producing mathematical narratives that can be endorsed by the mathematical community. Ritual routines are routines consisting of a rigid procedure and whose success is depending on others (e.g., the teacher or another more well-versed participant in the discourse). Deeds are routines consisting of transforming objects (discursive or primary) to new, transformed or re-arranged objects (discursive or primary), e.g., transforming animated cars visualized in a digital tool (see Figure 1). We want to contribute to this line of research, focusing on engineering students' learning opportunities related to deeds.

Deeds are divided in two types: practical deeds and discursive deeds. Practical deeds serve as a catalyst for a transformation of primary objects (objects existing outside of the mathematical discourse), while discursive deeds have a goal of a change in discursive objects (e.g., manipulation of an equation to get the unknown value to become a certain numerical value) (Sfard, 2008).

Ritualistic, explorative and deed-oriented learning

Learning mathematics is viewed as “a process of routinization of learners’ actions” (Lavie et al., 2019, p. 153). The initial routines are most likely implemented as rituals. That is, for a learner to enter a new discourse, he or she is most likely to imitate the actions by participants who are more well versed in the discourse. Thereafter, these routines are expected to gradually become explorations through a de-ritualization process. The learner is expected to move from asking oneself ‘How do I proceed?’ as focused on in ritual routines to ‘What is it that I want to achieve?’ as focused on in explorations (and deeds). The de-ritualization process may be slow and gradual, where the move from one type of a routine to another is depending on the learners’ awareness of its practical application (Lavie et al., 2019).

There are two levels of learning: object-level and meta-level learning (Sfard, 2008). Object-level learning involves “endorsing new narratives about familiar objects” (Nachlieli & Tabach, 2019, p. 256), while meta-level learning involves a use of keywords in a different way, leading to a transition between two incommensurable discourses (Sfard, 2008). In the latter, the ‘rules of the game’ changes, in which students are engaged in a discourse about their discourse.

Some researchers focus on designing tasks that support a particular routine use, in particular explorative routines (Cooper & Lavie, 2021). Baccaglini-Frank (2021) shows how dynamic interactive mediators can foster high school students’ explorative routine use in mathematical discourse.

Nachlieli and Tabach (2019) identify ritualistic teaching goals while studying 11 lessons in an eight-grade classroom from a TIMSS study (acronym for Trends in International Mathematics and Science Study). They conclude that a ritual-enabling opportunity to learn (teacher’s actions providing students with tasks that can be solved using a ritualistic routine) may act as a departure point for exploration-requiring opportunities to learn (teacher’s actions providing tasks that can be solved successfully only by participating exploratively). An opportunity to learn was categorized as ritual-enabling when the students were offered tasks that can be successfully solved in a ritualistic manner (performing a rigid use of procedures previously learned). An opportunity to learn was categorized as exploration-requiring when the tasks require an explorative participation to be solved successfully (focusing on producing an endorsed narrative). In this study, we focus on a deed-oriented opportunity to learn as when the task can be successfully solved by directly transforming mathematical or primary objects. Our research question is:

What are the learning opportunities from deed-oriented participation using an animation tool?

METHODS

Context of study

Our participants were 1st year engineering students enrolled in an elementary physics course at a public university in Norway. They had previously finished an elementary calculus course, including integral as an area below a curve. We give the students the fictive names Erik, Sam and Tom. They participated in the study outside of regular classes. They engaged in a new task situation consisting of four questions and with access to a digital animation tool called Sim2Bil. Sim2Bil can be seen in Figure 1.

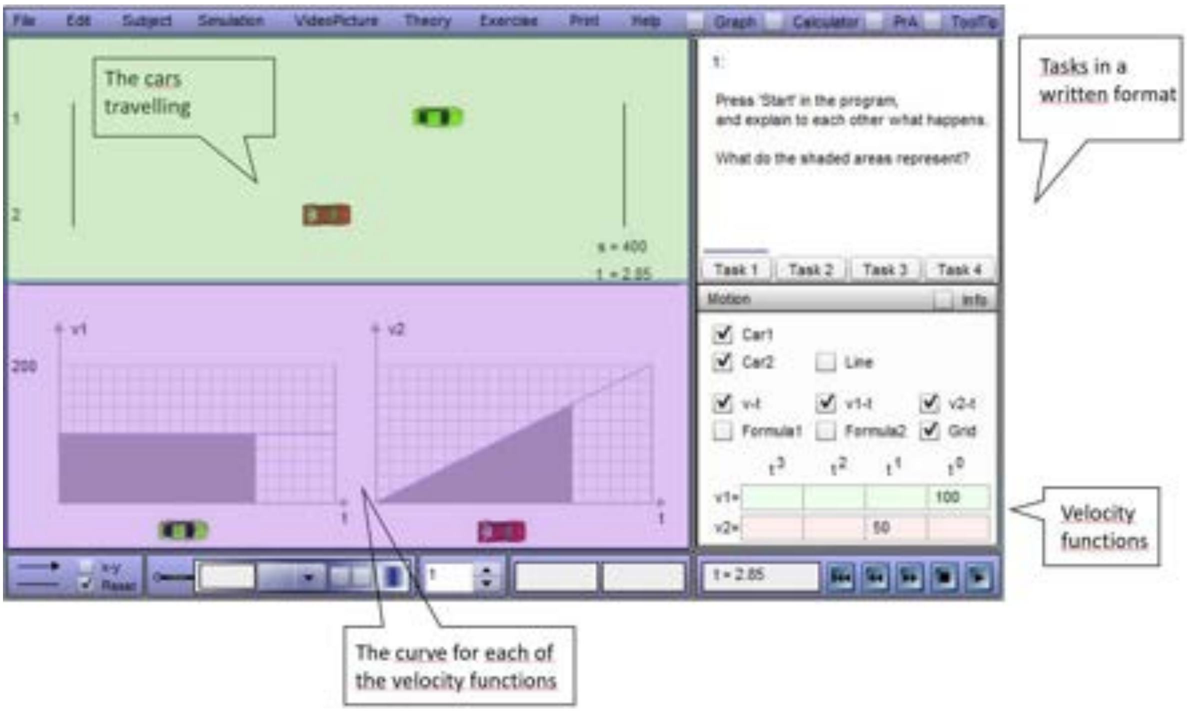

Figure 1: The interface of Sim2Bil

Sim2Bil offers an animation of two cars driving in a straight line from a start line to a finish line. The cars' velocity functions can be inserted in the bottom right corner of the interface. The velocity-time curve for each of the cars can be seen in the bottom left corner. The shaded regions beneath the two curves will be shaded as the animation runs. The students can realize the shaded regions as the cars' distance travelled. The pedagogical tool is designed to realize integral as an area under a curve and an accumulation function when running the animation. The students were offered the following four tasks:

T1: Press 'Start' in the program and explain to each other what happens. What do the shaded regions represent?

T2: Determine other values in the table so that the cars drive with different velocities and arrive at the finish line simultaneously.

T3: What can you do to make the green car only halfway when the red car reaches the finish line?

T4: Find the velocity to the green car and the red car so that v2 is half of v1 when they arrive at the finish line simultaneously at four seconds. Can you prove that your answer is correct?

In the first task (T1), the question of what the shaded regions are, the students are giving an opportunity to realize the definite integral as areas under curves and integrated velocity functions as distance covered. The cars' velocities can be integrated with respect to time to find the distance travelled at any specific time. These realizations can be derived by realizations of velocities as the derivative of positions with respect to time and the integral as an accumulation function. Several velocity functions can be a solution to each of the tasks 2-4, and the students are expected to determine the degree of polynomial functions and further use integration procedures. They might also reason about the areas under the two curves. In T4, the students are asked to take a meta-perspective on their discourse when asked to verify their answer.

Discursive analysis

The students' task situation was videorecorded and later transcribed. For our analysis, we took into account the students' word use (words spoken), visual mediators operated upon, narratives produced (students' stories about the mathematical objects involved which can be endorsed by the mathematical community) and routines established (repetitive patterns for how to solve a task where the task to be solved is determined by the task performer). Further, we used the questions in Table 1 (see the second column) to identify the types of routines the students were engaged in.

We used a modified version of the methodological lens by Nachlieli and Tabach (2019), presented in the form of a table (Table 1), and included deed-oriented OTLs (opportunities to learn) as task situations that invite students to perform deeds. The rows in Table 1 relate to routine stages and concern the "when" (initiation and closure) and "how" (procedure) of a routine. The cell directly to the right of each of these three routine stages includes questions to describe each of these stages. The cells in the other columns to the right, include descriptions of what to look for to determine whether the task situation offers ritual-enabling, exploration-requiring or deed-oriented learning opportunities.

We kept the original meaning of the terms ritual-enabling and exploration-requiring as participants' actions that provide tasks that could be successfully performed in a ritualistic or explorative manner without necessarily actually performing the ritual and/or explorative routines. We considered the teacher to be implicitly present through the questions and opportunities to learn through the animation tool. By deed-oriented learning opportunities we mean actions that were oriented towards providing tasks that could be performed by a change or re-arrangement of primary or discursive objects.

Thereafter, we used the questions in Table 1 to identify opportunities to learn offered in the task situations as ritual-enabling, exploration-requiring or deed-oriented.

Routine stages \ OTL types		Ritual -enabling	Exploration -requiring	Deed-oriented
1. Initiation	What is the question posed to or raised by the students?	How do you proceed?	What do you want to achieve?	What do you want to achieve?
2. Procedure	How is the procedure of the routine determined?	Students are expected to apply a rigid procedure that was previously performed by others in similar situations.	Students are expected to choose from alternative procedures.	Students are expected to change primary objects (existing independent of discourse) or discursive objects (originate in discourse) to new, transformed, or rearranged objects.
	What is the agency of the students?	Students are not expected to make independent decisions.	Students are expected to make independent decisions.	Students are expected to make independent decisions.
3. Closure	What type of answer does the students expect?	A final answer. If reasoning is provided, it details the steps of the applied procedure.	Stating the new narrative produced. If reasoning is provided, it details the mathematical reasoning involved.	Stating the outcome (or an expected outcome) of the change of primary or discursive objects. If reasoning is provided, it details the steps of the physical manipulation.
	Who determines the end conditions (to indicate the task has ended)?	Others (e.g., the teacher).	The student (based on mathematical reasoning).	The student (based on the outcome).

Table 1: Methodological lens – ritual-exploration-deed OTL's.

RESULTS

In this section, we first introduce a summary of the students' work during their 45 minutes session with the four tasks. Thereafter, we highlight some excerpts from our data and discuss our results on what can be achieved by deed-oriented OTL's.

A summary of the students' work

The students started to get familiar with the task situation. They immediately recalled the shaded regions as the cars' distance travelled when answering T1. Then, Erik began to explore the animation tool by inserting velocity functions and play the animation. Tom started to mathematize the given problem by suggesting setting up an equation set. Sam paid attention to what the previous two were doing. The way the students approached the task situation was repeated during the whole session. While Tom repeatedly tried to draw Erik's attention towards mathematizing the problem and integrating the chosen equation set to find velocity functions, Erik was more drawn towards Sim2Bil to find an answer there. Once Erik did not succeed in finding appropriate velocity functions through Sim2Bil to meet the requirements in the tasks (T2-T4) after several attempts, he turned towards Tom and tried to keep up with his mathematizing. Tom continued to explore, but when he sees that Erik does not have the means to continue on his own nor follow Tom's explorations, Tom took a different approach. He turned to Erik, explaining step-by-step how they can proceed and thus offered Erik a ritualistic way of acting. We interpret Tom's attempts as offering Erik a ritual-enabling opportunity to learn for inviting him to his more explorative way of acting.

Deed-oriented learning opportunities enabling explorative routines

In the following excerpt, the students work on the task T3 in which Tom suggests integrating two velocity functions but does not follow the requirement in the task T3 (distance s1 should be half of s2).

Tom:	Ehm… Then we have two things to integrate. We shall have when t equals four… then s1 equals s2.
Erik:	No, no, no.
Sam:	No, it should be…
Erik:	The green should be halfway when the red car arrives at the finish line.

The students are talking about that they are now seeing the outcome to be the same.

Erik:	Isn't it just to multiply one of them [the velocity function] with two?
Tom:	Then, then it can be something like…Hm…Yes, or if we set it equal to two hundred then of course.

In this excerpt, the students start a discussion where they find out that they see the outcome to be different. At this stage, Tom thinks the cars should travel the same distance as in T2. The students continue to discuss and agree on the movements of the cars to be in accordance with the requirements of the task (T3) and agreeing on how they should proceed. Once the students have derived two velocity functions and played the animation where they see the cars' movement towards the finish line, they start to comment on what they see:

Erik: It certainly looks halfway, but can we measure it in some way?

Sam: Integrate again?

Tom: Now we have actually calculated it.

The students find an interactive arrow (in the bottom left corner of the interface of Sim2Bil). They place the end of the arrow close to one of the cars and then extend the length of the arrow until the head of the arrow is placed at the finish line. The magnitude of the length of the arrow appears at the screen. In the following excerpt, Tom draws their attention to what they previously have done.

Tom: Do you see… Previously, right… we decided that just after four seconds, it should have gone four hundred meters. And then we in fact inserted in that after four seconds, that one should have gone two hundred meters, so…

We interpret that the students had different ways of convincing that their answer can be substantiated. For Tom, it was enough with the narratives that have been developed, while Erik seeks a substantiation through the animation tool (either by watching the cars' movement or measuring the distance travelled for the cars).

In the following excerpt, the students work on T4. After they decided that v1=100m/s, Erik starts by saying that they need to find a function that has a certain value (50m/s, half of v1) at the finish line.

Erik: And then we just have to find one or another function that makes that one…

Tom: That makes that there fifty [pointing at the end of the curve in Sim2Bil].

Erik: It is fifty there when it hits. And then… it is this one [the green car] … starts in hundred.

In the above excerpt, the students agree on what the numerical value of the velocity function should be at a certain time ($v_2(4) = 50$). We interpret that deed-oriented learning opportunities within Sim2Bil offer the students with something to explore, in which Tom takes the leading role. In the next excerpt, the students talk about how they can substantiate their answers and thus are engaged in meta-level rules.

Tom: We have proven it with those there [points at his writings].

Erik: But we do see that it stops at half of the other [points at the end of the curve].

Sam: Oh yes, and you see that there is a relation between the area under the curve which is the integral of velocity.

Further, Tom states that they can calculate the velocities at a certain time.

Tom: We can calculate the velocities, right. So, if t equals four at both….

Erik: Mhm.

Tom: And then we get half there [points at one of the velocity functions].

Lastly, Erik reflects on what they have done during the session.

Erik: I have never thought that we could use the area under the curve and just set it equal to…and then play around.

DISCUSSION AND CONCLUSION

In the following section, we discuss what can be gained from deed-oriented learning opportunities. In the task situations, the students had an opportunity to engage in deeds but were not successful in using only deeds to solve the tasks. The tasks the students were engaged in were exploration-requiring tasks (i.e. the students needed to participate exploratively to successfully solve the tasks) and thus provided explorative-requiring opportunities to learn. In our analysis, we observed deed-oriented learning opportunities served as a common ground for the end results the students could agree upon. This further might serve as a catalyst for engaging in more explorative routines and explorative-enabling opportunities to learn. The animation of the shaded regions beneath the curves in Sim2Bil gives the students an opportunity to realize integral as an area and helping them to recall a previously endorsed narrative of velocity as the derivative of the position with respect to time. In this situation, the learning is at the object level, where students are giving opportunities to develop new narratives about known mathematical objects involved, such as integrals and functions, or to remember or connect already endorsed narratives about these objects.

In the task situations, we also observe opportunities to learn at a meta level. The animation tool enabled the students to verify their answers, and for Erik, the tool served as the 'ultimate substantiator' for convincing that their narrative can be endorsed. For Tom, the calculations and their reasoning were enough to convince that their narrative holds. For him, narratives are endorsed by deriving new velocity functions based on integration procedures. Even at the end of the session, when Tom repeats how their narrative can be endorsed by the last question in T4, Erik still turns to Sim2Bil and says that they can see it on the appearance of the curves. Changing the meta-rules of a discourse, as in this case is about changing how they substantiate their narratives, seems to be a demanding task. For a change in the meta-rules to happen, new mathematical objects have to be introduced (Sfard, 2012). The students' different approaches during the session reveals that there is an opportunity for meta-level learning already from the beginning of the session. To understand what can be done to accomplish the tasks, to understand the 'rules of the game', offers opportunities for meta-level learning. The students can start by choosing polynomial functions to integrate, which does not seem legitimate for Erik, and offers an opportunity to engage in a new discourse about integrals.

More research on engineering students' work on tasks they can engage with in different ways and how they negotiate discourses are needed. More, specifically, we interpret that more research is needed on students' engagements with deeds in their engineering studies and in their professions and whether these engagements differ from each other. In our data, we see also how Tom invites Erik into his more explorative discourse by offering a more ritualistic way of engaging with the tasks (offering step-by-step procedures). This corresponds with the results by Nachlieli and Tabach (2019): ritual-enabling OTLs may act as catalysts for explorative-requiring OTLs. However, by also focusing on deeds, we gain more insights into engineering students' learning processes.

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Bridging Course: Why, How, and First Impressions

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The knowledge gap between high school and university level mathematics is a persistent issue that hinders students in their academic career. Freshman Civil Engineering students at the University of Twente, Netherlands, struggle with passing entry level Calculus courses. In 2022, a workshop was introduced to help students with their prerequisite knowledge; still, many students could not pass these courses. Capitalising on the idea behind the workshop, a fully digital course was introduced in 2023. In this research we dive into the design of the contents of this course. Furthermore, we investigate its impact on student performance with respect to previous years using a qualitative approach: interviews with second year students provide, to this avail, a valuable comparison.

Keywords: transition to university mathematics, digital resources in university mathematics education, gap between high school and university

INTRODUCTION

The knowledge gap between high school and university mathematics is a reoccurring issue which heavily impacts students. This effect is rather well documented in literature, with different countries and institutions trying to ease this transitio. In the past years the Netherlands, for example, has tried to combat and bridge this gap by reforming and adjusting the high school mathematics curriculum to better develop basic skills and understanding of mathematics (Rijksoverheid, n.d.). This was done in an attempt to motivate students to put more effort into studying the subject as, compared to their peers in other countries, these students show less interest in mathematics: they often do not find the subject important enough, but are concurrently more confident in their knowledge than peers in other countries (SLO, 2023). The importance of mathematics within other subjects and society has been underlined to encourage students to perform better as correlation has been found to exist between the students' results in their final national high school exam, their GPA during their first year of university and their eventual graduation from a Bachelor study programme (De Winter & Dodou, 2011).

At the University of Twente in Enschede, the Netherlands, students from a variety of engineering programmes struggle with finding their footing when it comes to mathematics subjects. Experience shows that students falter when met with the introductory Calculus course which is named as Calculus 1A. This struggle is experienced even in spite of the course being structured to be, at least for the Dutch curriculum, a direct successor of high school mathematics.

In an attempt to aid the students' learning, we focused on the Bachelor programme of Civil Engineering (CE), which in recent years has seen an increase of freshmen per cohort while also experiencing an overall decrease in the passing rate for Calculus 1A, as can be seen from Table 1. As a consequence of this, many students repeat the subject multiple times, and may continue to struggle with their mathematical basis. Many of those who pass, seem to do so barely as partially indicated by Table 1.

Year	**Average points out of 22**	**Pass rate: Main exam**	**Pass rate: Resit exam**	**Total pass rate**
2020	12.03	61.54%	35.71%	71.69%
2021	11.98	54.41%	40%	68.11%
2022	10.64	54.69%	28.57%	65.63%

Table 1: Overview of Civil Engineering results for Calculus 1A across three years.

In 2018, CE switched from being an international programme, further increasing the diversity in the mathematical background and present within the classroom. This made interventions by lecturers to cater to knowledge gaps of all students more difficult and less effective. Students also indicated that they found themselves struggling with topics they assumed to already have mastered, such as working with fractions; this made them feel left behind compared to their peers, further differentiating the level of understanding within the classroom.

In 2022, a mathematics workshop, in the form of a lecture accompanied by exercises, was given, just before the start of Calculus 1A, to assist CE students in bridging this knowledge gap. The intention was also to assess the students' prerequisite knowledge: this would give the lecturer a much clearer picture of which topics may require additional attention. This workshop also provided students with a soft introduction to the subject of Calculus 1A.

The abovementioned workshop served as the inspiration for the first iteration of the Bridging Course: a fully online course which simultaneously tests and supports students in their pre-university mathematics knowledge by providing them with immediate feedback. This course was developed throughout 2023 and was first utilised during the academic year 2023/2024. In this study we elaborate on how the contents of this first iteration of the Bridging Course were structured.

Theoretical framework

For more than three decades, scholars have closely examined the challenges students face in transitioning from secondary-level mathematics to tertiary-level mathematics. This focus has intensified due to concerns about enrollment and dropout rates in tertiary STEM (Science, Technology, Engineering and

Mathematics) programs (Hernandez-Martinez, 2016) including mathematics (Rach & Heinze, 2017; Pinto & Koichu, 2023). This phenomenon has been observed to be consistent across different countries and time periods (Higgins & Belward, 2009; Luk, 2005; Silius et al., 2011).

This phenomenon is notably prevalent in Europe, where mathematics learning outcomes often lag expectations. The alignment of Eastern European educational systems with those of the West worsens this gap, hindering efforts to increase STEM enrollments and leading to higher dropout rates (Mustoe & Lawson, 2002; Pinxten et al., 2015). First-year engineering students encounter increasing challenges in effectively completing foundational mathematics courses, essential for their subsequent mathematical and scientific advancement, which significantly influences students' confidence in their academic abilities (Parsons, 2004; Rylands & Coady, 2009). Additionally, educators face the formidable task of determining an instructional level that accommodates the diverse learning requirements of their students within the classroom setting (Metje et al., 2007).

Initial research in this field, explored the epistemological contrasts between school mathematics and professional mathematical practices (Tall, 1991). Subsequently, scholarly focus has transitioned from individual student perspectives to encompass sociocultural, institutional, and affective dimensions influencing the transition process and its implications for student learning outcomes (Artigue et al., 2007; Clark & Lovric, 2009; Di Martino & Gregorio, 2019; Gueudet, 2008). Some of these transition courses were structured by means of a blended learning approach (Bardelle & Di Martino, 2012) taking the transition course from a hybrid aspect.

Within the literature, no transition course was identified that combines online testing and instruction while offering immediate feedback through embedded questions within instructional videos, akin to the approach adopted in the Bridging course.

The research question of this study is twofold: first, introducing the design of the Bridging course along with its rationale; second, investigating the enhancement of students' learning with the Bridging course compared to previous years.

METHODOLOGY

The Bridging Course has been developed keeping both the background and learning goals of students in mind. To this effect, we detail below the structure of the Bridging Course as well as how this was achieved: to this avail, both the required pre-knowledge, learning goals of Calculus 1A and contents of the Bridging Course are laid out. We also include how and when the Bridging Course was implemented and how our preliminary results were achieved.

In order to be able to provide students with immediate feedback, the Bridging Course has been developed in CANVAS, the learning management system at the

University of Twente. The course consists of three main topics, which are subdivided in 19 subtopics. For each subtopic, skills are defined (each attached to a learning goal) that the students need in order to approach Calculus topics with more ease. For each of these skills, three parts were developed: a test question, an interactive explanatory video and a post explanation question. The test question aims to test whether the student already owns the skill at hand. If the student successfully completes this question, they may move on to the next skill, otherwise they are redirected to an explanatory video. All videos are made interactive by the presence of embedded questions that enable students to check their understanding of the explanation. After completing the video, students are further directed to the post explanation question: this wraps up the skill by testing whether the student has gathered on it. The post explanation question is designed to be of equivalent level as the test question. Because of this structure, depending on the student's level of mastery of a certain topic, the course changes in length: it effectively adapts to the needs of the individual student.

The topics and questions of the Bridging Course were chosen and designed keeping in mind both the prerequisite knowledge that students are meant to have based on the high school curriculum, as well as the mathematics knowledge required for the Calculus 1A, and teachers' experiences by means of the topics that students typically struggle with.

Contents of the Bridging Course

Students from many different nationalities, both from within and outside Europe, join the CE programme, leading to a wide variety of pre-knowledge levels. Accounting for each of these is beyond the scope of this research. The mathematical level required of students is determined based on the level expected for Dutch high school mathematics (Wiskunde B) or the equivalent. Considered were then the mathematics competencies and learning goals that the Dutch curriculum imposes students to have achieved by the end of high school for their final national exam. In doing so, we also considered our own experiences with teaching mathematics at secondary school. This proved to be a valuable tool in determining which of the learning goals, that are expected to have been achieved and mastered, students typically struggle with.

According to SLO (n.d.), by the end of high school students must have mastered skills across five different domains, A through E. Table 2 offers an overview of relevant (parts of) domains A through E.

The candidate:

A: Skills	• masters mathematical thinking activities including modelling, algebraising, ordering, structuring, analytical thinking, problem solving, manipulating formulas, abstracting, logical reasoning and proving.

B: Functions, graphs and equations	• Formulas and functions: can interpret and edit formulas and draw a graph in a coordinate system for a relationship between two variables • Standard functions: can draw and recognise graphs of the following standard functions: power functions with rational exponents, exponential functions, logarithmic functions, trigonometric functions and the absolute value function and can name and use the characteristic properties of these different types of functions. • Inverse: can conceptually handle, draw up and use the inverse of a function. • Equations and inequalities: can solve equations, inequalities and systems of two linear equations and interpret the solutions.
D: Goniometric functions	• can draw up and edit formulas for periodic phenomena, draw the associated graphs, solve equations and use periodicity with insight.

Table 2: Overview of relevant domains for the Dutch high school curriculum.

The subject Calculus 1A is taught over the course of four weeks, during which the global topics of 'Vectors', 'Limits', 'Differentiation' and 'Multivariate Analysis' are discussed. This is done according to the learning goals listed in Table 3.

After completing this course, the student is able to:

Work with vectors and elementary properties of functions, especially with the rules of differentiability • apply elementary vector operations • calculate dot product and cross product • determine equations of lines and planes in space • apply elementary properties of functions • calculate derivatives using differentation rules and the derivatives of elementary functions
Work with limits and the definitions of continuity and differentiability and applications, for functions of one variable • calculate limits • state and apply the definition of (left, right) continuity • work with limits involving infinity • state and apply the definition of differentiability • calculate and apply linear approximations and differentials • calculate the absolute extreme values on a closed bounded interval • apply l'Hôpital's rule to indeterminate forms of limits

Investigate functions in two variables • plot graphs and contour lines • investigate continuity and differentiability • calculate partial derivatives • calculate the tangent plane and linearisation

Table 3: Overview of the learning goals of Calculus 1A for Civil Engineering.

These learning goals are tested at the end of the course through an exam comprising final and open answer questions. These questions challenge the students to portray mastery of all learning goals as well as of mathematical reasoning.

Domains A through E (see Table 2) match the contents and end goals of Calculus 1A (see Table 3). Students, however, struggle still due to a lack in fundamentals: experience shows that students are under the impression that their algebraic skills are a solid basis they can rely on, and discover, while following Calculus 1A, that this is not the case. This falls in line with our experiences: in high school students often struggle with the introduction of topics such as fractions, decimals or percentages and proceed to build their mathematical knowledge upon unstable fundamentals. It is precisely for this reason that the Bridging Course also contains such topics, as can be seen from Table 4.

1: Numbers	Fractions, Decimals, Ratios/Percents
2: Functions	Definition, Linear functions, Quadratic functions, Cubic functions, Root functions, Exponential functions, Logarithmic functions, Absolute value functions, One-to-One and inverse functions
3: Trigonometry	Degrees and radians, Graphs, Unit circle, Right triangle, Trigonometric equalities, Double angle formulas, Inverse functions

Table 4: Overview of the subtopics for each main topic of the Bridging Course.

Topic 1 (see Table 4) has been designed to support students whose algebraic skills need revision or improvement. The subtopics constitute the basis upon which students revise topics 2 and 3. They ensure, therefore, a review of domain A (see Table 2). In this topic, students must perform a series of calculations (addition, subtraction, multiplication and division) and comparisons with fractions, decimals and ratios. This offers them the opportunity to discover potential issues in their algebraic skills and to make amends for those before the start of Calculus 1A. This benefits their ability to follow along with calculations of the lecturer and their own algebraic precision.

Topic 2 in Table 4, offers a revision of topics from domain B (see Table 2) and equips the students with a complete overview of the skills (such as graphing, equating and solving) and of the standard functions and relative graphs and characteristics (such as asymptotes) necessary to complete Calculus 1A. Once the cases for one variable are clear in the students' minds, applying differential calculus to them and moving on to functions of two variables feels like a smaller step. In addition, the topics of one-to-one and inverse functions have been added to slowly introduce students to the mathematical correct definition of an inverse function, which is not necessarily given at high school level: this ensures that damaging preconceptions are taken care of. This topic also allows students to become familiar with standard graphs without the aid of a graphic calculator as this tool may have been allowed in high school.

Topic 3 (see Table 4) is directly derived from domain D in Table 2 and aims to support students in handling trigonometric functions and equations, as these return very often throughout their study of differential calculus.

The questions that make up the Bridging Course are closed answer type questions (partly due to limitations we faced when using CANVAS): multiple choice, true/false and drag and drop. All answers provided to students to choose from have been designed based on common misconceptions reported in literature and on our teaching experience in secondary schools.

Implementation and interviews

The Bridging Course in its current iteration was administered to CE students to work through in September of 2023, before the start of Calculus 1A. During a four hours session, students were started off on the Bridging Course in the presence of a lecturer and teaching assistant and were given a full week to complete it at their own pace.

To valuate the impact of the Bridging Course on students' performance, interviews were held with a panel of four second-year students, who all repeated Calculus 1A in 2023. As these students experienced both the Bridging Course and the workshop in 2022, they could elaborate on the qualities of either approach: this allowed to (partly) qualitatively, evaluate the improvements brought about by the Bridging Course.

RESULTS AND DISCUSSION

Below, the comparison of the Bridging Course with the previous years' attempts in mathematics transition as well as the implementation of the Bridging Course will be presented and discussed through the results of the held interviews.

The Bridging Course was well received by the students, who showed great appreciation for it. In particular, students indicated that this course was a great improvement compared to the workshop of 2022. The four interviewed students felt that the Bridging Course caters to the individual student through a unique experience. Students shared that, while useful, the workshop suffered from the same issues as the lectures: not everybody within the classroom was on the same level, students did not feel comfortable asking questions and, in some cases, also felt left behind and that their questions would be judged to be trivial or unintelligent. This was the case for a student who realised that they had not mastered how to multiply and divide fractions: this student thought they could rely on this being part of their skillset and felt ashamed to admit to the lecturer that this was not the case. Following along with calculations and steps had become a challenge, so this student could not make up for this gap in knowledge and ended up not passing the course: their final mark in 2022 was about 3.0 out of 10. The Bridging Course suited this student's need much better, providing them with ample exercise for the topics they struggled with. This student saw incredible improvement. In 2023, they passed Calculus 1A on their first attempt with around a 6.5: a difference of over 3.5 points compared to 2022. The other three also reported passing the exam on the first try with an improvement of about 3.0 points upon their grade in 2022 and thought that the Bridging Course had played a role in this by helping them move forward in their learning. One student explained that the goal of the workshop in 2022 felt aimed at testing their pre-knowledge, rather than teaching them. This demotivated them greatly, as it made them insecure about their knowledge. They could appreciate that the Bridging Course had a clear focus on having students learn from their mistakes.

Two students indicated that they experienced the size of the Bridging Course to be daunting: for each of the 19 subtopics up to four different skills could be tested. This gave the impression that the course could take up a considerable amount of time. Students, however, admitted to later finding that the course could be worked through with relative ease as no open questions are included in the course. This also initially created a false sense of security, as all four students indicated that they thought they would easily be able to solve many of the questions, only to be surprised by their knowledge gaps and at the trickiness that the course managed to maintain. Students appreciated that the Bridging Course could provide them with a reliable indication of which topics and skills needed extra revision and liked that the length of the course would vary based on their own performance: the experience felt like one tailored to them. Students, however, also indicated that they only partially appreciated the kick-off session of the Bridging Course: three recognised that, while this session was a great starting point to immediately resolve any accessibility or technical issues with the tool, it also meant that students faced the judgement of their peers. As contact hours are clearly still necessary, in 2024 the session could be made non-compulsory and be aided by the addition of office hours.

This is a preliminary study that faces limitations. The sample size of interviewed students is limited: while the feedback was quite positive, we are currently working on painting a full picture of the benefits and disadvantages of the Bridging Course. This will be topic for further research, where the results from students both for the Bridging Course and Calculus 1A will be quantitively evaluated, providing further insight into the impact of the course. However, one may already take into consideration that the Calculus 1A exam this year achieved about a 75% passing rate (a clear improvement with respect to previous years). Additionally, during the design phase attention was paid into how teachers can be supported with the information gathered during the Bridging Course. The collection of this data culminated in a heatmap for the teachers to use: from here the teachers could conclude on which (groups of) subjects (groups of) students scored poorly. The design and results of the teacher support will be discussed in future publications.

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Building on slope: results of a second research cycle on the differential calculus of two-variable functions

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This study applies Action-Process-Object-Schema (APOS) theory in a second research cycle investigating how an approach based on the explicit consideration of slope in three dimensions and local linearity contributes to students' learning of the differential calculus of two-variable functions. We compare the problem-solving tendency of students from two sections that were taught differently. We find that students' conceptualizations of slope were different in the two sections and that their understanding of slope was reflected in their problem-solving and the justifications of the relations that they constructed between different basic notions of the differential calculus. Overall, we show that the students who based their constructions on slope and local linearity obtained a deeper understanding of the differential calculus.

Keywords: APOS, functions of two variables, slope, differential calculus, multivariable calculus.

INTRODUCTION

Slope is a basic notion that is commonly studied for the first time in the middle school curriculum, then is revisited in secondary school, in courses such as algebra, trigonometry, and pre-calculus, before being revisited again in the context of the calculus of one-variable functions (Nagle et al., 2019). So, it is reasonable to attempt to base students' understanding of multivariable calculus on the notion of slope. However, students can show difficulty generalizing the notion of slope from two to three dimensions (Moore-Russo et al., 2011; Martínez-Planell et al., 2015). So, attempts to build the differential calculus of two-variable functions based on the notion of slope require the explicit consideration of slopes in three dimensions when teaching the course (McGee & Moore-Russo, 2015). This is done in this study, in which classroom instruction of an "activity section" started with an explicit discussion of slope in three dimensions (3D), then slopes were used to have students construct a notion of vertical change on a plane ($dz = dz_x + dz_y$; see Figure 1) which subsequently served as a basis from which to develop other ideas of the differential calculus of two-variable functions, including the tangent plane, directional derivatives, and the total differential. In the study, we investigate how this approach to multivariable calculus affects students' ways of thinking about slope and, more importantly, how students who used this approach tended to establish more relations between different notions of the differential calculus of two-variable functions when problem-solving justifying the relations in terms of slope, in comparison with students that did not use this approach.

There is an increasing number of studies dedicated to the didactics of multivariable differential calculus (e.g., Borji et al., 2023a, 2023b; Harel, 2021; McGee & Moore-Russo, 2015; Martínez-Planell et al., 2015, 2017; Lankeit & Biehler, 2019; Tall, 1992; Thompson et al., 2006; Trigueros et al., 2018; Weber, 2015). But there is still much to learn, particularly as it pertains to how can one best help students to interrelate the different notions of the differential calculus of two-variable functions.

THEORETICAL FRAMEWORK

We use the Action-Process-Object-Schema theory (APOS). For more details, see Arnon et al. (2014). In APOS, an Action is a transformation of a previously constructed mathematical object that the individual perceives as external. An Action may appear as a rigid application of an explicitly available or memorized fact or procedure. When an Action is repeated, and the student reflects on the Action, it might be interiorized into a Process. A Process is perceived as internal. A student with a Process conception will show characteristics like justifying the Process, discussing it in general terms, thinking of it as independent of representation, and generating dynamical imagery of the Process. A Process may be reversed or coordinated with other Processes to form new Processes. When an individual is able to think of a Process as an entity in itself and can apply or imagine applying actions on this entity, then the Process has been encapsulated into an Object. The important thing about an Object is being able to do Actions on a (previously encapsulated) Process. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas that are interrelated in such a way that the individual can determine if it applies to a particular problem situation. Although the complexity of the differential calculus of two-variable functions suggests the use of Schemas to model student understanding, in this report we focus on slope and its role in establishing connections between different component structures of the differential calculus of two-variable functions. We will not need to use Schemas to model this.

Another important idea in APOS is that of a genetic decomposition (GD). This is a model of constructions a student could do in order to understand a particular mathematical notion. The GD is expressed in terms of the structures (Action, Process, Object, Schema) and mechanisms (interiorization, coordination, reversal, encapsulation, de-encapsulation, etc.) of the theory. A GD is not unique and is not meant to be the best way a student may come to understand a notion. It is only a hypothesis that may be improved by research results. After proposing a GD, classroom activities are designed to help students do the proposed constructions. They are class-tested, and data is obtained from students with an instrument based on the GD. The obtained data can suggest improvements to the GD and the activities. The new GD and activities may be tested in further cycles of research.

Reflection is the key ingredient allowing students to go beyond an Action conception. To foment reflection, in APOS one typically uses the ACE pedagogical strategy. This means that the specially designed activities are worked in collaborative groups of three or four students, there are general class discussions, and exercises for the home.

GENETIC DECOMPOSITION (GD)

We adapt some ideas of Tall (1992) to base the development of the differential calculus of two-variable function on the notion of slope and local linearity. The construction is suggested by Figure 1 and is developed in much detail in a GD given by Martínez-Planell et al. (2017), and for the total differential, in Trigueros et al. (2018). We must omit all details for reasons of space. Essentially, we start by the explicit consideration of slope m in three dimensions (3D). Since we treat a surface as locally linear, we start by considering planes and use the slopes in the x and y directions, m_x and m_y, to construct Processes of vertical change on a plane in the x and y directions, $dz_x = m_x dx$ and $dz_y = m_y dy$ respectively, and coordinate them to obtain a Process of total vertical change on a plane $dz = dz_x + dz_y$. From here, the point-slopes equation of a plane follows immediately and if the plane happens to be the tangent plane, we also obtain its equation, were the slopes in the x and y directions are now the partial derivatives. As Figure 1 suggests, the notions of total differential and directional derivative can also be obtained based on this idea.

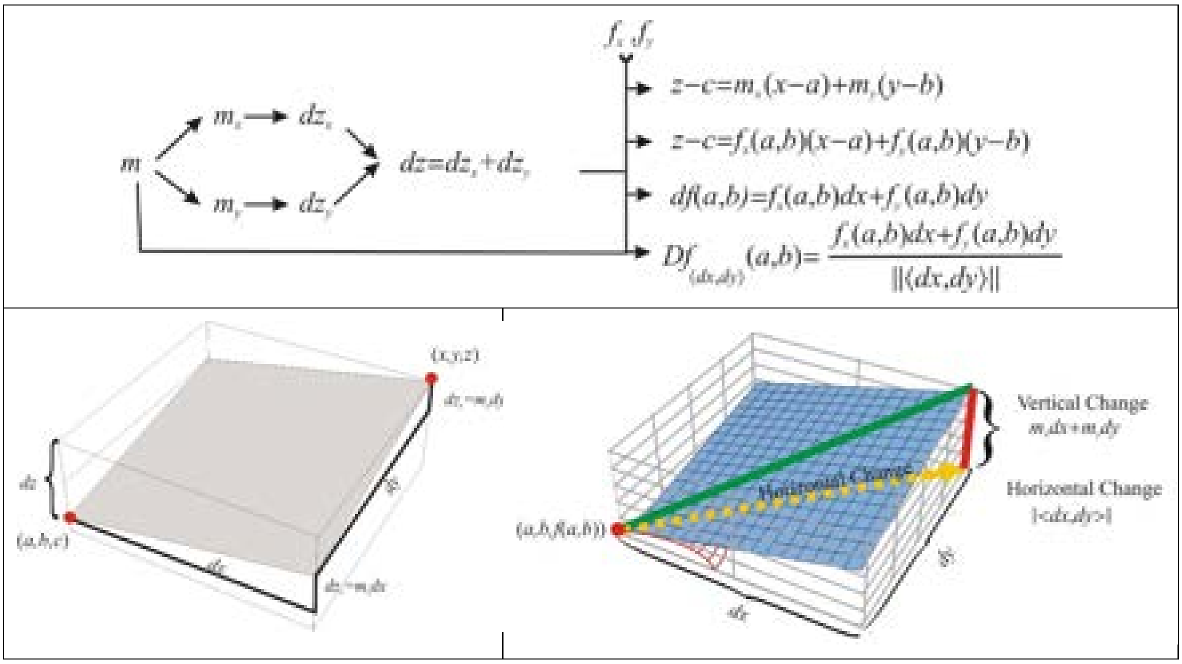

Figure 1. Diagram and figures suggesting the main constructions of the Differential Calculus (*m* is slope).

FIRST RESEARCH CYCLE AND METHODOLOGY

This study is a result of a second research cycle investigating students' understanding of the differential calculus of two-variable functions. The first research cycle and the resulting GD are described in Martínez-Planell et al. (2015, 2017) and in Trigueros et al. (2018). The results of the first cycle showed that students seemed to have an Action conception of the main ideas of the differential calculus; many students did not construct slope in 3D and they seemed constrained to the rigid application of memorized formulas which they could not justify geometrically. The problem is that visualization is not possible when working at the Action level. Only one student (out of 26) constructed a Process of directional derivative, and none constructed a Process of total differential. The data suggested that the notions of tangent plane, total differential, and function remained isolated in the minds of most students.

After that first cycle, the GD was revised and the activity sets were redesigned in order to better help students do the constructions proposed in the GD and use slope and the tangent plane to interrelate partial and directional derivatives, the point-slopes equation of a plane, and the total differential in different representations as described before. The GD was now more detailed, thus introducing many changes in the activities. The activities for the differential calculus and other areas can be downloaded in the link https://www.researchgate.net/publication/373990320_Activities_for_Multivariable_Calculus.

In this second cycle, we compared students' inferred mental constructions in an activity section that used the newly improved activity sets and the ACE pedagogical strategy, and a regular section that followed very closely the textbook (Stewart, 2012), used problems from the textbook, and was lecture-based. Having a regular section allowed us to recreate conditions similar to those of the first research cycle so that the types of constructions students in the regular section make can serve as a baseline with which to compare the constructions of the students in the activity section. It also enables us to verify that the results of the first research cycle are reproduced with those of the regular section. We underscore that this is a qualitative rather than a quantitative study in which we look for the general tendency of students of the regular and activity sections when constructing different structures (Actions, Processes, Objects) in their problem-solving.

Each professor chose 11 students so that three were over-average, five average, and three under-average according to the professor's criteria. This was done in order to be able to observe as many different types of constructions as possible. The students were comparable in the sense that they took the previous single-variable calculus course with the same professor (of the regular section), and it was verified that they had comparable grades in that course. Both professors had ten years of experience teaching the course. Semi-structured interviews took place after the semester was over; each interview had two parts, which were held on separate days, and each part lasted approximately one hour. The interviews were audio and video recorded, transcribed, and translated into English. The data analysis compared the structures (Action, Process, Object) that students gave evidence of having constructed with those proposed in the GD, and also took note of unconjectured constructions. The analysis was done individually by the researchers and then discussed as a group until a consensus was reached.

The interview instrument had a total of 20 questions in its two parts. For the purpose of this article and for lack of space, we only show the four questions below.

2. a. In the plane given below, find the slope of the line in bold [Figure 2 left].
6. The plane in the figure below [Figure 2 right] is tangent to the graph of a differentiable function $z = f(x, y)$ at the given point.
   a. What can you say about the change in the value of the function if *x* increases 0.02 units and *y* decreases 0.02 units?
   b. Find the differential of *f* at the point (1, 2), $df(1,2)$. If it is not possible, explain why.
   d. Use the graph of the given tangent plane to find $D_{\langle 1,1 \rangle} f(1,2)$.

RESULTS

As could be expected, classroom activities that emphasize a geometric interpretation of slope and the notions of differential calculus had an effect on the students' conceptualizations of slope (Moore-Russo et al., 2011; Nagle et al., 2019) with most students in the activity section (10 of 11) giving evidence of a geometric ratio conceptualization of slope while a majority of students in the regular section (7 of 11) showed an arithmetic ratio conceptualization. Of course, as argued in Nagle et al. (2019), students with a Process conception of slope can exhibit either conceptualization as needed in a problem situation. The following two examples show the difference between the geometric and arithmetic ratio conceptualizations of slope. Student A1 is from the activity section and R2 is from the regular section.

A1: The slope of this line will be this vertical change which is 5 minus 2 and it's 3 umm over this horizontal change which is umm from 1 to 2 so the horizontal change is 1. The slope is $\frac{3}{1} = 3$.

Student A1 shows a geometric ratio conceptualization of slope in 3D while, in the following example R2 shows an arithmetic ratio conceptualization.

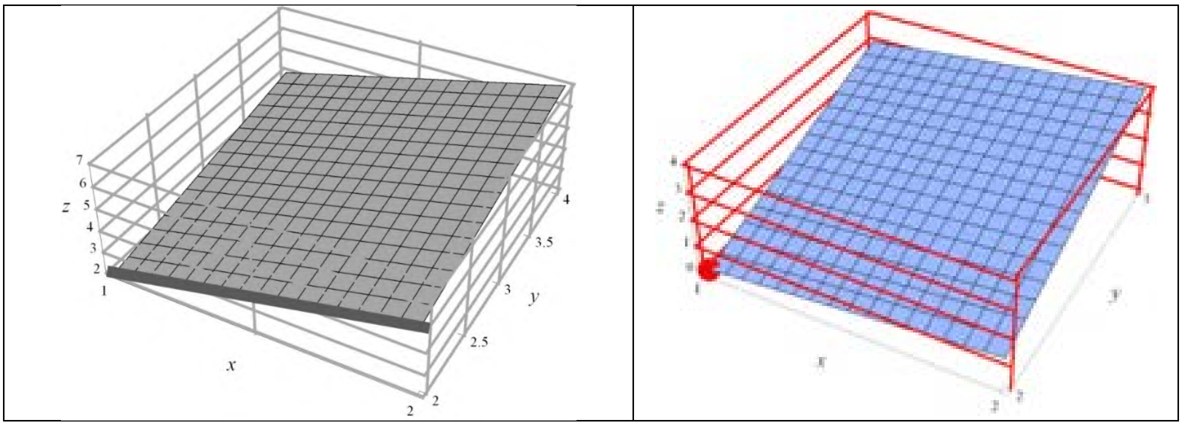

Figure 2. Figures for questions 2a and 6a, 6b, 6d, respectively.

R2: It's a line in $3D$ umm I don't know how to compute the slope of a line in $3D$ because we have three variables x, y, and z in $3D$… The y coordinate is fixed at 2 in both points … So I ignore y in my computation, and I use the formula $\frac{z_2 - z_1}{x_2 - x_1}$ …

Later, when asked about the slope in the y direction:

R2: Okay, I have a line in $3D$ which is in the y direction … I see the x coordinate is fixed as 2 in two points so I ignore it in my computations, and maybe the formula for slope in this case can be $\frac{z_2-z_1}{y_2-y_1}$ … in part 2a [the x direction] the y coordinate was fixed, and in part 2c [the y direction] the x coordinate was fixed, but I am thinking how can we find the slope of a line in $3D$ if all the three coordinates x, y, and z change from first point to the second point. I have no idea for finding the slope of such points because in these cases the formulas $\frac{z_2-z_1}{x_2-x_1}$ and $\frac{z_2-z_1}{y_2-y_1}$ don't work.

Overall, R2 seems to be more dependent on formulas. The statement of R2, "I have no idea for finding the slope of such points because in these cases the formulas $\frac{z_2-z_1}{x_2-x_1}$ and $\frac{z_2-z_1}{y_2-y_1}$ don't work", is an example of "the two change problem," observed early by Yeruschalmy (1997) and explored and named as such by Weber (2015). It anticipates some of the challenges students face if learning directional derivatives by using an entirely algebraic perspective.

Some students in the Regular section showed a geometric ratio conceptualization of slope, like R1.

R1: The line in bold goes 3 units up and moves 1 unit in the x axis so its slope is $\frac{3}{1}$.

Overall, as shown in Table 1, all 11 students in the activity section showed their understanding of slopes in 3D by computing the slopes in the x and y directions, while five of the 11 students in the regular section were also able to compute slopes in 3D, something which, as observed by Moore-Russo et al. (2011) and as seen in the first research cycle (Martínez-Planell et al., 2015), is not generalized on their own by some students. Table 1 also shows relations established between the notions of tangent plane and function (TP-F; problem 6a), total differential (TP-TD; problem 6b), and directional derivative (TP-DD; problem 6d) when problem-solving.

Students showing construction	Slope in 3D	TP-F	TP-TD	TP-DD
Activity section	11	8	8	9
Regular section	5	2	1	3

Table 1. TP=tangent plane, F=function, TD=total differential, DD=directional deriv.

We now consider some examples of these relations. In question 6a, students are given a graphical representation of the tangent plane in order to approximate a change in function values. Student A1 uses slope in his argument, interrelating different notions of differential calculus. He interprets the tangent plane in terms of the total differential, showing awareness that the total differential at the point will give vertical change on the tangent plane as a function of the horizontal change (dx, dy). Further, he relates the partial derivatives to the slopes of the plane in the x and y directions and ends up by relating total differential to function in order to produce the requested approximation.

A1: I know $df = f_x dx + f_y dy$. Here we have $dx = 0.02$ and $dy = -0.02$. I have to find the values of f_x and f_y at the point (1,2,0). Since it is a tangent line to the function f at the point (1,2,0) so f_x is m_x and f_y is m_y. Based on the figure m_x is 1 over umm 2 minus 1 which is 1 so it will be 1, so m_x is 1, and m_y is 3 units to the up over 3 minus 2 which is 1 umm it will be 3 over 1 which is 3, so m_y is 3. The change in the value of the function is 0.02 times 1 plus -0.02 times 3 and umm the answer is -0.04.

Like A1, eight of the 11 students in the activity section also used slope to relate tangent plane and function in question 6a, with some also relating these notions with the total differential. On the other hand, only two of the 11 students in the regular section could do the problem, and none used slopes. Consider R1:

R1: I think this is like Question 2 but here it's tangent plane to the function f. I can find the change in the z coordinate on the plane. Looking at the plane we see when x increases 1 unit umm from 1 to 2 then the z coordinate increases 1 unit to the up umm now if x increases 0.02 units we have the proportion $\frac{1}{0.02} = \frac{1}{\Delta z}$ so it will be $\frac{0.02\times 1}{1}$ which is 0.02. Based on the figure if y increases from 2 to 3 which is 1 unit then the z of the plane increases as 3 units, so for $\Delta y = -0.02$ because y decreases it's negative, so we have the proportion $\frac{1}{-0.02} = \frac{3}{\Delta z}$ and from this we have $\Delta z = \frac{-0.02\times 3}{1}$ which is -0.06. So the final change in z is umm 0.02 minus 0.06 which is -0.04. It's the change of the value of the z of the plane.

Note that R1 relates the tangent plane with the function, showing awareness that one can be used to locally approximate the other. She does this without explicitly recurring to slopes. Instead, she uses proportions.

Question 6b gave students the graph of the tangent plane at a point and asked for the total differential at the point. As shown in the previous problem, A1 had constructed a relation between function, total differential, and tangent plane.

A1: It's $df(1,2)$ equal to 1 times dx plus 3 times dy

Like A1, eight of the 11 students in the activity section could relate tangent plane with total differential. The only student of the regular section to do so was R1.

R1: I know the formula of the differential of f is $df = f_x dx + f_y dy$ and for the point (1,2) it will be $df(1,2) = f_x(1,2)dx + f_y(1,2)dy$. But I don't know how to find $f_x(1,2)$ and $f_y(1,2)$.

Interviewer: Use the figure of the tangent plane.

R1: Since it's the tangent plane to the function f at the point $(1,2)$ umm it seems that I can find the derivatives based on the figure. In Question 2 I found the slope in the x and y direction, now the slope in the x direction is $\frac{1}{1}$ which is 1, and the slope in the y direction is $\frac{3}{1}$ which is 3. If I consider f_x equal to 1 and f_y equal to 3 then the differential will be umm $d_f(1,2) = 1d_x + 3d_y$.

Note that with a hint from the interviewer, R1 was able to find the total differential, perhaps as an Action, a memorized formula, since she did not justify it on her own. In doing so, she showed the need to construct slope as an Object she can flexibly use to relate the graphical representation of tangent plane and partial derivatives. The most common response of students in the regular section, as those in the first research cycle, was similar to that given by R2:

R2: I don't know how to find $df(1,2)$. I don't know what the differential means on the graph umm neither know its formula.

Question 6d gave students the same graph of a tangent plane and this time asked for a directional derivative. Note that the notion of slope is central to A1's argument.

A1: It's the directional derivative. The direction vector is $\langle 1,1\rangle$ so the horizontal change is $\sqrt{1^2 + 1^2}$ which is $\sqrt{2}$. The vertical change is 1 times 1 plus 1 times 3 and umm is 4. So the slope or umm I mean the directional derivative is 4 over $\sqrt{2}$.

Student A1 seems to think of a directional derivative as a slope, as proposed in the GD. Like A1, 9 of the 11 students in the activity section related tangent plane to directional derivative (eight of them used slope). In the regular section, three of the 11 students constructed that relation; they all used a formula based on the gradient vector, like R1. This formula seems to have been used as an Action, a memorized procedure, since geometric understanding of the formula would require a Process of vertical change on a plane, which R1 did not give evidence of having constructed.

R1: It's the directional derivative of f at the point (1, 2) in the direction of vector. The magnitude of the vector is $\sqrt{1+1}$ which is $\sqrt{2}$. I know the directional derivative $Df_{\langle a,b\rangle}(x,y)$ where $\langle a,b\rangle$ is a unit vector, is equal to $a\cdot f_x(x,y)+b\cdot f_y(x,y)$, so the directional derivative is $Df_{\langle 1,1\rangle}(1,2) = 1\times\frac{1}{\sqrt{2}} + 3\times\frac{1}{\sqrt{2}}$ which is $\frac{4}{\sqrt{2}}$.

DISCUSSION AND CONCLUSIONS

This study examines a second research cycle investigating students' understanding of the differential calculus of two-variable functions. The results of the first cycle suggested that the notions of tangent plane, total differential, and function remained isolated in the minds of most students. The results of the second cycle now show that it is possible to help students interrelate these notions based on the slope and local linearity approach. That is, by having students work collaboratively and discuss in class activities that use slope as a base to construct vertical change on a plane, and from there, exploring and interrelating the notions of tangent plane, total differential, function, and directional derivative in different representations. A contribution of this study is showing that students can succeed in this construction. The approach to the differential calculus of two-variable functions, based on the geometric understanding of slope and vertical change on a plane, is another contribution of this study.

The study's results suggest that students can obtain a deeper understanding of the differential calculus with this approach. In question 6a, we saw that slope can play a role in fomenting the interrelation of tangent plane, total differential, and function, notions which on the first research cycle seemed to remain isolated in the minds of most students. The results dealing with question 6b suggest that the construction of slope and vertical change on a plane, helps students relate tangent plane and total differential, thus showing an improvement on the results of the first research cycle (Trigueros et al., 2018), where no student showed to construct total differential as a Process. Directional derivative, as seen in question 6d, is another notion that students can relate to tangent plane, giving geometric meaning to the usual formula that students mostly tend to memorize, and thus understand as an Action conception. This is suggested by the study (Borji et al., 2023b), which shows results about directional derivative that improved from the first research cycle, where only one of 26 students constructed a Process of directional derivative. All these differential calculus ideas are held together by the notions of slope and the derived vertical change on a plane, as suggested by the genetic decomposition and the results of the study.

The interview instrument in its entirety involves several components of the differential calculus of two-variable functions, including slope, function, vertical change on a plane, point-slopes equation of a plane, tangent plane, total differential, partial derivative, directional derivative, and gradient. Thus, the complexity of the corresponding Schema requires an investigation that takes advantage of tools like a GD stated in terms of the schema components and relations between components, the types of relations between Schema components, and the triad of stages of Schema development (Arnon et al., 2014). This is future work.

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Complex analysis task design through the notion of confrontation

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The notion of confrontation is conceived theoretically and methodologically as a tool that allows to describe a possible way in which historical subjects did mathematics, through the idea of building knowledge against previous knowledge. However, this paper presents the notion of confrontation as a tool for designing a sequence of tasks in complex analysis that allow to create a scenario similar to the one identified in a study of an original work of Cauchy. Through this sequence of tasks, it is intended to characterise how the different ways in which Cauchy did mathematics are nuanced in contemporary scenarios dealing with complex analysis.

Keywords: Epistemology philosophy and history in university mathematics education, Teachers' and students' practices at university level, Complex Analysis, Confrontation, Task design.

INTRODUCTION

The theory of complex functions, also known as complex analysis, is a branch of mathematics that permeates several scientific disciplines. For example, according to Conway (2012), the theory of complex functions is the ancestor of other areas of pure mathematics such as homotopy theory and the theory of manifolds, while according to Nahin (1998), some concepts of complex analysis allow to explain some phenomena in physics and in electrical engineering.

The fact that complex analysis is present in a variety of scientific disciplines means that its introduction into the university education system is not without its own complications. Authors such as Garcia and Ross (2017) report that one of the difficulties with complex analysis courses is the level of rigour required to prove theorems within this branch of mathematics. According to the authors, mathematical proofs can act as an incentive for undergraduate mathematics students, but can also act as an obstacle for those who are more interested in the applications of complex analysis.

In mathematics education there have been various efforts to understand and resolve some conflicts in the contemporary school scenario of complex analysis through different types of studies. For example, works such as Dittman, et al. (2016) and D'azevedo-Breda and Dos Santos (2021) have introduced digital technologies with the aim of making the concept of complex-valued functions more accessible to a group of teachers and students respectively. With the aim of providing a geometric meaning to algebraic expressions, there have been studies on how professional mathematicians and undergraduate students work with the concept of complex integrals (Hanke, 2020; Soto and Oehrtman, 2022) and the concept of complex differentiation (Troup, et al., 2017; Soto-Johnson and Hancock, 2019). There are studies in the discipline that have explored different ways of working with the concept of complex numbers

(Dananhower, 2000; Panaoura, et al., 2006; Nemirovsky, et al., 2012), and there are even research studies in mathematics education that have analysed original mathematical works (Cantoral and Farfán, 2004; Piña-Aguirre and Farfán, 2022; Hanke, 2022; Piña-Aguirre and Farfán, 2023a), with the aim of recovering different ways in which historical subjects did mathematics that may have been overlooked in the configuration of school discourses on complex analysis.

As part of an ongoing Ph.D. research aimed at contributing to the configuration of a *reference epistemological model* (Gascón, 2014) of complex analysis, informed by historical evidence and supported and enriched by empirical data, in this paper we take as a starting point the results of a historical-epistemological study (Piña-Aguirre and Farfán, 2023a) to present a set of tasks configured with the aim of answering the question: how does a community of undergraduate mathematics students make use of different ways in which historical subjects did mathematics in complex analysis? We hope that by understanding how these different ways of doing mathematics are nuanced in contemporary school scenarios of complex analysis, we can enrich the theoretical results reported in Piña-Aguirre and Farfán (2023a), which describe a way in which complex analysis can be attended through the gradual incorporation of figures, in addition to the use of purely algebraic arguments as a means of mathematical justification.

THEORETICAL ASPECTS

Piña-Aguirre and Farfán (2023a) assume that school mathematical knowledge is the result of a process of didactic transposition (Boch and Gascón, 2006), which transforms mathematical knowledge by detaching it from its scenarios of origin and transforming it into teachable knowledge. Therefore, Piña-Aguirre and Farfán analysed original works related to complex analysis with the aim of recovering different ways in which historical subjects did mathematics that might have been overlooked by acts of didactic transposition.

One of the main results of the study by Piña-Aguirre and Farfán (2023a) is the conformation of three categories that describe how historical subjects did mathematics in what we now call complex analysis. With the aim of revealing what kind of didactic phenomena are related to the different ways in which historical subjects did mathematics in complex analysis, this paper presents a series of tasks that hypothetically allow two of these three categories to be brought into play.

In particular, Piña-Aguirre and Farfán (2023a) report that in the *Mémoire sur les intégrals définies prises entre des limites imaginaires* (1825), Cauchy deals with concepts related to complex integration through an interplay between algebraic expressions (such as equations or functional relations) and the use of figures (conceived as two-dimensional drawings) through the following two categories.

The first category, called *geometric formulations as means of representation*, refers to the fact that Cauchy did mathematics relying only on the use of algebraic expressions, without the need to incorporate any explicit use of figures. This implies that one way in which Cauchy did mathematics is characterised by the use of algebraic symbolism as the only means of mathematical justification, and therefore, it is conjectured that in this category the use of figures could be associated (at most) as a means of representing algebraic expressions.

The second category, called *geometric formulations as means of construction*, alludes to the fact that Cauchy did mathematics by relying on algebraic symbolism and by incorporating figures accompanied by their counterpart via algebraic expressions as means of mathematical justification. Apart from the fact that in this second category figures are indispensable for the act of doing mathematics, these figures cannot exist without their analogous representation by algebraic expressions.

It is worth noting that Piña-Aguirre and Farfán (2023a) use the notion of *confrontation* to show that the transition from the first to the second category occurred because Cauchy had to incorporate the use of figures (via narrative expressions) in addition to his purely algebraic arguments to support his mathematical procedure. More generally, the notion of confrontation is based on the idea of conceiving that historical subjects had to *confront* the way they did mathematics in complex analysis in order to develop it further, since their successful ways of doing mathematics eventually had to be complemented by other ways of doing mathematics because they could not convey their mathematical ideas. For a further explanation of how this idea comes into play in the analysis of original works, we recommend reading Piña-Aguirre and Farfán (2023a).

Although the notion of confrontation has points in common with Brousseau's (2006) epistemological obstacles, Piña-Aguirre and Farfán (2023b) specify that the notion of confrontation distances itself from epistemological obstacles because it does not rely on the notion of error. For a more detailed explanation of this difference, we recommend reading Piña-Aguirre and Farfán (2023b). In what follows, we will show how the notion of confrontation comes into play in the elaboration of a set of tasks that hypothetically allows students to move from the category geometric formulations as means of representation to the category geometric formulations as means of construction.

METHODOLOGICAL ASPECTS

The notion of confrontation was conceived to explain a possible way in which historical subjects did mathematics in what we now call complex analysis. However, for the purposes of this paper, the notion of confrontation will allow us to design a set of tasks with the aim that, in order to solve these tasks, a group of undergraduate mathematics students from a Mexican university will gradually incorporate the use of figures in addition to the use of purely algebraic arguments. We envision that by gradually incorporating the use of figures, we can create a scenario similar to the one identified in Cauchy's memoir of 1825, in order to understand how students use figures as a means of mathematical justification.

The set of tasks presented in this paper consists of three types of tasks. Type I tasks are configured in order to create a scenario in which the answers to the questions rely solely on the use of algebraic symbolism as a means of mathematical justification. We conjecture that if students use figures to solve these types of tasks, they will use them as a means of representing some algebraic expressions, but the answer to these tasks will be supported only by algebraic symbolism. These types of tasks are configured based on the category geometric formulations as means of representation.

Type II tasks are designed to show that, in addition to algebraic symbolism, it is necessary to consider additional means (in this case the use of figures) to address different concepts in complex analysis. That is, these tasks are designed to make students *confront* the idea that algebraic symbolism is the only way to provide an answer to the tasks. It should be noted that in these types of tasks the need to recognise that other means of mathematical justification than algebraic symbolism are needed is not a consequence derived from the fact that algebraic symbolism leads to errors.

Type III tasks are structured with the aim of understanding how students relate algebraic expressions to a particular type of figure (straight-line segments). In contrast to Type I tasks, answers to Type III tasks require arguments based on figures. These types of tasks are based on the category geometric formulations as means of construction in that they relate specific figures to algebraic expressions.

That is, Type I tasks were designed to show the productive potential of algebraic symbolism, while Type II tasks were designed to show that it is necessary to incorporate the use of figures in order to work further with concepts related to complex analysis. Finally, the Type III tasks were designed with the aim of understanding how students use figures to answer some of the tasks. In this respect, we hope that the design will allow students to move through the categories identified in the work of Cauchy (1825).

THE SEQUENCE OF TASKS

The following seven tasks, which are part of a larger set of tasks, are presented in order to address the concept of integral comprised between complex numbers by a process of extension used by Cauchy in his *Mémoire sur les intégrals définies prises entre des limites imaginaires.* Although the tasks were designed on the basis of this memoir, it is important to note the following. On one hand, the historical awareness of the participants in the study will not be an explicit objective addressed by the design. On the other hand, we do not claim that Cauchy gave a meaning to the concept of complex integral as depicted in the tasks. These tasks will allow us to understand how students gradually incorporate the use of figures to their algebraic arguments.

Each task is followed by its objectives and the heading of each task indicates the type of task (I, II or III). If the tasks are of Type I or II, the type of mathematical activity that the task is expected to elicit is indicated, whereas if the task is of Type III, we also present some questions that we hope to answer once we have conducted a pilot study.

1.- *Task 1* (Type I)

In 1825, in his *Mémoire sur les intégrals définies prises entre des limites imaginaires*, Cauchy gives the following definition for the computation of integrals of real functions comprised between real numbers.

In order to stablish in general terms the meaning of the notation

$$(1) \quad \int_{x_o}^{X} f(x)dx$$

where x_0, X denote real limits, and $f(x)$ denotes a function of the variable x, it is sufficient to consider the definite integral represented by this notation equivalent to the limit of the sum

$$(2) \quad (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1})$$

when the elements

$$(3) \quad x_1 - x_0, \;\; x_2 - x_1, \dots , \;\; X - x_{n-1}$$

are getting smaller and smaller.

In contemporary notation, this definition can be rewritten as follows:

$$\int_{x_0}^{X} f(x)dx = \lim_{n\to\infty}[(x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1})]$$
$$= \lim_{n\to\infty}\sum_{k=0}^{n-1}(x_{k+1} - x_k)f(x_k)$$

Use this last expression to compute the value of the integral

$$\int_1^5 x\,dx$$

if a partition of the interval [1,5] into n equal parts is considered.

Aim of the task: This task will allow us to know how students work with the definition proposed by Cauchy. In particular, this task is designed with the intention that the students will notice that the definition proposed by Cauchy makes it possible to calculate the value of a given real integral through symbolic manipulations involving the concept of limit and sigma notation.

The type of mathematical activity involved: It involves a uniform partition of the interval [1,5] by expressions of the form $x_k + \frac{4}{n} = x_{k+1}$. The calculation of the integral therefore involves algebraic manipulations in the form of a limit and sigma notation.

2.- *Task 2* (Type I)

Based on Cauchy's definition of the concept of integral, obtain a symbolic expression, in terms of a limit and sigma notation, to address the following integral

$$\int_{x_0+iy_0}^{X+iY} f(z)dz$$

Aim of the task: To find a symbolic expression associated with an integral comprised between complex numbers.

The type of mathematical activity involved: Students are expected to extend the expression used in Task 1 to obtain an expression of the form

$$\int_{x_0+iy_0}^{X+iY} f(z)dz = \lim_{n\to\infty}[(x_{k+1}-x_k)+i(y_{k+1}-y_k)]f(x_k+iy_k)$$

This type of mathematical work is based on identifying what is changing in the symbolic expression used in Task 1, and therefore the type of mathematical activity that is expected to answer this task is related to algebraic manipulations.

3.- *Task 3* (Type I)

How would you use the expression you obtained in Task 2 to obtain the value of the following integral?

$$\int_{1+1i}^{5+5i} z\,dz = \int_{1+1i}^{5+5i} (x+iy)dz$$

Aim of the task: To unveil the forms of mathematical work that students use for a particular case of a generalisation.

The type of mathematical activity involved: The limits of integration $1+1i$ and $5+5i$ are expected to suggest that the variables x, y take values in the interval [1,5]. The effect of this is that the answer to this task is based on the answer to Task 1, and therefore it is expected that algebraic manipulations similar to those used in Task 1 will produce an expression such as the following.

$$\int_{1+1i}^{5+5i} z\,dz = \lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\frac{4}{n}+i\frac{4}{n}\right)\left[\left(1+k\frac{4}{n}\right)+i\left(1+k\frac{4}{n}\right)\right]$$

This type of mathematical activity is based on the idea that one way to proceed in a new scenario is to proceed through ways of working that are already known in other known scenarios.

4.-*Task 4* (Type II)

Locate the end points of the following integral on a cartesian plane and draw four different ways of connecting these points.

$$\int_{1+1i}^{5+5i} z\,dz$$

Aim of the task: To show that, in contrast to working with integrals in $\mathbb{R}$, there is no single way to connect the endpoints of a complex integral. In this way, students are made aware that they need to consider additional means to algebraic symbolism in order to address the concept of complex integral.

The type of mathematical activity involved: Students are expected to locate the points $1 + 1i$ and $5 + 5i$ in the first quadrant of the Cartesian plane and then to connect them by various freehand curves.

5.- *Task 5* (Type III)

If you want to calculate the integral of an arbitrary function $f(z)$, using the expression you obtained in Task 2, over the straight-line segment that connects point $3 + 4i$ with point $5 + 1i$, how does this line segment relate to the expression in Task 2?

Aim of the task: To identify the type of mathematical work that students use to give meaning to purely algebraic expressions by means of a specific figure.

Expected mathematical activity: Students are expected to draw the straight-line segment connecting the points $3 + 4i$ and $2 + 1i$. Do they need to find an algebraic expression describing the line segment connecting these points? If so, how do they find this expression? If not, how do they work with the freehand curve?

6.- *Task 6* (Type III)

What is the curve that connects points $1 + 1i$ and $5 + 5i$ and is associated with the expression you obtained in Task 3.

Aim of the task: To identify the integration curve associated with the algebraic expression obtained in Task 3.

Expected mathematical activity: Students are expected to locate the points $1 + 1i$ and $5 + 5i$. Do they identify that the expression $\left(1 + k\ \frac{4}{n}\right) + i\left(1 + k\ \frac{4}{n}\right)$ is associated with a straight-line segment because the real and imaginary parts are the same? do they describe the curve via an algebraic expression or solely through a figure?

Note that the symbolic expression obtained by integrating $f(z) = z$ over the line segment connecting the points $1 + 1i$ and $5 + 5i$ corresponds to the expected expression in task 3. In this way, it is not necessary for algebraic symbolism to lead to errors (thus, the sequence of tasks is framed by the notion of confrontation rather than by the notion of epistemological obstacle), but it is necessary to incorporate other forms of mathematical activity (in this case related to the use of figures) that allow working with the concept of complex integrals.

7.- Task 7 (Type III)

If you want to use the expression in Task 2 to calculate the integral

$$\int_{x_o+iy_o}^{\mathrm{X+iY}} \frac{dz}{z}$$

through the curve defined by $\beta(t) = (1 - t) + it; t \in [0,1]$, what is the curve described by this algebraic expression?

Aim of the task: Unlike the previous tasks, where the integration paths were associated via a figure, the intent of this task is to identify how students work with the expression obtained in task 2 when the integration path is given via an algebraic expression.

Expected mathematical activity: Students are expected to recognize that the integration path begins at $1 + 0i$ and ends at $0 + 1i$. What curve do they conceive that connects these points? do they draw the curve that they conceive? how do they justify that the curve they are conceiving is the integration curve? do they identify that the values of x_k and y_k of the expression obtained in task 2 belong, respectively, to the real and imaginary parts of the curve β?

DISCUSSION AND CONCLUSIONS

Although the seven tasks presented above are part of a larger design, these seven tasks show that the way in which the concept of complex integral is treated does not start from the assumption that the complex integral is based on integration paths. In the sequence of tasks, integration paths come into play as a means of making sense of purely algebraic expressions, but as Hanke (2022) shows in his review of fifty textbooks of complex analysis, integration paths are the starting point for defining the concept of complex integral.

For the purposes of this doctoral research in progress, it is assumed that the introduction of the concept of complex integrals through integration paths is the result of an act of didactic transposition, insofar as, according to Gascón (2014), acts of didactic transposition disrupt mathematical knowledge to the extent that different codes are configured that dictate the ways of doing and conceiving mathematics in contemporary teaching and learning scenarios. Gascón therefore suggests that in order to achieve emancipation from these codes, which he calls *dominant epistemological models*, it is necessary to configure *epistemological reference models* that provide evidence of ways of doing and conceiving mathematics that are not usually recognised by dominant epistemological models.

Based on the ideas of Gascón (2014), it is considered that the categories reported in the study of Piña-Aguirre and Farfán (2023a) can be conceived as part of an epistemological reference model of complex analysis, since these categories allow us to recognise different ways of doing mathematics that allow us to approach the concept of complex integral without having to start from integration paths privileged in complex analysis textbooks.

Furthermore, we acknowledge that the study by Soto and Oehrtman (2022) has some similarities to our approach in that they present a study in which a group of undergraduate mathematics students solve a series of tasks related to the concept of complex integral. However, a key difference in our studies is that Soto and Oehrtman give students a course prior to their study of the complex integral that allows them to explore a geometric interpretation associated with the arithmetic of complex numbers. In contrast, in our study we want to understand how students incorporate the use of figures into their arguments without prior instruction on how to use them.

Finally, since the concepts addressed in the task sequence require that the participants in the study be familiar with algebraic operations of complex numbers and that they be unfamiliar with the parametrisation of curves, as future work the task sequence will be applied with a group of undergraduate mathematics students from a Mexican university who have only taken a course in calculus of a real variable and who have not yet studied complex analysis. It is hoped that from the students' responses it will be possible to identify how the different ways in which Cauchy did mathematics are nuanced in contemporary complex analysis scenarios.

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Didactic transposition of the fundamental theorem of calculus-Part 1: A comparative study of the knowledge to be taught at university and the taught knowledge in YouTube learning resources

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Textbooks and YouTube videos are among resources undergraduate students use to learn mathematics. In this paper, a prominent calculus textbook and two highly viewed YouTube videos related to the fundamental theorem of calculus-part 1 are analysed through a multiple case study. For this purpose, the theories of didactic transposition and praxeological analysis are utilised. The findings indicate that the knowledge to be taught at university as presented in the textbook emphasizes both the praxis and logos blocks, with more focus on techniques in exercises. In contrast, the YouTube learning resources have different purposes, one prioritises contextual tasks for developing the logos block, while the other focuses on teaching the techniques necessary for differentiating integrals, in line with the taught knowledge in several universities.

Keywords: didactic transposition, praxeology, YouTube learning resources, textbook analysis, fundamental theorem of calculus.

INTRODUCTION

Textbooks and YouTube videos are among the learning resources that university students utilize when studying mathematics (Pepin & Kock, 2021). In contrast to the long history of mathematics textbooks as supporting teaching and learning materials, textbook research has a significantly shorter history. However, it has experienced rapid growth in recent decades, including several studies in the past couple of years in undergraduate mathematics education (e.g., González-Martín, 2021). Regarding YouTube videos, they are identified as one of the most popular sources university students use for assistance with their mathematics courses (e.g., Aguilar & Esparza Puga, 2020). Previous studies have reported that these learning resources are used to recall certain mathematical concepts (Kanwal, 2020), understand mathematical concepts or problem-solving processes (Aguilar & Esparza Puga, 2020), and are even utilized by university students when engaging in challenge-based projects (Pepin & Kock, 2021). YouTube videos are also among the learning resources suggested by postgraduate tutors to university students (Grove & Croft, 2019). Despite the influential role of YouTube videos in the mathematical learning of a large body of university students, it appears that the content of these videos has not been the primary focus of past research in mathematics education, and their content has not been thoroughly analyzed. Over the past year, I have started to examine these learning resources both individually and in collaboration with my colleagues, using different theoretical frameworks (e.g., the realization tree from commognition (Radmehr & Turgut, 2024) and the framework of advanced mathematical thinking—a combination of action-process-object-schema (APOS) theory and Tall's three worlds of mathematics (Radmehr, 2024)). In this study, I am focusing on two components of the Anthropological Theory of the Didactic (ATD) that are particularly suitable for this purpose: *The theory of didactic transposition* and *praxeological analysis*. It is worth highlighting that this appears to be the first attempt to use these two components of ATD for analyzing the content of YouTube videos in mathematics education. Therefore, this study could also contribute to the exploration of how ATD could serve as a framework in mathematics education studies. The mathematical knowledge chosen for this study is the first part of the fundamental theorem of calculus (FTC1), a core theorem in calculus that its importance "hardly needs justification" (Swidan & Fried, 2021, p. 1). The following research question is considered in this study, reflecting on the aforementioned components of the ATD: *What transpositions have been made on the FTC1 during the didactic transposition from knowledge to be taught at university to the taught knowledge in YouTube learning resources?*

THE THEORY OF DIDACTIC TRANSPOSITION

The theory of didactic transposition describes how a body of knowledge is transposed from the instance it is created by a scholar to the point at which it is taught and learned in an educational institution (Chevallard & Bosch, 2020). This theory goes beyond what happens in a classroom or lecture, and in empirical studies, calls for the incorporation of data from beyond the confines of these learning environments (Strømskag & Chevallard, 2022). In more detail, the ATD postulates that what is taught in schools and universities originates from *scholarly knowledge*–the knowledge developed by mathematicians, in our case, at universities or other scholarly institutions. Furthermore, when a body of knowledge is to be transposed from where it is originated to another institution, certain adaptations "should be carried out to rebuild an appropriate environment with activities aimed at making this knowledge 'teachable', meaningful and useful" (Chevallard & Bosch, 2020, p. 214). Several actors, referred to as the *noosphere*, "those who 'think' about teaching", such as mathematicians, teachers, lecturers, and curriculum designers, participate in this *transpositive work* (Chevallard & Bosch, 2020, p. 214). Their role is to preserve the main elements of the scholarly knowledge while negotiating, managing, and addressing the demand imposed by the society on the educational system (Chevallard & Bosch, 2020). The didactic transposition process is summarized in Figure 1.

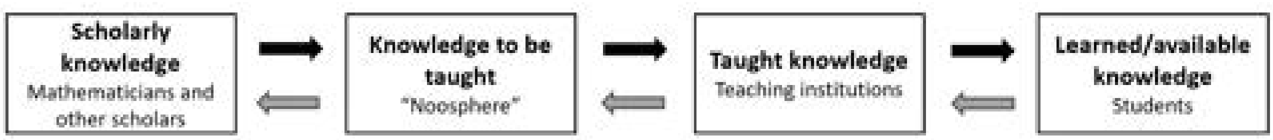

Figure 1: A summary of the didactic transposition process (adapted from Strømskag & Chevallard, 2022, p. 120).

When the sub-theory of didactic transposition is used as a lens, researchers investigate the transposition across the four mentioned instances (Strømskag & Chevallard, 2022): (a) scholarly mathematical knowledge; (b) the mathematical knowledge to be taught as it appears in curricula and textbooks, (c) the mathematical knowledge teachers/lecturers impart in classrooms/lectures; and (d) the mathematical knowledge students learn (Strømskag & Chevallard, 2022). In the following, I discuss praxeological analysis, an important component of the ATD that is often used when investigating the didactic transposition of a body of mathematical knowledge (e.g., Strømskag & Chevallard, 2022). When examining the didactic transposition process, YouTube channels can be regarded as *teaching institutions*, and the content of YouTube learning resources can be seen as *taught knowledge*. Consequently, these can be compared to other instances in the didactic transposition process.

PRAXEOLOGICAL ANALYSIS

In ATD, knowledge can be studied using praxeological analysis (Strømskag & Chevallard, 2022). A praxeology is a model of human activity consisting of a quadruplet [$T/\tau/\theta/\Theta$]. These components include a specific task (T) to be accomplished, a corresponding technique (τ) enabling task completion, a rationale (θ) that provides an explanation and justification for the technique, and a theory (Θ) that encompasses and substantiates the rationale. The first two components [T/τ] constitute the *praxis* block, often referred to as *know-how*, while the latter two [θ/Θ] constitute the *logos* block, which serves to describe, elucidate, and justifies the actions taken (González-Martín, 2021; Strømskag & Chevallard, 2022). A praxeology, denoted as $\mathcal{P}$, is typically the result of the work of an institution or a group of institutions, referred to as $\boldsymbol{I}$. Often, it originates within another group of institutions, denoted as $\boldsymbol{I}^*$, and undergoes *institutional transposition* to adapt to the conditions and constraints of $\boldsymbol{I}$. In many instances, through didactic transposition, $\mathcal{P}$ becomes a simplified version of $\mathcal{P}^*$. For instance, during this process, certain task types may lose their relevance and certain elements of the *logos* block may become implicit or repressed (Strømskag & Chevallard, 2022). The final point to highlight is that in comparative studies conducted using the theory of didactic transposition and praxeological analysis, a reference praxeological model (RPM) is often constructed (e.g., González-Martín, 2021). This serves the dual purpose of distancing the researcher from the institutions under investigation (Bosch & Gascón, 2006) and elucidating their perspective on the knowledge at stake (Topphol, 2023). Due to the word restrictions of this paper and the complexity of the FTC, where a few pages are needed to discuss an RPM, such an RPM is not presented here. However, my personal perception of this body of knowledge is in line, not completely the same, with the recent RPM proposed by Topphol (2023).

THE FUNDAMENTAL THEOREM OF CALCULUS

The FTC connects differential and integral calculus. It enables the computation of integrals using antiderivative instead of relying on the limits of Riemann sums (Stewart et al., 2021). In many classical calculus textbooks, it is presented in two parts:

> **Part 1** If f is continuous on $[a, b]$, then the function g defined by $g(x) = \int_a^x f(t)\, dt \quad a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$. (Stewart et al., 2021, p. 400)

Part 2 If f is continuous on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f, that is, a function F such that $F' = f$. (Stewart et al., 2021, p. 403)

Previous studies (e.g., Radmehr & Drake, 2017; Thompson & Silverman, 2008) have reported that many students face challenges in developing a conceptual understanding of the FTC, or from an ATD perspective, they struggle in developing the logos block for this body of mathematical knowledge. For instance, in Radmehr and Drake's study (2017), many students could not understand that f is the function that describes the rate of change of the accumulated area function $g(x)$ and struggled with comprehending the symbols embedded in the FTC. Previous studies have also offered several suggestions to enhance the teaching and learning of the FTC, such as emphasising accumulation functions over the traditional approach of calculating a number representing the enclosed area over an interval (e.g., Thompson & Silverman, 2008), and integrating digital technology into teaching (e.g., Swidan & Fried, 2021).

METHODOLOGY

This study is conceptualized as a multiple case study with three cases, each presenting a body of knowledge on FTC1. Case 1 is part of a well-known calculus textbooks used for teaching calculus in many countries, written by Stewart et al. (2021). For YouTube learning resources, I searched YouTube using the keyword "fundamental theorem of calculus" on October 10, 2023, and sorted the results based on view counts. The first two videos were from a channel entitled, *3Blue1Brown* with over 5.5 million subscribers. This YouTube channel was founded by Grant Sanderson who received a communication award from the American Mathematical Society for his contribution to mathematics teaching and learning. The first YouTube video serves as an introduction to a series of videos discussing the main concepts of calculus, rather than specifically the FTC. Therefore, I decided not to include it in the analysis. The knowledge discussed in the second video[1], considered as Case 2, is dedicated to integration and, more specifically, the FTC. The knowledge covered in the third video[2], considered as Case 3, is from a channel with 1.36 million subscribers, titled *patrickJMT*. PatrickJMT, the founder of this YouTube channel, mentioned in a video[3] on his channel's homepage that he holds a master's degree in mathematics and has taught at several universities and colleges. He describes his goal as creating "clear and effective videos" for helping students "to get their homework done" as "some extra supplements". Due to the word restriction of this paper, I have exclusively focused on two aspects of didactic transposition: the *knowledge to be taught* at university and the *taught knowledge* in YouTube learning resources. To achieve this, I used praxeological analysis of the ATD.

FINDINGS

Praxeological analyses of the three cases are discussed below.

Cases 1: Stewart's calculus textbook (Stewart et al., 2021)

In this case, both the *praxis* and *logos* blocks of FTC1 are well unpacked. The FTC is presented as part of the *integrals* chapter. In this chapter, before introducing the FTC, topics such as *the area and distance problems* and the definition of *definite integral as a limit of Riemann sums* are discussed. The FTC section begins with a brief introduction, emphasising the significance of the FTC in calculus and providing a short historical account for it. Then, the accumulation function is introduced to build up elements in the *logos* block:

> The first part of the Fundamental Theorem deals with functions defined by an equation of the form $g(x) = \int_a^x f(t)\,dt$ where f is a continuous function on $[a, b]$ and x varies between a and b. (p. 399)

Further explanations for this function in a plain language is provided and readers are encouraged to pay close attention to what g depends on:

> […] g depends only on x, which appears as the variable upper limit in the integral. If x is a fixed number, then the integral $\int_a^x f(t)\,dt$ is a definite number. If we then let x vary, the number $\int_a^x f(t)\,dt$ also varies and defines a function of x denoted by $g(x)$. (p. 399)

The authors continue by relating the accumulation function to area and providing a graphical realization of this function (Figure 2a).

> If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of g as the "area so far" function; see Figure 1 [Figure 2a]). (p. 399)

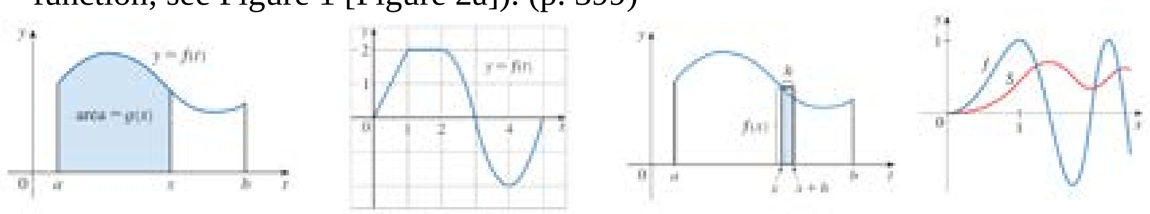

Figure 2: From left to right Figure 2a to Figure 2d (Stewart et al., 2021, p. 399–402).

Then, an example is provided to help readers develop a better understanding of the accumulation function by asking them to calculate the value of the accumulation function $g(x)$ for a number of values and then sketching the graph of $g(x)$.

> **Example 1** If f is the function whose graph is shown in Figure 2 [Figure 2b in this paper] and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g. (p. 399)

In solving the example, the graph of $g(x)$ for each value are drawn which could be useful for readers to develop their *logos* block. Then, another example is provided to prepare readers to be introduced to FTC1. This time an algebraic representation of $f(t)$, i.e., $f(t) = t$ is provided, and $g(x)$ is calculated using a task from the definite integral section, i.e., "Prove that $\int_a^b x\,dx = \frac{b^2-a^2}{2}$" (p. 396):

> If we take $f(t) = t$ and $a = 0$, then using Exercise 5.2.47, we have $g(x) = \int_a^x t\,dt = \frac{x^2}{2}$. Notice that $g'(x) = x$, that is $g' = f$. In other words, if g is defined as the integral of f by

> Equation 1 [$g(x) = \int_a^x f(t)\, dt$], then g turns out to be an antiderivative of f, at least in this case. And if sketch the derivative of the function g […] by estimating slopes of tangents, we get a graph like that of f […]. (p. 400)

Afterwards, the authors make an effort to intuitively address the *logos* block before presenting the theorem by assuming that $f(x) \geq 0$, associating $g(x) = \int_a^x f(t)\, dt$ to the area under the graph of f, and using the derivative definition to compute $g'(x)$:

> … from the definition of a derivative we first observe that, for $h > 0$, $g(x + h) - g(x)$ is obtained by subtracting area, so it is the area under the graph of f from x to $x + h$ (the blue area in Figure 5 [Figure 2c in this paper]). For small h you can see from the figure that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h: $g(x + h) - g(x) \approx hf(x)$[,] so $\frac{g(x+h)-g(x)}{h} \approx f(x)$ [...]. (p. 400)

Then, the FTC1 is provided as shown earlier, and the authors provide an explanation of the theorem in plain language: "In words, it says that the derivative of a finite integral with respect to the upper limit is the integrand evaluated at the upper limit" (p. 400). Then, a full proof of the theorem is provided. The proof begins with the calculation of $g(x + h) - g(x)$, and along the way, two properties of integrals (e.g., "$\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx$" (p. 393) [1]), the extreme value theorem and the squeeze theorem are utilized (see p. 401). The authors continue by presenting the FTC1 using the Leibniz notation for derivative: "$\frac{d}{dx}\int_a^x f(t)\, dt = f(x)$" (p. 401). They point out that this equation "roughly" says that "if we first integrate f and then differentiate the result, we get back to the original function f" (p. 401).

Then, three examples are provided (Examples 2–4) wherein Examples 2 and 4 focus on the *praxis* block, while in Example 3, the focus is on the *logos* block. Example 2 is about finding the derivative of $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$ where it is solved by considering $f(t) = \sqrt{1 + t^2}$ and using the FTC1. Example 3 is a contextual illustration. The authors start with pointing that an accumulation function "may seem like a strange way of defining a function" (p. 402), but such functions appear in many STEM fields. Here, they provided the Fresnel function, i.e., $S(x) = \int_0^x \sin\left(\pi \frac{t^2}{2}\right) dt$, as an example, mentioning from where it is originated and where it has been used (i.e., in the design of highways). Then, they calculate the $S'(x)$ using the FTC1 and labelled it $f(x)$. In addition, among other things, they sketch $S(x)$ and $f(x)$ in one graph (Figure 2d) to probably help readers realize the connection between an accumulation function and its rate of change, and/or in their words, "give a visual confirmation of Part 1 of the Fundamental Theorem of Calculus" (p. 402). Example 4 (i.e., "Find $\frac{d}{dx}\int_1^{x^4} sect\, dt$" (p. 402)) is similar to Example 2, however, here the chain rule is also required to be used as part of the *technique* for solving this task. Here the authors do not discuss why using chain rule is needed. After this example, the second part of the FTC is discussed. Before moving to the next case, it is worth noting that in the exercises following this section, there are five tasks similar to Example 1 addressing the *logos* block, and 12 tasks that are similar to Examples 2 and 4. These tasks ask readers to calculate the derivative using FTC1 addressing the *praxis* block, where for some, the chain rule should be used.

Case 2: Integration and the fundamental theorem of calculus

In this case, FTC1 is discussed by utilising a contextual task, and the focus is mainly on the *logos* block. Early on, Grant mentions that he wants viewers to realize that the integration and differentiation are inverse process. Then, a contextual task is presented for discussing the integral and both parts of the FTC:

> Imagine that you're sitting in a car, and you cannot see out the window; all you see is the speedometer. At some point, the car starts moving, speeds up, then slows back down to a stop, all over the course of 8 seconds. The question is, is there a nice way to figure out how far you've travelled during that time, based only on your view of the speedometer? Or […] can you find a distance function $s(t)$ that tells you how far you've travelled after a given amount of time, t, somewhere between 0 and 8 seconds.

To approach this task, Grant assumes that the velocity at each second is provided in this example, plot those coordinates on a v-t plane, and assumes that the function that model velocity is $v(t) = t(8 - t)$. Among other things, Grant focuses on how to find the area bounded by the velocity graph and the concept of integral. Focusing on FTC1 comes later by returning to the velocity example and addressing the *logos* block:

> […] Think of this right endpoint as a variable, T. So, we're thinking of this integral of the velocity function between 0 and T, the area under this curve between those two inputs, as a function, where that upper bound is the variable. That area represents the distance the car has travelled after T seconds, right? So, in reality, this is distance versus time function, $s(T)$ [Figure 3a]. Now ask yourself: What is the derivative of that function? On the one hand, a tiny change in distance over a tiny change in time, that's velocity [...] But there's another way to see this [...] A slight nudge of dT to the input, causes that area to increase, some little ds represented by the area of this sliver. The height of that sliver is the height of the graph at that point, $v(T)$, and its width is dT. And for small enough dT, we can basically consider that sliver to be a rectangle [Figure 3b]. So, this little bit of added area, ds, is approximately equal to $v(T)dT$. And because that's an approximation, it gets better and better for smaller dT, the derivative of the area function $\frac{ds}{dT}$ at this point equals $v(T)$, the value of the velocity function at whatever time we started on [...] The derivative of any function giving the area under a graph like this is equal to the function for the graph itself.

Furthermore, toward the end, Grant also discusses the situation where the velocity function is negative and how it could impact the accumulation function:

> What if the velocity function was negative at some point? Meaning the car goes backwards. It's still true that the tiny distance travelled ds on a little time interval is about equal to the velocity at that time multiple by the tiny change in time [on the screen, $ds = v(t)dt$], it's just that the number you'd plug in for velocity would be negative, so that tiny change in

distance is negative. In terms of our thin rectangles, if the rectangle goes below the horizontal axis […] its area represents a bit of distance travelled backwards, so if what you want in the end is to find the distance between the car's start point and its end point, this is something you gonna want to subtract […] Whenever a graph dips below the horizontal axis, the area between that portion of the graph and the horizontal axis is counted as negative […]

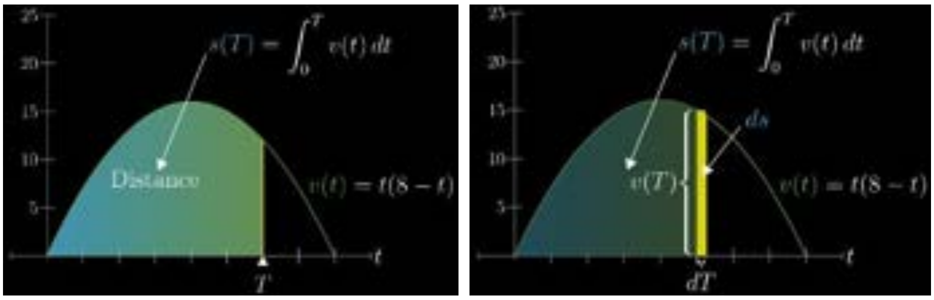

Figure 3: Left: (Figure 3a); Right: (Figure 3b), screenshots used with permission.

Case 3: Fundamental theorem of calculus–Part 1

In this case, the *praxis* block is the focus as opposed to the previous case. At the outset, PatrickJMT highlights that tasks related to this topic "typically involve taking the derivative of integrals". A definition of the FTC1 is presented on the screen right from the beginning of the video, closely resembling Stewart's definition. After the introduction, he reads the FTC1 aloud. Following this, PatrickJMT focuses on the *technique* that can be used for such a task: "so really all that happens is, it says this variable [x], […] this upper limit gets plugged in [on the screen, he uses a blue pen to show this ], is what it amounts to". The rest is dedicated to a single task type, finding the derivatives of integrals using FTC1. Four examples are solved here "$g(x) = \int_1^x (t^2-1)^{20}\,dt$, $h(x) = \int_x^2 [\cos(t^2)+t]\,dt$, $g(x) = \int_1^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds$, [and] $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$". When solving these tasks, the *logos* block is only briefly touched upon. He only refers to the chain rule when solving the third and fourth tasks and [1] is used for the fourth task. Here is what he discussed regarding the chain rule:

> Technically you are using the chain rule on all these problems [...] You can write this, do a substitution, let u equal this [pointing to $\sqrt{x}$] and justify what I am about to do using the chain rule, but the basic idea is the following [and he continued with the *technique*].

DISCUSSION AND CONCLUSIONS

The FTC1, as described in Case 1, is transposed differently in Case 2 and Case 3. From a praxeological analysis standpoint, FTC1 in Case 1 encompasses both the *praxis* and *logos* blocks. Both components are discussed quite comprehensively. The main focus in the text was the *logos* block; however, the *praxis* block is given more emphasis in the exercises at the end of the section, as evidenced by the presence of 12 tasks focused on finding the derivative of an integral, while tasks that emphasize the *logos* block are less frequent. When comparing the teaching of FTC1 in Case 2 with Case 3, the analysis reveals that different types of tasks are discussed in these two cases. Case 2 places its focus on contextual tasks that aid students in developing the *logos* block, whereas Case 3 prioritizes the *praxis* block to differentiate various types of integrals. It is worth noting that Case 2 does not delve into the *techniques* for solving these tasks and instead primarily focuses on the *logos* block. This emphasis on the *logos* block is consistent in other videos produced by this channel. In contrast, Case 3 prioritizes the *techniques* that students need to learn to successfully complete their homework, as discussed earlier in the methodology section. Such a focus on finding the derivative of integrals does not come as a surprise to me. In one of my previous research projects (see Radmehr, 2016, Chapter 6), I observed that such tasks were the central focus in the lectures, tutorials, and assignments on the FTC1 topic. Furthermore, these tasks also appeared in the midterm exam of the calculus course. In conclusion, it seems that the *taught knowledge* in YouTube learning resources serve different purposes and address various aspects of mathematical praxeology based on the intentions of the content creators in different *teaching institutions* (here, YouTube channels). As undergraduate mathematics educators, I believe we are responsible for investigating the opportunities that these *teaching institutions* provide for our students and considering their possible inclusion in the support we offer, aligning with our intended learning outcomes.

NOTES

1. https://www.youtube.com/watch?v=rfG8ce4nNh0&t=491s
2. https://www.youtube.com/watch?v=PGmVvIglZx8&t=48s
3. https://www.youtube.com/watch?v=HflALiqXJDo

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INTRODUCTION OF THE NOTION OF ORDINARY DIFFERENTIAL EQUATION FROM THE INFINITESIMAL APPROACH IN CALCULUS COURSES FOR ENGINEERING STUDENTS

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In the introduction to a Differential Equations course, students face the fact that the approach promoted in the first units is not the same as that they were taught in their previous courses, associated with Calculus. In this way it is considered that a conceptual and procedural break exists between these courses. The purpose of this project is to promote a learning trajectory based on the infinitesimal approach within the ideas developed by Leibniz, where the differential played a fundamental role. In this summary, one of the designed is shown, wherein students can have an encounter with the notion of differential, which will help to introduce the notion of ordinary differential equation in subsequent objectification processes.

Keywords: learning of Calculus, infinitesimal approach, differential, ordinary differential equation, objectification.

BACKGROUND

Professors participating in the training of engineers have a common concern in the learning process of Differential Equations —the way in which they are presented to students in the courses in which they are first introduced and the prior knowledge that the students are required to have. Among the requirements, students are expected to have knowledge associated with Calculus, but they encounter a big problem: the approach promoted in the introduction to Differential Equations course is not the same approach they were taught in their previous courses. Specifically, in the formulation of a differential equation represented as a quotient of differentials, they are required to conceive the differential as a quantity that can be manipulated, that allows modeling phenomena associated with reality, and that the same manipulations allow you to solve them. This totally breaks with the conception that they may have formed about the differential, because in a Differential Calculus course it has been presented to them as a quotient that is restricted to being a limit, and that, therefore, cannot be manipulated as required.

As a historical fact, Napoles and Negron (2002) point out that the concept of differential equation remains related to the concept of differential until around 1821, when Cauchy created the name derivative. In current textbooks, this version established by Cauchy remains in force, "although when the methods for solving first-order ordinary differential equations are presented, the first conception is used without making it explicit (that is, the derivative is no longer the derivative, but a quotient between differentials)" (p. 47). In that sense, we can foresee difficulties associated with the notion of differential given the incorporation of these two versions in textbooks, and therefore in how the idea of differential is presented to students in different university courses, as mentioned by Recalde and Henao (2018). They maintain that it is common for a course associated with mathematics to present the version established by Cauchy, but in courses that are related to physics usually the infinitesimal strategies proposed by Newton and Leibniz are preferred.

The above allows us to think that the orientation of the courses prior to Differential Equations, such as those in Calculus, should promote the understanding of a notion as important as that of differential. In that sense, Recalde and Henao (2018) mention that "the teaching of differential equations should include reflection and discussion of the concept of differential" (p. 68), adding that usually in Calculus and Differential Equations courses this concept is not taught.

THE PROBLEM

The historical development of ordinary differential equations (ODE) shows that the way in which they emerged was from real-life phenomena, and that geometric analysis was what allowed them to be proposed and solved. This contrasts with what is presented today in "traditional" courses on Differential Equations for engineering, since –as Carranza (2019) mentions– when analyzing the study plans and the bibliography presented in them, we can find that working with algebraic methods is the preferred method proposed from the beginning, and once these are developed in the classroom, it is when we begin to work with real-life phenomena, which predetermines the fact that the applications are influenced by the strategies followed in the methodical presentation of the ODEs. Even if we look a little further back in the curriculum and study plans, where in theory (although not explicitly) ordinary differential equations first appear, that is, in a Differential Calculus course, it is evident from the version of the derivative that is presented (Cauchy's), based on limits, it is difficult for ODEs to be introduced, since this approach restricts the manipulation of differentials.

Continuing with the previous idea, it would seem contradictory that in a Differential Calculus course the derivative is presented based on limits –a method that imposes strong restrictions on the manipulation of differentials– and that in a Differential Equations course –in whose first units the solution methods are presented– the differentials are manipulated operationally, which is actually the essence of the very first method presented in the course and the first method in history —separation of variables (Napoles and Negron, 2002).

In view of this, the need to establish a solid conceptual bridge regarding the concept of differential between Differential Calculus and Differential Equations courses is clear, in such a way that students establish a relationship between said courses, understanding that Calculus, seen under the infinitesimal approach where the differential plays a fundamental role, provides the tools that allow modeling real-life situations by posing an ordinary differential equation.

From the above, a key element in the development of ODEs is the concept of differential, since it is directly involved in the derivation and solution of ODEs, in terms of how Newton and Leibniz proposed it, that is, operationally. In this regard, it is important that both Calculus and Differential Equations courses delve into the definition of differential, as mentioned by Recalde and Henao (2018), that is, that reflection and discussion about the phenomenological meaning of this concept be promoted in the classroom, since this would allow the translation from physical to mathematical language of ODEs to be more understandable for students.

It is clear that the version of the derivative from the perspective of limits is inadequate to give the differential the relevant role that it played during the rise of ODEs. Therefore, it could be said that engineering students are being deprived of the possibility of delving deeper into a concept that could be fundamental (Recalde and Henao, 2018), especially to model real-life situations closest to them in mathematical terms, and from there they can think of plausible solution strategies. Experimentation within an extra-mathematical context where differentials are involved, and their algebraic manipulation, could result in a path that helps them reflect and understand the entire process of mathematical modeling and, in turn, see the usefulness of the derivative in their own practices.

The version of differentials established by Leibniz could help students understand notions that are complex for them (Ely, 2020; Veron et al. 2022). It is mentioned in Veron et al. (2022) that the strategies used by students to respond to some questions associated with differentials can be related to the approach established by Leibniz, which reinforces the promotion of his approaches in this project. Furthermore, Ely (2020) points out that the flexibility of working with differentials would allow finding a differential equation, which in turn could answer several questions about the situation that is evolving, among them, those associated with its solution for which the integral has a fundamental role, interpreted from the perspective of the infinitesimal approach.

THEORY OF OBJECTIVATION

In the Objectivation Theory (OT), the development of learning is conceived as a result of collective processes that, as Radford (2020) emphasizes, "are rooted in the social, cultural and historical" (p. 17). To reinforce the differences between OT and other theoretical approaches, Radford (2023) points out that "knowledge is not something that the teacher transmits to the child. Nor is knowledge something that the child constructs on his or her own" (p. 16). He adds that in OT it is assumed that mathematical knowledge already exists, that it is rooted in the historical and cultural contexts from which it has been established, and that ways must be sought instead to organize the student's encounter with said knowledge in the classroom. These are seen as teaching and learning processes. That said, what is sought in this project is that the student's encounter with the notion of differential be based on what historically gave rise to it, as well as the contexts and phenomena that allowed its introduction and development. For this reason, it is considered enriching that OT is a social theory, according to which the students and the teacher, by assuming the commitment to the activity, are able to capitalize from this encounter, trying to make it natural, not just problematic, and even less traumatic.

One of the first concerns was to clarify to what extent OT would allow the construction and harmonization of conceptual and procedural bridges between the Calculus (Differential and Integral) and Differential Equations courses. To do this, it has been necessary to delve deeper into the relevance of designing an intervention project from the point of view of OT (especially since it has only been used for research projects). In this regard, Radford (2023) points out that in education it is important to provide optimal conditions so that the encounter with knowledge is as rich as possible. Since the infinitesimal approach is different from that traditionally promoted in the calculus classroom, to talk about the success of this approach, or that it can be considered a viable route in learning Calculus and Differential Equations, it is important that from the work Together, students are precise in their verbal and gestural arguments about the notions that are being addressed. In that sense, OT considers precision as a fundamental aspect; Radford points out that to achieve the encounter with knowledge (objectification), students must be precise in the way of expressing themselves.

By proposing an approach different from the traditional one, the types of activities that are promoted in the objectification processes will be fundamental for the notions of differential and ordinary differential equation to be introduced.

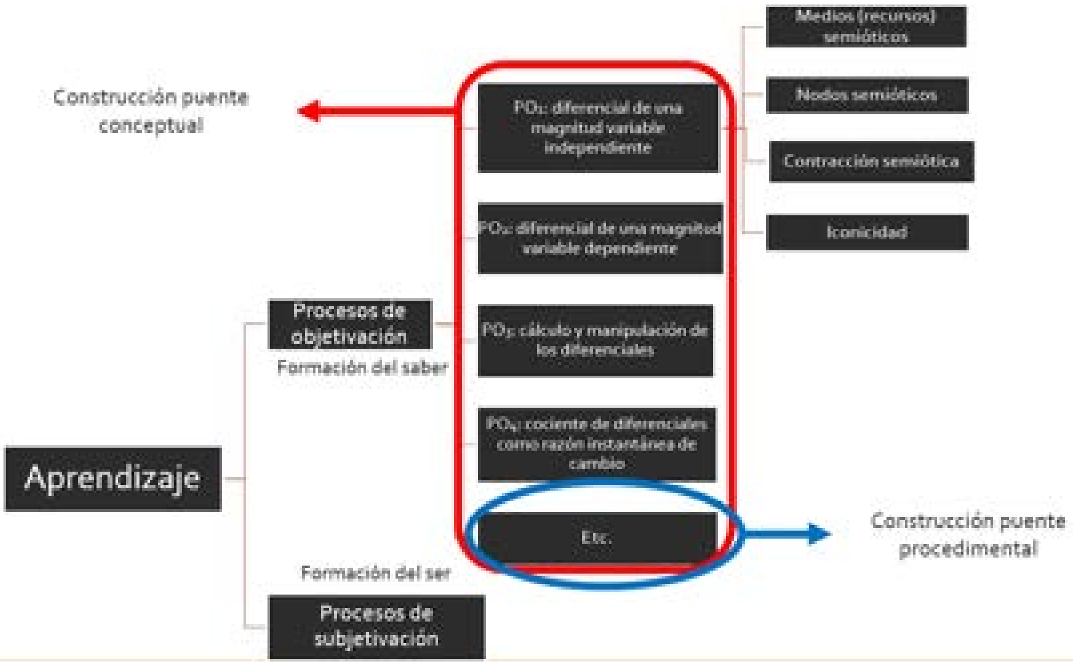
It is precisely in the processes of objectification, where Radford (2003, 2005, 2008, 2010) strongly involves Vygotsky's semiotics. In particular, the actions that are developed in the objectification processes, and specifically in the activities that are promoted in the proposed intervention, are associated with semiotic means, for which Radford (2003) points out that he refers to the objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of consciousness, to manifest their intentions, and to carry out their actions to achieve the objective of their activities (p. 5).

Such means are used by students to express themselves or convey certain ideas. Relying on joint work, in which they must be involved as active participants alongside the teacher, and where discussions are generated in various directions, these media are refined and connected with each other for the emergence of what Radford (2005) defines as semiotic nodes, which are “a piece of students' semiotic activity where action and various signs (for example, gestures, words, formulas) work together to achieve the objectification of knowledge” (p. 2). Therefore, in OT those are considered as a prior step to achieving objectification. The intention of these nodes should be that students move towards a precise way of expressing themselves and arguing, using a smaller number of semiotic means. This reduction of semiotic means, which is compensated by the concentration of meanings to express their ideas, is what Radford (2010) calls semiotic contraction. Furthermore, when students use their previous experiences to guide their actions in a new situation to achieve an encounter with knowledge, it is called iconicity (Radford, 2008).

To represent the above, Figure 1 outlines the considered elements of the theory and specifies some of the objectification processes that are involved.

Figure 1.

The objectivation processes for the learning of the concept of ODE



EXAMPLE OF THE PROPOSED ACTIVITIES

In this section, one of the designed activities is presented as an example, belonging to the objectivation process 2 (PO_2), called "differential of a dependent variable magnitude." It is important to note that for this process there are three activities designed in a preliminary phase, in such a way that it is expected that these will be sufficient to enable students to encounter the notion of the differential of a dependent variable magnitude. Considering the objectivation developed in PO_1 as a fundamental element in this process, so that with this, they are able to identify the relationship between the independent and dependent variable magnitudes. Below are the details of the activity.

Activity 2. The algebraic calculation of the infinitesimal variations of a dependent variable magnitude. Case 2.

Taking advantage of what was done in Activity 1 of PO_2, this activity begins by proposing three possible scenarios to calculate the differential of a dependent variable magnitude (Bezout, 1770), these are:

$$du = u(z + dz) - u(z)$$

$$du = u(z) - u(z - dz)$$

$$du = \frac{u(z + dz) - u(z - dz)}{2}$$

In the activity, students are asked to work with the relationships previously established in the course, about the problem of filling a conical container. Specifically, they are asked to work on the calculation of the differential of the direct proportionality relationships found in the problem. As an example, the case of the relationship between the radius and the height of the liquid of the circular surface is shown.

$$h_l = \frac{H}{R} r$$

$$0 \leq r \leq R, \quad 0 \leq h_l \leq H.$$

Its differential could be calculated within any of the three scenarios above. The case where the current moment and the immediately previous one is considered is presented here.

b) $dh_l = \frac{H}{R}(r) - \frac{H}{R}(r - dr)$

$$dh_l = \frac{H}{R}(r) - \frac{H}{R}(r) + \frac{H}{R} dr = \frac{H}{R} dr$$

The above is intended to indirectly bring students closer to the application of the operational rules of differentials, which will be explored in greater detail in the objectification process 3. In addition, they are asked to perform this calculation for five other relationships, for which it is expected that while performing the work (first in teams and then as a group), each of these differentials will be precisely calculated, hoping that they use the three previous scenarios. It is desirable that this activity be carried out in at most one hour of class.

DESCRIPTION OF THE MISE-EN-SCENE

It is important to note that a first trial of the first two objectivation processes has already been carried out. We worked with 28 Industrial Engineering students who are taking the course “Differential and Integral Calculus 1” within their first semester and who had made significant progress in the course, which was developed using the infinitesimal approach. The students were taken to a classroom that allowed them to have adequate workspace, especially to be able to work in teams. Teams of three people were formed and one student from each team was asked to record what their classmates did, in order to compile most of the actions that were carried out in the classroom by them.

Furthermore, it is worth noting that since we are working with a group exposed to the variational approach from the beginning of the Differential and Integral Calculus 1 course, the activities were adapted to the work pace that the students had. Therefore, the activities are part of the continuity of the course and were considered a fundamental part of the evaluation of the students; thereby it was expected that they would assume commitment, responsibility, and care for one another in the development that each of them had in the classroom.

For the development of the activities in the classroom, the phases of joint work indicated by Radford (2020) were considered. They are: presentation of the activity by the teacher, work in small groups, teacher-student discussions, group discussions, and general discussion. In that sense, the intention at this point is that both teacher and students must be active participants throughout the activity.

Specifically, Activity 2 was carried out in sessions 8 and 9, which corresponds to two hours of class. By then, students had gone through several introductory readings on infinitesimals and activities corresponding to the first and second objectivation processes that were carried out in the previous sessions. Furthermore, for the activities they were provided with only one worksheet per team, so they had to have a joint response in each requested section.

To collect information, the students video-recorded the sessions and took notes of what they did, which they uploaded to the work group formed on the Microsoft Teams platform. Also, the students handed in the worksheets to their teacher. In addition, the author was counted as an observer of the sessions and notes were taken to contrast with what was reported by the students.

ANALYSIS OF ACTIVITY 2

A preliminary analysis of Activity 2 has been carried out based only on the notes collected by the session observer. At this time, the experimentation is still being carried out, so these observations still need to be compared with the worksheets and video recordings that the students uploaded to the Microsoft Teams platform.

At the beginning of the Activity in session 8, the teacher asked the students to pay attention to the three solution schemes that the students had previously constructed based on the formulations made by Bezout (1770). With this, the expectation was that, for each case of the proportionality relationships established in the problem of filling a conical container, the students would be able to determine the differential of the dependent variable magnitude with the three schemes and reach the same result.

When beginning the work with the relationship between the volume of the liquid with the flowrate and time, most of the teams had difficulties in identifying the independent variable magnitude, since in the relationship $V_l = F \cdot t$, the students used the flowrate as that magnitude so that the teacher had to intervene in the teams that encountered this difficulty, trying to remind them that, in the video of the filling of a conical container, the flowrate acted as a constant magnitude. Furthermore, the functional notation of the three schemes caused difficulty in algebraically representing the differential of the volume of the liquid, so that the teacher's intervention had to be aimed at first trying to represent the magnitudes and their changes separately, and then move on to the calculation of that differential.

Something that was not present in the students' mobilizations is the conception of the differential of a constant magnitude, which they had previously worked on, so that the operational rule of multiplication, which should be used for the cases of proportionality relationship, was not considered by any team.

Table 1 is presented to summarize the theoretical elements detected in Activity 2, including the semiotic means used by the students, the semiotic nodes, and a first version of a semiotic contraction.

Table 1.

Theoretical elements present in Activity 2

Semiotic media	Verbal expressions	Students used phrases like "remember that the differential represents an infinitely small change." Furthermore, in the relationship between magnitudes "the differential of the independent variable magnitude influences the dependent one"
	Deictic gestures	This type of gesture appeared when students specifically pointed out on the worksheets the formulas, they should adhere to in determining the differential.
	Physical iconic gestures	This type of gesture appeared when the students remained attached to filling the conical container to try to conceive the differential of the dependent variable magnitude, from the differential of the independent variable magnitude.
	Symbolyc iconic gestures	This type of gesture appeared when the students remembered from Activity 1 the algebraic representation associated with the magnitudes, which later helped them to represent the differential of the dependent variable magnitude.

Semiotic nodes	It became noticeable that the students conceived the differential as a manipulable quantity so that, in the case $$dV_l = F(t) - F(t - dt)$$ $$dV_l = F(t) - F(t) + F(dt),$$ students divided by the time differential in the ratio to determine the instantaneous rate of change. $$\frac{dV_l}{dt} = F$$
Semiotic contraction	Although this is not the activity that closes the objectivation process 2, it was possible to glimpse in the students' arguments that they were close to objectivation, since they were able to identify that the differential of a dependent variable magnitude does not change uniformly the way that an independent variable magnitude does.

CONCLUSIONS

The proposed intervention is still in its initial phase, so that only a first experimentation has been carried out and not all the processes that have been considered. However, despite being an early stage of the intervention, it has been shown that promoting the infinitesimal approach as an alternative route for learning in an Engineering Calculus course is a viable option, since it allows students to take ownership of the concepts associated with infinitesimals and to connect them with some real-life phenomena. It is assumed that the fact that they have previously exposed to this approach predisposes them to develop their language in terms of what is expected. Furthermore, the theoretical elements indicated allow that in the group work done by the students, conjectures can be proposed that through the collaboration between teams and the general discussion end up either materialized or reconceived, so that this allows them to achieve the encounter with knowledge. .

In order to contrast the above approach, the intention is to use the proposed intervention with a group of students who have been exposed to the traditional teaching approach.

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What is important in undergraduate mathematics? Revisiting covariation through functional equations

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In undergraduate mathematics, calculus and analysis appear as related but often separate courses that both take the importance of certain objects for granted. While both constructivist and epistemological research programmes in mathematics education had important early agendas related to the questioning of mathematical contents to be taught, both seem to have later focused almost exclusively on contents delivery, leaving the selection and design of contents to scholarly tradition. In this paper, we take the "elementary functions" of secondary and undergraduate course in Calculus as an example. When, if ever, should students encounter rationales for the choice of these functions as "basic"?

Keywords: Teaching and learning of specific topics in university mathematics ; Curricular and institutional issues concerning the teaching of mathematics at university level

AN INTRIGUING PROLOGUE

In 1966, Walter Rudin (1921-2010) wrote the seminal and still widely used textbook *Real and Complex Analysis*, which as the title says is characterized by an attempt to integrate two otherwise often separate fields of study: real analysis (based on measure theory) and complex function theory. The book is typical for its genre: it has few worked examples, while elegant proofs of central theorems of analysis take up most of the text. Curiously, it begins with a "prologue" on exponential functions, that we shall also take as our point of departure here. The very first paragraph reads:

> This is the most important function in mathematics. It is defined, for every complex number z, by the formula
>
> $$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1}$$
>
> The series (1) converges absolutely for every z and converges uniformly on every bounded subset of the complex plane. Thus exp is a continuous function. The absolute convergence of (1) shows that the computation
>
> $$\sum_{k=0}^{\infty} \frac{a}{k!} \sum_{m=0}^{\infty} \frac{b}{m!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} a^k b^{n-k} = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!}$$
>
> is correct. It gives the important addition formula

$$\exp(a)\exp(b) = \exp(a+b) \tag{2}$$

valid for all complex numbers a and b. (Rudin, 1986, p. 1)

The book is clearly not written for readers with no mathematical preparation. According to Rudin (1986, p. xiii), "the prerequisite for this book is a good course in advanced calculus (…) The first seven chapters of my earlier book *Principles of Mathematical Analysis* furnish sufficient preparation". It is doubtful that many universities, today, offer courses labelled "advanced calculus" that teach what it takes to follow the "computation" cited above, but of course the notions of pointwise, uniform and absolute convergence of complex power series do appear in such courses or at least in the transition to real analysis. What we shall focus on here is the more informal notion "important", used twice in the above quote. The first use is a strong claim: "This is the most important function in mathematics", merely pointing at (1). There is, perhaps, a kind of justification in the second use: "important addition formula" (pointing at (2)). However, the rest of the preface merely derives other properties of the (complex) exponential function, most certainly met by students in previous calculus courses. Later in the book, exponential functions do appear here and there, both in the text itself and in exercises - most prominently perhaps in the development of Fourier analysis, based on Hilbert space theory (Chapter 4), and of course in the chapters on classical complex analysis. But the fundamental importance of (1) and (2) remains opaque.

RETHINKING OR QUESTIONING MATHEMATICAL CONTENTS AS A KEY DIDACTIC TASK

Confrey and Smith (1994, p. 135) pointed out a general challenge in research on mathematics education, while referring to the agenda of constructivism (more dominant 30 years ago than now):

> Constructivists have effectively documented that student errors are seldom random or capricious - they have a rationality and functionality of their own. In this regard, constructivists have documented that teachers and researchers must pay close attention to how a mathematics problem is conceptualized, worked on and evaluated by students. (…) reform efforts which attempt to open up and rethink the mathematical content are targeted mostly at the elementary grade levels, while secondary educational reform is more typically limited to pedagogical approaches as the content is assumed to be well-secured in its expert structure.

In the paper, which these considerations preface, the authors in fact examine and question the meaning and importance of functions, and in particular the fundamental properties of linear and exponential functions that make these turn up as models in both secondary mathematics, its uses in other school subjects, and even outside of schools. Developing a covariational view of functions, as an alternative to the more conventional correspondence approach, the meaning of (what is essentially) property

(2) is interpreted in terms of “rates of change” in two covariant quantities. In the next sections, we shall develop an analogous idea for the undergraduate level, related to functional equations. We do so as a case for a more general hypothesis: what the authors propose for secondary level reforms (understood as systematic efforts to improve mathematics education) is no less relevant for the university level. Many efforts of reform are, also at this level, focused on generic pedagogical approaches like flipped classroom or other supposedly student-oriented forms of teaching, rather than on “efforts which attempt to open up and rethink the mathematical content”. Little or no effort is put into a student experience of the rationales – we could say, of the *mathematical value* – of what is taught, probably from the assumption that the content will speak for itself. The more or less tacit assumption that “content is assumed to be well-secured in its expert structure” is likely to be even more part and parcel of the institutional contract at universities, with the frequently praised collaboration between what one terms “mathematicians” and “educators”: the former are in charge of the content and the latter are the supposed experts of delivery to more or less challenged students.

Several mathematicians and didacticians (most commonly, hard to classify uniquely as either) have challenged this division of labour, at least for the secondary level, also much before Confrey and Smith. Klein’s (2016) work on “elementary mathematics from a higher standpoint” advocated and demonstrated how both modern and historic approaches to elementary concepts such as functions should be made available to future teachers. He considers that “to instruct scientifically can only mean to induce the person to think scientifically, but by no means to confront him, from the beginning, with cold, scientifically polished systematics” (Klein, 2016, p. 292). Several decades later, Klein’s proposals concerning the centrality of functional thinking in secondary mathematics were superseded by the “New Math” reforms, which certainly went much beyond pedagogy and form in their attempt to rebuild mathematics teaching at all levels on modern foundations such as set theory and logic. The most lasting effects of these changes occurred at universities (Bosch et al., 2021). Indeed, present-day students are likely to be treated to texts like Rudin’s book. In the heydays of New Math, texts for the secondary level also confronted students with “cold, scientifically polished systematics” with no traces of their historical origin or motivation. One can interpret the early works of Freudenthal (1968) and Chevallard (1985) as problematising this situation, calling to put fundamental questions and phenomena to the forefront of school mathematics, and thus to rethink not merely its delivery but also its contents.

Chevallard (1985) introduced the fundamental distinction between *external* and *internal* didactic transposition, in order to emphasize the link – but also the difference – that exists, in modern school institutions, between the act of forging official educational programs for the school, and their day-to-day implementation within the school. Teachers are mainly (if not only) in charge of the latter. Universities function in this regard – much more than in the epoch of Klein – as schools (Verret, 1975).

While university teachers may exercise some course-level influence on the external transposition, the remarkable stability and similarity of the overall structure of university mathematics programmes suggest that this influence remains in practice very limited (Bosch et al., 2021). Certainly, there exist documented examples of undergraduate mathematics programmes which deviate substantially from the common forms and contents, not least in smaller and newer college-type institutions (see eg. Niss, 2001; Buteau et al., 2016), where they may both arise and disappear more easily. But the overall homogeneity remains, as does the question of what forms and means a deeper questioning of well-established contents could take in mainstream university institutions.

The general question of selecting and organizing mathematics contents in university programmes is a difficult and complex one, which involves not only the international scholarly community of mathematicians, but also more subtle sociological features of university institutions, like those documented by Verret (1975) in his analysis of didactic transposition in this context. The modular structure of many university programs – with each module being responsible to teach a clearly delimited and strongly coherent collection of theoretical knowledge – leads to a sequence of student encounters with dense packages of given, long established *answers*. Students get no or at best very limited opportunities to experience mathematics as a problem solving, questioning activity, especially as the amount of material to cover in each module tends to increase, to satisfy more advanced needs. In the sense of Bouligand (see Bosch and Winsløw, 2016), the emphasis on syntheses is much stronger than on questions or problems.

As a result, students are not given deliberate opportunities to reflect on the importance or motivation of fundamental mathematical constructs, such as the exponential function, even if applications may be quite abundant (while often appearing at quite distant moments of study). Especially for future researchers and teachers of mathematics, one can argue that undergraduate programmes should offer such opportunities for students to work on such questions, possibly as extensions of, or complements to, standard modules. Indeed, both professions are concerned with selecting, formulating and engaging with mathematical problems, and with developing explicit and deep knowledge of how certain fundamental constructs and syntheses contribute and combine to solve them.

THEORETICAL CASE STUDY: WHAT MAKES THE REAL VALUED EXPONENTIAL FUNCTION IMPORTANT?

A first observation, concerning the opening quote, is that Rudin's definition (1) is explicit and based on analysis (power series and their convergence), while (2) is at first sight merely a derived property of the object defined by (1), and moreover, a purely *algebraic* property: it states that exp is a homomorphism from $(\mathbb{C},+)$ into $(\mathbb{C},\cdot)$. Other properties are derived later in the prologue, like $\frac{d}{dz}\exp z = \exp z$, but in this short case study we will just focus on (2), and mostly restrict our attention to *real* (rather than complex) variables. It is a main purpose of this paper to exemplify the kinds of questioning that undergraduate programmes do not (but, as we shall argue, ought to) include, in order for students to evaluate or appreciate the meaning and value of central mathematical constructs.

In the language of Confrey and Smith (1994), (2) then means that exp defines a covariation of variables with a *multiplicative rate of change* : adding a fixed value k to the first variable leads to multiplying the second variable by a fixed value k'. In functional notation,

$$\exp(x + k) = k' \cdot \exp\,(x)$$

where $k' = \exp\,(k)$. Similar assumptions on rate of change occur in many models of familiar phenomena, from compound interest on savings accounts to radioactive decay. A priori, the particular construction (1) could be just one example of a function satisfying the more abstract equation

$$f(x + y) = f(x)f(y),\ x, y \in \mathbb{R}. \tag{3}$$

To evaluate the importance of the function defined by (1), it is therefore reasonable to ask if there are other functions satisfying (3) than the one defined by (1). In fact, $f_c: x \mapsto \exp(cx)$ defines such a function, for any real c. But then – are there other than these?

Before answering this question, we could first think about what makes f_c work just as well as exp, in the sense of having the homomorphism property given by (3). In fact, any *linear function* (defined, for any $c \in \mathbb{R}$, by $x \mapsto cx$) is a homomorphism from $(\mathbb{R}, +)$ into $(\mathbb{R}, +)$. Composing it with exp yields a homomorphism from $(\mathbb{R}, +)$ into $(\mathbb{R}, \cdot)$, and in fact for any homomorphism h from $(\mathbb{R}, +)$ into $(\mathbb{R}, +)$ we could get a function satisfying (3) by definition $f(x) = \exp\big(h(x)\big)$. So we might as well begin by investigating the possibilities for having functions h that satisfy

$$h(x + y) = h(x) + h(y),\ x, y \in \mathbb{R}. \tag{4}$$

Readers (and some undergraduate students) will no doubt recognize this as part of the condition for a linear map (on the vector space $\mathbb{R}$). To Confrey and Smith, (4) is an advanced or somewhat technical formulation of what it means for a covariation to exhibit an *additive* rate of change. Of course, at the undergraduate level, the use of functions must be supposed to be well-established, although its actual meaning to them will depend on whether the pre-university experience has been informed by thorough content questioning.

It is time to point out that (3) and (4) are *functional equations*, that is, they enounce a property that functions could have, and come (like ordinary equations) with the natural question: what functions actually satisfy the equation? It is a singular irony that current external didactic transpositions make it unlikely that undergraduate students have investigated this question for (3) and (4), which are both simple and significant as argued by Confrey and Smith – while they are routinely exposed to differential equations, a technically more complicated type of functional equation. They may even answer the question in the title of this paper by saying that exp is the unique solution to $f' = f$ with the boundary condition $f(0) = 1$. This is not bad, of course, but still way less fundamental than a possible characterisation in terms of the purely arithmetic property in (3), if it exists.

Students' investigation of (3) and (4) could of course be organised in many ways, following a thorough reflection on what makes the properties they enounce fundamental and important (for instance based on elements of Confrey and Smith, 1994, as outlined above). Teacher students, who have worked thoroughly with Brousseau's puzzle situation, may realize the fundamental importance of (4) in the theory of similarity in the plane (see, for instance, Winsløw, 2007). This, then, would make the solutions equally important.

If undergraduate students investigate (4) further, based on literature or the internet, they will soon discover that the equation is in fact named after Cauchy – since in fact, it was Cauchy (1821, pp. 104-106) who first proved that linear functions are the only *continuous* functions on ℝ which satisfy (4). Working with some form of this proof, they will realize that continuity is not required for the simpler case of functions defined only on ℚ. In fact, one easily derives from (4) that $h\left(\frac{m}{n}\right) = \frac{m}{n}h(1)$ for all $m \in \mathbb{Z}, n \in \mathbb{N}$, and only then uses continuity to conclude $h(x) = xh(1), x \in \mathbb{R}$. Further study of literature could lead students both to discover a number of alternative conditions which together with (4) ensure linearity (see for instance Jung, 2011, p. 21), and to the fact that weird non-continuous solutions to (4) do exist (Hamel, 1905), at least if one assumes some form of the axiom of choice.

In most applications, whether in geometry or in natural sciences, assuming continuity for solutions may in fact be quite natural. With that assumption, the solution for (3) can be derived much as for (4). Here is one way: if f is any solution to (3), then $f(x) = f(0)f(x)$ holds for all $\in \mathbb{R}$, so either $f = 0$ or $f(0) = 1$. Certainly, $f = 0$ solves (3); to investigate other solutions, we assume $f(0) = 1$. But then, for all $x \in \mathbb{R}$, continuity at 0 implies that there is some $n \in \mathbb{N}$ so that $f\left(\frac{x}{n}\right) \neq 0$ and thus $f(x) = f\left(n \cdot \frac{x}{n}\right) = f\left(\frac{x}{n}\right)^n \neq 0$. So if $f \neq 0$, f has no zeros at all, and as $f(x) = f\left(\frac{x}{2}\right)^2 \geq 0$ we see that $f > 0$. Then with $h(x) = \ln\big(f(x)\big)$, (4) holds and so by the above, h is linear and continuous. It follows from Cauchy's result that $f(x) = \exp(cx)$ where c can be any real number.

All in all, the importance of exp can be explained, at least in the real variable case, by the fact that its composition with linear functions give all non-zero continuous solutions to the functional equation (4). In courses for advanced undergraduate students, we have experimented various designs of assignments which allow students to work on some of the above arguments, and found that reconstructing or looking up such arguments contribute to their appreciation of the fundamental mathematical importance of exp. In the next section, we complete the picture by looking at two other functional equations which complement (3) and (4).

COMPLETING THE REAL PICTURE: WHAT OTHER FUNCTIONS ARE IMPORTANT?

In terms already used above, which are likely to be more familiar to undergraduate students than the idea of covariations and their various rates of change, we can interpret the above as classifying the continuous homomorphisms $(\mathbb{R},+) \to (\mathbb{R},+)$ and $(\mathbb{R},+) \to (\mathbb{R}^+,\ \cdot)$. They are, respectively, all linear functions (no constant term) and all exponential functions. Students may then ask (or be asked) about the two remaining cases: can we also determine all continuous homomorphisms $(\mathbb{R}^+,\cdot) \to (\mathbb{R},+)$ and $(\mathbb{R}^+,\cdot) \to (\mathbb{R}^+,\cdot)$?

From secondary school, students know that the real exponential function exp is injective, with inverse ln. Now, if $f\colon \mathbb{R}^+ \to \mathbb{R}$ satisfies $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$, then if we let $g(x) = f(e^x)$ for $x \in \mathbb{R}$, we get $g(x+y) = g(x) + g(y)$ for $x, y \in \mathbb{R}$. If f is also continuous, then so is g, and by Cauchy's 1821 result, we have $c \in \mathbb{R}$ so that $g(x) = cx$ for all $x \in \mathbb{R}$. But then $f(x) = f\left(e^{\ln x}\right) = g(\ln x) = c \cdot \ln x$. Certainly, this *is* a solution for any $c \in \mathbb{R}$, and if $c \neq 0$ the solutions can also be written in the form $f(x) = \log_a x$ where $a = \exp(\,1/c)$ is any positive number different from 1. Thus, continuous homomorphisms $(\mathbb{R}^+,\ \cdot) \to (\mathbb{R},+)$ are the logarithms and 0. By a similar argument, continuous homomorphisms $(\mathbb{R}^+,\cdot) \to (\mathbb{R}^+,\cdot)$ are the power functions $x \mapsto x^a$ where $a \in \mathbb{R}$.

Continuous homomorphisms from $\downarrow$ to $\rightarrow$	$(\mathbb{R},+)$	$(\mathbb{R}^+,\cdot)$
$(\mathbb{R},+)$	Linear functions	Exponential functions
$(\mathbb{R}^+,\cdot)$	Logarithms and 0	Power functions

Table 1. Continuous homomorphisms on additive and multiplicative groups of real numbers.

To sum up, Table 1 shows four classes of "important functions", deriving their importance from being the continuous functions that satisfy the four arithmetical functional equations.

Investigating the four functional equations up to establishing the picture in Table 1 could, in fact, be an important topic in an undergraduate capstone course, also in view of looking forward to graduate mathematics (hints in the next session). The topic could and should, especially for future secondary teachers, also be related to the more elementary conceptualisationss of Confrey and Smith (1994), considering covariations generated by simple arithmetic progressions, and their practical uses.

GOING COMPLEX

One can ask similar questions about the homomorphisms of any other ring or field, and as in the real number case, further conditions may ensure a nice classification. At the undergradate level, a natural first step is to ask whether something like Table 1 holds for the complex number field. There is no difficulty in extending the result to continuous homomorphisms $(\mathbb{C},+) \to (\mathbb{C},+)$, that is, they are simply linear functions. The rest of the table cannot be generalised directly for the simple fact that $\mathbb{C}^+$ does not make sense (in fact, $\mathbb{C}$ cannot be made into an ordered field). Still, to establish a similar importance of trigonometric functions (about the only transcendental functions students know but besides the three non-linear ones in Table 1), it would be worthwhile to consider at least the case of continuous homomorphisms $(\mathbb{C},+) \to (\mathbb{C},\cdot)$, to which exp belongs according to (2). As in the real case, $f(z) = \exp(cz)$ indeed defines more examples (now with complex constant c and variable z), and one can then show that besides 0, there are no more than this, by reducing to the real case already treated above. Alternatively, one can replace the assumption of continuity by the existence of a complex derivative at 0, and use the complex version of (4) to derive that is f entire with $f'(z) = f(0)f(z)$, $z \in \mathbb{C}$, from which the result follows readily. Whether one derives this as a special case, or it appears as an intermediate step to prove the general case, we also get that all continuous homomorphisms $(\mathbb{R},+) \to (\mathbb{T},\cdot)$, where $\mathbb{T} \subseteq \mathbb{C}$ is the unit circle, are of form $f(x) = e^{icx} = \cos cx + i \sin cx$ for some $c \in \mathbb{R}$. Besides reminding students that cosine and sine can be constructed as a derivate of (1), this characterization shows their relative importance, as the real and imaginary parts of continuous homomorphisms of the real line onto the circle. For future teachers, students can also revisit the high-school interpretation of cosine and sine as "coordinates on a circle" (where the variable is interpreted as an angle, a highly informal notion at pre-university level (Winsløw, 2016).

One can naturally also investigate the remaining two functional equations in the complex case, and in fact for other fields, Banach algebras, and so on. The study of homomorphisms (often endomorphisms) appears with additional hypotheses in many areas of graduate mathematics, such as algebraic topology, Lie algebra theory and functional analysis - often with some of the elementary settings considered above as illustrative special cases.

OUTLOOK: QUESTIONING AND VALUING IN UNIVERSITY MATHEMATICS

The value (or importance) of a mathematical construct cannot itself be defined or determined mathematically. It is not even what Chevallard (1985, p. 49-51) calls a *paramathematical* object, unlike notions like equation or proof, as one can do mathematics without ever referring to the value of the objects that one treats. It is quite different when teaching: it is part of the institutional contract that universities (and other schools) do not merely teach *any* mathematics, but that the external didactic transposition must somehow select and prescribe topics of considerable potential value to the learner. Even to university students, both future teachers and scientists, it does not (practically and ethically) suffice to *claim* value: it must, at least for the most fundamental constructs, be part of the teaching to allow students to question and evaluate these constructs, mathematically or otherwise. In the special case of future teachers, it is related to the recently very active research theme of "making university mathematics matter for secondary teacher preparation (Wasserman et al., 2023).

Again in Chevallard's terms, mathematical value is a *protomathematical notion* (informal notions that cannot be mathematized): constructs derive their mathematical value from the questions they serve to solve, whether they are strictly mathematical or not. We have moved from the value of objects or answers, to valuing questions: what questions are worth solving? This, indeed, depends on the learners' foreground. But there is still a considerable advance from merely claiming that certain mathematical constructs or answers are important, to organising an experience of how they serve to solve specific questions.

Questioning and valuing thus go hand in hand, in any attempt "to induce the person to think scientifically" (Klein, 2016, p. 292), and requires teachers to "pay close attention to how a mathematics problem is conceptualized, worked on and evaluated by students" (Confrey and Smith, 194, p. 135). Realizing the proposals above, for questioning and valuing the meaning of elementary functions based on functional equations, depends ultimately on just that.

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COMPRENSIÓN DEL SIGNIFICADO CAMPO DE DIRECCIONES ASOCIADO A UNA ECUACIÓN DIFERENCIAL ORDINARIA

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Keywords: Ecuaciones diferenciales ordinarias, campo de direcciones, modos de pensamiento.

DE LA ENSEÑANZA TRADICIONAL AL ENFOQUE GEOMÉTRICO

En México, la asignatura de ecuaciones diferenciales ordinarias (EDO) suele ser obligatoria en la universidad. Los libros de texto empleados en su enseñanza (e.g., Edwards et al., 2021; Elsgolts, 1977) abordan los diferentes métodos de solución: analítico, numérico y gráfico, desde una perspectiva algorítmica para los primeros dos, mientras que relegan los métodos cualitativos a un segundo plano. De acuerdo con Bajpai et al. (1970) los estudiantes, con la práctica, adquieren experiencia para aplicar diversas técnicas y procedimientos para encontrar la solución o soluciones de algunas EDO. Sin embargo, detrás de estas resoluciones correctas, subyacen ideas erróneas y lagunas conceptuales (Arslan, 2010). En contraparte, la perspectiva geométrica y el estudio gráfico de las EDO promueven un acercamiento conceptual y visual (Artigue y Gautheron, 1983; Rasmussen, 2001). En ésta, el estudio del campo de direcciones constituye una herramienta didáctica, que permite a los estudiantes visualizar y experimentar con conceptos abstractos, y contribuye al desarrollo de habilidades de pensamiento crítico. Con base en lo anterior, se propuso la siguiente pregunta de investigación: ¿Cuáles son los niveles de comprensión del campo de direcciones asociado a una EDO de primer orden? Para abordarla, se consideró el enfoque cognitivo de los modos de pensamiento (Sierpinska, 2000), los cuales se especificaron para el estudio de este concepto. En este póster se ilustran elementos del modo de pensamiento sintético-geométrico.

MARCO TEÓRICO Y METODOLOGÍA

Sierpinska (2000) establece tres modos de pensamiento asociados al estudio de un concepto: Modo sintético-geométrico (SG), Modo analítico-aritmético (AA), Modo analítico-estructural (AE). Su identificación permite establecer una caracterización consciente de los niveles de comprensión existentes en los estudiantes. Estos modos pueden usarse como heurísticas al resolver una tarea, eligiendo un orden en su uso, no único, pero el tránsito entre estos modos lleva a tener diferentes significados del objeto matemático. Para este estudio, se han hecho adaptaciones pertinentes a estos modos de pensamiento para estudiar el objeto matemático: el campo de direcciones asociado a una EDO de primer orden.

La metodología elegida fue cualitativa de carácter exploratorio y descriptivo (Maxwell, 2013). Participaron 4 estudiantes universitarios, que habían cursado la asignatura de EDO, seleccionados mediante un muestreo no probabilístico por conveniencia (Maxwell, 2013). Con el objetivo de caracterizar los modos de pensar que tienen los estudiantes respecto al campo de direcciones asociado a una EDO de primer orden, se diseñó e implementó un cuestionario (Cohen et al., 2007) con ítems enfocados específicamente en situaciones en donde aparecen los campos de direcciones asociados, respectivamente, a diferentes EDO. Una vez que los estudiantes realizaron el cuestionario fueron entrevistados.

PRIMEROS RESULTADOS

El modo de pensamiento sintético-geométrico se identifica en dos actividades del cuestionario. Por ejemplo, cuando se pide trazar curvas solución para la ecuación $\frac{dy}{dx} = x - y$, que pasen por puntos específicos (condiciones iniciales), sobre un campo de direcciones ya marcado, en el cual aparece una curva solución. Únicamente un participante realizó la actividad correctamente. En la entrevista, él reconoció que cada curva solución de la EDO debe parecerse a la curva ya trazada, pasar por el punto señalado y ser tangente a esos "pequeños trozos" de rectas en la trayectoria en donde está el punto de interés. Es decir, reconoce que las curvas así trazadas son soluciones de la EDO que el analizó. De forma general, los modos de pensamiento identificados ilustran diferentes niveles de comprensión del campo de direcciones asociado a una EDO de primer orden y constituyen una base teórica para el diseño de una propuesta didáctica.

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“THINKING CLASSROOM”-BASED ACTIVITIES IN CALCULUS TUTORIALS: BARRIERS TO IMPLEMENTATION

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Keywords: Teachers' and students' practices at university level, Novel approaches to teaching, Active learning, Transition to university, Teaching assistants.

INTRODUCTION AND THEORETICAL BACKGROUND

It is well-documented that students can experience the secondary-tertiary transition in mathematics as demanding, and one of the discontinuities highlighted by Gueudet and Thomas (2020) is that universities traditionally give students fewer opportunities to be involved in mathematical activity within organized teaching. In a Norwegian context, Goodchild et al. (2021) found indications of a preference for teacher-centred approaches over student-centred approaches, and infrequent use of approaches they considered having a potential for promoting active learning. However, while teacher-centred approaches may be prevalent in large-enrolment courses with auditorium lectures, there have been attempts to transform tutorials towards more discussion-based teaching and student-active learning (e.g. (Borge, 2019)).

Inspired by the work of Borge, Goodchild and others, we designed activities for first-semester calculus tutorials that are based on the "Thinking classrooms"-framework originally developed in the context of K-12 mathematics classrooms (see for example (Liljedahl, 2016)). We focused on two elements of mathematics teaching discussed within this framework: the type of tasks the students are given, and how the students work on these tasks. In our intervention, each weekly tutorial starts with a task designed to promote student engagement in core mathematical activities such as problem solving, reasoning and communication, and in line with the "Thinking classrooms"-framework we intend the students to collaborate in small groups, working on vertical non-permanent writing surfaces (such as small whiteboards).

In this study, we will investigate how the “Thinking classrooms”-based approach was enacted by the teaching assistants (TAs) responsible for the calculus tutorials, with the aim of identifying barriers to implementation and factors that may guide future adaptations of the design. Our study can be classified as implementation-related research (Koichu et al., 2021), as we are interested in the enactment of an approach outside of the context in which it was originally developed.

METHODS AND DATA

The TAs have written brief descriptions of how the tutorials have progressed. These descriptions have formed the basis of semi-structured interviews with TAs, and we have collected student questionnaire responses to supplement the interview data.

PRELIMINARY RESULTS

The TAs appear to have made only minor modifications to the intended design, and all report that - from their perspective - integrating "Thinking classroom"-activities in the tutorials have been meaningful. However, this appears to contrast with preliminary findings from the student questionnaire, which indicate barriers to implementation such as (1) students perceiving the tutorial activities to be irrelevant for the intended learning outcomes of the calculus course; (2) students' reluctance to engage in group work; or (3) other elements of the course design being perceived to compete with the tutorial activities, such as students wanting to focus on mandatory assignments.

CONCLUSION

Many undergraduate mathematics courses are large-enrolment courses, involving a set of teachers and TAs. New teaching approaches must be implemented at different levels, and a threat to implementation may occur if the different elements of the course design are not perceived to be in alignment. More actively involving both students, lecturers and TAs in discussions about design decisions may reduce this threat to implementation.

POSTER CONTENT

The poster will depict the "Thinking classrooms"-approach with illustrations and an example task, elaborate on the overall design of the calculus course and how the TAs have been supported throughout the semester, as well as presenting and discussing barriers to implementation and factors that will guide future adaptations of the design.

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Didactic variables in pre-structuralist praxeologies: the case of continuous functions

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Keywords: Teaching and learning of analysis and calculus, transition to, across and from university mathematics, anthropological theory of the didactic.

INTRODUCTION

In my recent work I have analysed the structuralist aspects in the teaching and learning of real analysis in the first year of a Bachelor course (Laukert et al., 2023). For this, I examined the course material. For the analyses presented on this poster, I will use the same material but will focus on the role of the didactical variables that appear in the type of tasks of pre-structuralist praxeologies in the case of continuous functions. Thus, the research question is: What is the role of didactical variables in the development of pre-structuralist praxeologies in the case of continuous functions?

THEORETICAL FRAMEWORK

The Anthropological Theory of the Didactic (Chevallard & Bosch, 2020) offers a 4T-model of *praxeologies* (task, technique, technology, theory). The notion of didactical variables relates to the praxis block of a praxeology. Chaachoua & Bessot (2019) consider a generic type of task and a system of variables. Then the values of the variables generate more specific types of tasks. An additional point of view with regard to the 4T-model is gained by mathematical structuralism. In praxeological terms, the structuralist method consists in the passage from a praxeology $\mathbf{P} = [T/?/?/\Theta_{\text{particular}}]$ where it is unclear which technique to apply, to a structuralist praxeology $\mathbf{P_s} = [T^g/\tau/\theta/\Theta_{\text{structure}}]$ where, modulo generalisation of the type of tasks (T^g), the theory of a given type of structure guides the mathematician in solving the problem. Furthermore, Hausberger (2018) distinguishes several levels of structuralist praxeologies: at level 1, structures appear mainly through definitions; at level 2, the technique mobilises general results about structures. While in the context of an introductory analysis course, structuralist praxeologies are not fully developed, the praxeologies there are rather pre-structuralist (Laukert et al., 2023). A pre-structuralist praxeology is contextualized within **R**, and it turns into a structuralist praxeology if the metric or topological structure of **R** is generalised in terms of a metric or topological space.

RESULTS

Didactic variables of a type of task play a role in the development of pre-structuralist praxeologies. I differentiate between two main variables. The first main variable refers to the mathematical objects in the tasks. They can be specific or generic. The second main variable refers to the mathematical structure. It appears either explicitly or implicitly in the tasks and it can also be specific or generic. In the course material, the type of task T appears: Show that a function f has a maximum or minimum. The function f is the object that constitutes a variable of T. In a problem set of the course, the variable takes the specific value $f(x) = x^2 e^{x^2} + 17x^3 + 2$ defined on the interval $[0, \sqrt{2}]$. Then, the value of the second variable is specific, because it refers to the ordered field structure of **R**. However, the role of the ordered field structure in T stays implicit. In upper secondary school, students searched for the critical points of f and evaluated the function at the endpoints of the interval. This contributes to establish a pre-structuralist praxeology of level 1 where derivatives and the concept of critical points are used as a technology. Now, let us generalise the value of the first variable such that f is a real continuous function on a compact interval. This choice turns the task T into a proving task of the extreme value theorem (EVT). In the course, proving the EVT is not a task, but rather done by the lecturer. Although the function f is now a generic object, the second variable is still specific referring implicitly to the ordered field structure of **R** and to the topological notion of compactness which appears merely as a vocabulary in terms of compact intervals in the course. The EVT provides a pre-structuralist theorem to the students. Now, they can specify the first didactic variable again by returning to $f(x) = x^2 e^{x^2} + 17x^3 + 2$ and apply the obtained general result (EVT). Thus, a transition from a pre-structuralist praxeology of level 1 to level 2 occurs. The pre-structuralist praxeologies that have been obtained here, turn into structuralist praxeologies if the second didactical variable becomes generic and if continuous functions between a compact metric space and **R** are considered. Then, a structuralist theorem is evoked, namely the general EVT. Thus, the role of the didactical variables is to point out the dialectic between objects and structures and between the generic and the specific. Since different values of the didactical variables do not change the type of tasks, they also play a role for the unification of praxeologies. Finally, they help to discuss the elaboration of pre-structuralist praxeologies. The poster will show additional examples in table form to account for all the pre-structuralist praxeologies in the sector of continuity developed in the course.

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Teaching double integrals using GeoGebra: normal domain and polar change of variables.

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Keywords: Teaching and learning of mathematics in other disciplines, Digital and other resources in university mathematics education, ACE, GeoGebra Applets.

There have been several attempts to improve students' learning and the teaching of mathematics in basic and multivariable calculus through moving away from remedial classes towards teaching to increase understanding using technological tools. Within the integration of two-variable functions, specific difficulties can arise, the students should know and correctly apply integration techniques, recognise, and describe the domain of integration, and evaluate using different available strategies. Limited research has been reported. Khemane et al. (2023) commented on students' difficulties in double integrals, which included changing the order of integration. Moore et al. (2014) describe three types of complexity that students encountered during the construction of the polar coordinate system: understanding the measurement of the angle in radians, overcoming the input conventions output and the differentiation of the pole. The present study was conducted to determine the effect of the ACE (Activities on the computer, Classroom discussion and Exercises done outside the class) cycle-based learning process (Arnon et al 2014) on students' learning the use of normal domain and polar change of variables in double integrals calculus. Is there an improvement in the understanding and use of these methodologies thanks to ACE cycle that includes GeoGebra Applets? The participants were 36 and 19 second-year undergraduate Civil and Envoirmental engineering students at the University of Udine for a second course in Analysis. The activities on the computer, as first part of ACE cycle, are the presentations of two interactive GeoGebra Applets, regarding normal domain and polar change of variables referred to the following examples:

1. Determine the area of the set $A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq \frac{x}{2} \leq y \leq 2x,\ y \leq -x + 3\right\}$

2. If $A = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1, x^2 + y^2 \geq 1 \}$, determine $\iint_A \frac{x}{x^2+y^2} dxdy$

The purpose of Applet 1. for 1. (https://www.geogebra.org/m/zpx5ypsv) is to point out, using the slider that, if we want to use normal domains, in this case with respect to x, it is necessary to divide the set into 2 parts. The "intersection" command was used to further highlight the change in representation once intersection point is crossed. The purpose of Applet 2. For 2. (https://www.geogebra.org/m/uhztazbu) is to compare the use of normal domains with that of changing in polar coordinates. Here sliders were also used. We wanted students to reflect that, when using polar coordinates, the radius depends on the angle and the domain is transformed into a normal domain, with the respect of the angle, and not simply into a rectangle. The subsequent discussions in class (second cycle) highlighted the students' difficulties in seeing functions and not numbers as extremes of integration, even more when a change in polar coordinates was made. In the last case students are tempted to take the smallest and largest values that the radius can have. By moving the sliders, the students were able to experience and convince themselves of what was happening. As third cycle we propose some homework that the students had to do and submit via email. The exercises were chosen to highlight the difficulties tested. After receiving the works, the best ones were chosen and proposed as a correction of the homework. Some observations have also been added to complete them. This was followed by a further discussion in the classroom to highlight the most common errors. An example of the exercises proposed is:

EX.1 Calculate in 3 ways $\iint_A \ x\, dxdy$ where $A = \{(x, y) \in \mathbb{R}^2 |\, x^2 + y^2 \leq 4,\ x \geq 1\}$.

Its purpose is to push students to think about the set as a normal domain with respect to x or y and change in polar coordinates. The result of this exercise has been compared with the same one proposed last year where no specific GeoGebra Applets were proposed. Table 1. shows, for both academic years, in the second column, the percentage of those who carried out the exercise correctly considering the set as a normal domain with respect to x and y, only with respect to x in the third column, only to y in the fourth, and finally with change in polar coordinates.

EX. 1	**Normal for x and y**	**Normal for x**	**Normal for y**	**Polar coord.**	**3 methods**
22/23	60%	18%	3%	5%	3%
23/24	30%	42%	13%	13%	8%

Table 1: Results of EX1

The use of polar coordinates is less immediate for students. Between the choice of considering the domain normal with respect to x or to y, the first is more immediate. However, this academic year, the second method had a greater choice. The use of the polar coordinate had a substantial increase, as did the number of those performed the exercise correctly in all three methods. With the inclusion of GeoGebra Applets and using ACE cycle, the results were undoubtedly better.

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Functions as solving tools illustrated by problems of simple relativity

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Keywords: functions, problem solving, relativity.

INTRODUCTION

Through this poster we would like to communicate a theme that points to the creative aspects of the concept of function and indeed to its fundamental mathematical significance. The presented didactical proposition, targeted at students of Mathematics Departments and mathematicians, aims:

a. To encourage students to work with functions with more imagination and flexibility.
b. To appreciate more the usefulness of the abstract form of functions.

We hope that the working on this kind of problems will give to students experience in:

i. Identifying the most suitable variable(s) to attack a problem genuinely cognitively (rather than 'blindly' converting into algebraic expressions).

ii. Adapting the context of a function into another where the same function is imbedded, but in such a way that properties of the function are more evident. (In problems analogous to the one presented in Monk, 1992).

WHAT ARE PROBLEMS OF SIMPLE RELATIVITY

A problem of simple relativity is thought of as an approach towards a particular type of problem (rather than a problem in itself). The type of problem involves (typically) two *objects* with *velocities* in some *space*; instead of considering the dynamics of the two objects separately, we regard the one as a stationary and add to the velocity of the second the 'opposite' velocity of the first. This yields significantly different outlooks for solving strategies to answer questions involving the simultaneous positions of the objects. We study:

a. How strong is the intuition about relativity?

b. How can we say in a more mathematical way what we are doing here?

EXEMPLARY PROBLEM (ELEMENTARY)

Adam and Eve are walking together, when Adam realized that he has dropped his apple. Adam turns around to find the apple, whilst Eve remains walking in the original direction. Now Adam walks 1.5 times faster than Eve. Compare the time in which Adam finds the apple with the time between finding the apple and catching up with Eve again.

Approach 1. Let time $t = 0$ represent the time Adam (A) and Eve (E) first separate and the position $d = 0$ represent the place where A and E separate. Let $t = T$ be the time when A picks up the apple. Then the movements of A and E are represented separately by forming the functions f_E and f_A where

$$f_E(t) = t \text{ and}$$

$$f_A(t)\begin{cases} -{}^3/_2\, t, & t \in [0, T] \\ {}^3/_2\,(t - T) - {}^3/_2\, T, & t \in [T, T_1] \end{cases}$$

where T_1 is the time when Adam catches up. Hence,

$$f_E(T_1) = f_A(T_1) \Rightarrow\ T_1 = {}^3/_2\, T_1 - 3T \Rightarrow T_1 = 6T \text{ and}$$

$$\frac{time\ \ between\ A\ finding\ the\ apple\ and\ catching\ up\ with\ Eve}{time\ A\ takes\ to\ find\ the\ apple} = \frac{T_1 - T}{T} = 5.$$

Approach 2. Consider the variable $'distance\ d\ between\ A\ and\ E\ '$. When A is on his way to collect the apple, d is increasing at a rate $1.5 + 1 = 2.5$, (some unit per time) until the maximum d_{max} of d is reached when A picks up the apple. Hence, d decreases at a rate $1.5 - 1 = 0.5$ from d_{max} down to 0. (i.e. when A catches up with E). The answer of the question is $\frac{2.5}{0.5} = 5$.

ISSUES TO CONSIDER

What will be the problems for students following either approach? Which is the easiest to think of and then to carry through? What be more readily accepted as a Mathematical argument? How do you think of basing the answer on variable *d*? Which is the 'closest in meaning' to the original context? Why can approach 2 be regarded as relational?

ENDNOTE

The poster points to a need in mathematics education for students to develop a well-founded 'sense of functions' (Eisenberg, 1992). We hope that by applying a constructive line of using functions as tools in problem solving, leads to the consolidation of the concept. A problem of simple relativity exemplifies the proposition as a starting point on this particular kind of problems.

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La acumulación como resignificación de la integral definida en la educación básica a partir de una revisión bibliográfica

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Palabras clave: Teaching and learning of analysis and calculus, categoría de acumulación, integral definida

La categoría de acumulación se refiere a cantidades que se acumulan cuando varían con respecto al tiempo $\left(F(b) - F(a) = \int_a^b f dx\right)$ (Cordero, 2005). En una investigación con estudiantes para profesores de matemática, Marcía-Rodríguez (2020) encontró dos formas del uso de esta categoría: la primera se manifiesta mediante la discretización de un fenómeno que está variando continuamente, es decir, los estudiantes toman valores separados y recurren a la suma de estos para determinar una cantidad acumulada. Esta primera forma del uso permitió llegar a la otra forma: la constantificación de lo variable y toma del elemento diferencial, es decir, donde un valor se considera constante en un lapso pequeño de tiempo. Llama la atención el orden en que se manifestaron las formas del uso, lo que motiva a cuestionarnos sobre la relación entre la categoría de acumulación y sus manifestaciones en los niveles educativos básicos, ya que la variación discreta está relacionada con las operaciones elementales de la suma y resta que aparecen en los primeros niveles del sistema educativo (Cordero, 2005).

Considerando lo anterior, surge el interés de indagar sobre esta categoría en la educación primaria. La revisión bibliográfica que se realizó busca responder ¿qué elementos de la categoría de acumulación subyacen en investigaciones de educación básica? La búsqueda se realizó en el repositorio *Google Academic*. Se utilizaron las siguientes palabras de búsqueda: children, elementary education, advanced mathematics, calculus, integral calculus, accumulation. Además, se utilizó las "referencias conectadas" mediante *Connected Papers* y *Google Academic*.

La literatura encontrada en esta línea es escasa. Se encontró que se han explorado ideas sobre la derivada mediante la velocidad en contextos de cinemática (Noble et al., 2001; Stroup, 2002). Asimismo, se destaca un estudio cualitativo e intuitivo del cálculo integral y el uso de lo discreto para el desarrollo de actividades (Nemirosvky, 1993; Stroup, 2002). Además, se identifica el rol del tiempo para justificar orden y rapidez en las actividades que se plantean (Tierney y Monk, 2008).

Los elementos de la categoría de acumulación que subyacen en la revisión de literatura realizada son: el *valor acumulado*, que se expresa como $F(b) = F(a) + \int_a^b f(x)dx$, es decir lo que se está agregando, ¿cómo está cambiando? Este valor acumulado lo identificamos en algunas de las actividades de los estudios de Nemirosvky (1993) y Stroup (2002), en donde se pregunta al estudiante ¿cuántos bloques (una cantidad) habrá al final? o ¿cuántos había inicialmente? En estas actividades también subyace la *comparación de dos estados*. Otro elemento identificado de la categoría de acumulación es la *constantificación de lo variable*, cuando se relaciona la velocidad a la que cambia la altura de un vaso de coctel en cierto momento con la velocidad constante de un vaso cilíndrico que tenga la misma abertura que el vaso de coctel en ese momento (de Beer et al., 2017). Un último elemento que identificamos es el *significado de área*: Araujo y Avelar (2022) proponen lo que denominan *pensamiento integral*, y que la introducción de este en actividades con niños propiciaría el desarrollo de diversas ideas de área en ellos, además los encaminaría en la cultura del concepto de integral y probablemente hacia pensamientos propios del cálculo diferencial e integral.

Finalmente, y dada la importancia del *tiempo* en la acumulación de cantidades y en la toma del elemento diferencial, a partir de esta revisión bibliográfica inicial nos planteamos otra pregunta para el desarrollo posterior del estudio: ¿qué otros roles desempeña el tiempo en las situaciones donde se presenta la categoría de acumulación en niños de educación básica?

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Intuitive arguments for asymptotic approximation of some famous sums in the university mathematics classroom

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Keywords: teachers' and students' practices at university level, transition across university mathematics, intuitive arguments, estimation of sums, asymptotic approximation of functions.

HISTORICAL INTRODUCTION TO THE THEME OF RESEARCH

Asymptotic approximations of functions (or sequences) $f(n)$, for large values of n, have played an important role in mathematics until current mathematical research (see for example Gowers, 2002, ch.7). In many cases mathematicians are satisfied by finding such an approximation instead of an exact solution. It seems reasonable to try to introduce mathematics students in the idea of such a practice, by using historical examples and especially the estimation of the sums $S_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$ for $k = 1, 2, 3$ and for large values of n, by a polynomial approximation. Here we present a university classroom experience with third year mathematics students about the approximation of the sum $S_2(n)$.

THE CLASSROOM DIDACTICAL RESEARCH PROJECT

A. Research questions and collection of data

Our research questions were the following: what are the implicit ideas of mathematics students about «infinitely great» or «infinitely small»? Do students conceive these expressions as absolute properties or as relations between two quantities? In particular, how do they compare two polynomials of one variable n for $n \to +\infty$?

All dialogues in the classroom were recording. Moreover, we asked the students to give us short written essays on their answers to the questions posed.

B. Analysis of some dialogues and students' arguments

A task was given initially to a group of students in the second year at the University of Patras during the course of Mathematical Analysis. The subject of this task was to estimate the sums $S_1(n) = 1 + 2 + 3 + \cdots + n$ and $S_2(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2$. The students were in acquaintance with geometrical representations of "triangular numbers" etc. (Mesquita et al., 2022) and estimated $S_2(n)$ geometrically by themselves without any previous information, using a pyramid made of small cubes in consecutive levels of 1, 4, 16, …, n^2 cubes respectively. By considering spontaneously these cubes as infinitesimally small, they identified them to points.

Then, we asked a class of third year students to examine the above result and criticize the method. According the previous model, we form a pyramid and we calculate its volume $V = \frac{1}{3} \cdot B \cdot h$, where B is the base area and h is the height of the pyramid. The base area B is a square with side equal to n, so $B = n^2$ and $h = n$. So, $V = \frac{1}{3} \cdot n^2 \cdot n = \frac{1}{3} \cdot n^3$. Our question to the students was: «How essentially different from $\frac{1}{3} \cdot n^3$ is the sum $S_2(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2$?». One of the students, Vangelis, used the formula $S_2(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$ and said that if n is very big, then $n + 1$ "*tends to* n", so $2n + 1 = 2 \cdot (n + \frac{1}{2})$ "tends to $2n$". Another student, Ilias, continued by manipulating the previous formula $S_2(n) = \frac{(n^2+n) \cdot (2n+1)}{6} = \frac{2n^2+3n^2+n}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$ and remarked that the first term $\frac{n^3}{3}$ is very big compared to the others. «Do you mean that if n is very big, then $\frac{n^3}{3}$ will be very big compared to the other terms?», we asked Ilias. Ilias agreed, but two other students, had a different view, saying that the term n^2 is not infinitely smaller than n^3. Then Ilias said: «I am going to correspond the volume of the pyramid to n^3 and the area to n^2 : this looks like a sheet of a paper». So «It is clear that n^2 is infinitely smaller than n^3, just as we think of the space and get a slice from it». Moreover, when we asked the students to compare $\frac{n^3}{3}$ and $S_2(n)$ for large values of n, Vangelis said that in order to compare these results, we would take the *ratio* $S_2(n) / \left(\frac{n^3}{3}\right)$ and this «would be equal to 1 or better would tend to 1». But in order to explain himself, he wrote that «for n=100, $\frac{S_2(100)}{100^3/3} = 1 + 0{,}015 + 0{,}0005$, which tends to 1», meaning that $\frac{S_2(100)}{100^3/3}$ is almost equal to 1.

CONCLUDING REMARKS

Our research results indicate that some of the students have spontaneously a good intuition of the "orders of infinity" as they appeared in Analysis, towards the end of the 19th century. However, they seem to make an abuse of the expression "tends". Also, as we have seen above, Vangelis used the word "tends" for expressing the asymptotic approximation of a function by another one. This indicates that a dialogue such as the one presented here could indeed introduce the students to the important topic of asymptotic approximation of functions.

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Triadic Notions of Signs and Abstraction of Infinite Series

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Keywords: abstraction in context, design-research-redesign, Peircean perspective, teaching and learning of specific topics in university mathematics, teaching and learning of analysis and calculus

INTRODUCTION & RESEARCH QUESTIONS

Mathematics education research has long focused on understanding how students construct abstract mathematical knowledge (Dreyfus et al., 2015), particularly in contexts like calculus where concepts are complex and abstract. Despite wide applications of infinite series, it remains an under-researched topic, with recent studies highlighting challenges in students' understanding of convergence. Martinez-Planell et al. (2012) found two conflicting cognitive constructs for infinite series: the infinite process of adding numbers, and infinite series as the limit of the sequence of partial sums. The notion of summing an infinite number of terms can be challenging for students to grasp. With this, the present study aims to explore how students form their mathematical knowledge about the convergence or divergence of infinite series using direct comparison test before formal instruction, focusing on their representations and interpretations of signs, and vertical reorganization of prior knowledge. Drawing on Peircean triadic notions of signs and Abstraction in Context (AiC), this study seeks to shed light on these processes, aiming to inform instructional design and improve students' conceptual understanding of infinite series. Specifically, this research seeks to answer the following:

a) What representations and interpretations exhibited by the students help them construct a mathematical structure in determining convergence or divergence of infinite series using Direct Comparison Test (DCT)?
b) How do students vertically reorganize their prior constructs to form a new mathematical structure, i.e., convergence or divergence of infinite series using direct comparison test?

THEORETICAL FRAMEWORKS AND METHODS

The poster presents how two theories, Peircean triadic notions of signs (Saenz-Ludlow et al., 2016) and Abstraction in Context (AiC) (Dreyfus et al., 2015), complement each other in exploring how learners develop their understanding of mathematical concepts. Peircean theory explains how individuals represent, interpret, and create symbols for mathematical objects through the triadic signs: the object, sign-vehicle, and the interpretant. The poster will show the directions of meaning-making and refinement of concepts using the triadic signs. AiC offers a framework for studying the construction of abstract mathematical knowledge within specific contexts, through vertical reorganization of prior constructs into a new mathematical structure in three stages: the need for a new construct, its emergence, and consolidation which are described and analyzed by the nested epistemic actions, recognizing, building-with, and constructing.

For this study, we investigated the learning processes of eight preservice mathematics teachers and seventeen mathematics majors in answering a mathematical task, i.e., determining convergence or divergence of infinite series prior to their formal instruction, and corroborated the data with the follow-up one-on-one interview. Employing a design-research-redesign framework (Prediger et al., 2015), we designed, executed, analyzed, and refined a domain-specific task that served as the point of investigation, all grounded by the insights derived from the two theoretical lenses. The study aimed to understand how these theories intersect to shed light on how learners form and refine their mathematical understanding.

RESULTS & DISCUSSION

The results of the study focused on two research questions concerning students' understanding of the convergence or divergence of infinite series using DCT. Qualitative analysis of participant responses and follow-up interviews revealed that while some students demonstrated correct constructs, others exhibited knowledge gaps or misconceptions that became apparent during the interviews. The study found that students who grasped the concept of DCT often did so through symbolic representations, particularly inequalities. Regarding the reorganization of prior constructs, the study identified instances where students recognized necessary prior knowledge but still failed to form the concept of DCT correctly because they failed to establish the connections of these relevant constructs. Hence, the study highlighted the importance of activities or tasks that promote reflection and critical thinking. The study also suggested a hypothetical learning trajectory of the students in studying convergence of infinite series, offering valuable insights for educators in designing tasks and scaffolding instruction of infinite series.

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TWG2: Teaching and learning of linear and abstract algebra

Developing a formal attitude within a blended learning environment

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This paper reports on the experimentation of a blended learning approach to mathematics at the University level. In order to overcome possible predominance of procedural knowledge in favour of developing conceptual knowledge. We set a personal learning portfolio as students' pivotal resource acting at cognitive and metacognitive level to foster a change of their attitude towards mathematics through becoming aware of the three components of mathematical knowledge: procedural, intuitive and formal. In this paper we present and discuss the findings of a qualitative analysis of learning portfolios authored by freshmen Information Engineering students attending a Linear Algebra course.

Keywords: teachers' and students' practices at university level, digital and other resources in university mathematics education, teaching and learning of linear algebra, learning portfolio.

INTRODUCTION

In the transition to University, students' face difficulties that have been widely studied and described (Clark & Lovric, 2008; Gueudet, 2008; Tall, 2008). In particular, as Ufer et al. (2017) clearly state, "the character of mathematics shifts from a school subject with a focus on calculations and applications to a scientific discipline based on explicit definitions, deductive proofs, and formal representations" (p. 397). This is even more true for University courses considering mathematics as a "service" subject, which often leads students to make the equivalence between learning mathematics and rote learning of mathematical procedures. In order to foster change in students' attitude towards mathematics and mathematics learning, some of us started to be interested in designing and exploiting digital activities, through which students can be engaged (Albano & Pierri, 2014; Albano, 2017), with the purpose of helping them in overcoming possible predominance of procedural knowledge in favour of developing conceptual knowledge (Hiebert and Lefevre, 1986). Such activities aim at engaging students outside the classroom and in the weekly tutoring sessions when collective discussions the tutor focus on the main difficulties that were observed through access to the digital platform. In this paper we report on an experience carried out with university students attending a regular course of "Linear Algebra" with a scenario of blended teaching, by means of the use of an e-learning platform. In the following, after presenting an outline of our conceptual background and the key elements of the didactical scenario, we analyse some exemplar protocols drawn from the data collected and discuss the first results arising for this experience.

THEORETICAL BACKGROUND

The terms procedural and conceptual, referred to mathematical knowledge, have become widespread in the mathematical education literature using as common reference classic definitions given by Hiebert and Lefevre (1986) that sounds as follows: procedural knowledge is characterised by managing rules or procedures for solving mathematical problems; at the same time, because many of the procedures consist in chains of prescriptions for manipulating symbols, procedural knowledge also consists in a managing individual symbols and the syntactic conventions of the representation system (pp.7-8). Conceptual knowledge, also referred to as relational knowledge, can be thought of as a connected web of knowledge, a networking in which the linking relationships are as prominent as the discrete pieces of information (pp. 3-4). It is widely recognized the role that both of them play in teaching and learning Mathematics Several research studies highlight a lack of equilibrium between the two modes, and evidence shows that often in school practice, the procedural can overcome the conceptual (Engelbrecht, Harding & Potgieter, 2005; Baroody, Feil & Johnson, 2007).

In this respect, this distinction seems appropriate to describe specific students' difficulties in the transition from high school to university, as said above. However, in order to fully grasp the complexity of the issue, and specifically the change of focus that is required entering the University, we need to elaborate further on the complexity of mathematical knowledge and its possible components. In particular, the *conceptual component* usually remains not completely described; such indeterminacy is based on the ambiguity of the term *knowledge* that can be referred to the individual as a cognitive agent, but also to the discipline per se, as a corpus of formally organised set of properties. In order to overcome this ambiguity, we found illuminating Fischbein's discussion (1994), where two perspectives - the cognitive and the formal - are clearly distinguished:

> Essentially speaking, mathematics should be considered from two points of view: (a) mathematics as a formal, deductive rigorous body of knowledge as exposed in treatises and high-level textbooks; (b) mathematics as a human activity (*ibid*, p. 231).

More specifically, the author describes mathematics as a human activity as the combination of three interrelated aspects: the formal, the algorithmic, and the intuitive. All these aspects represent the core of mathematics activity, as a formal science and as cognitive activity: each aspect must have an active part in the mathematical reasoning processes and for this actively used by the student.

The formal aspect refers to axioms, definitions, theorems, as well as proof as hypothetic-deductive construction, and the feeling of coherence and consistency. The formal aspect concerns mathematics as a formal science and as such must be taken into account in analysing students' cognitive processes. Specifically, the formal aspect has to be related to the other aspects: the algorithmic aspect consisting of standard procedures, shared and saved, to be applied in solving problems and the intuitive aspect that in spite of its possible discrepancy with mathematical theory, moves any solution process. According to Fischbein's model the duality between procedural and conceptual can be reinterpreted, in particular the conceptual component can be enriched considering both the formal and the intuitive aspect. Consistently, in the transition from high school to the University courses, students' difficulties mentioned above as the dominance of procedural approach can be interpret a disequilibrium between the three components: the dominance of algorithmic component can be interpreted as a break between the algorithmic and the formal aspect, leading to weaken the intuitive aspect. Specifically, we can interpret the difficulty in dealing mathematics subject from a theoretical perspective (Gaudet, 2008; Dorier, 2000) as a flaw in the formal aspect. Moreover, the weakness of the relationship between the formal and the algorithmic component can explain the students' difficulties in moving from the solution of a single problem, to the solution of a theoretical problem that is a problem addressing a class of problems.

In summary, we can assume that it is necessary to develop not only the interrelationships of the three components but also students' awareness of it. Following the work of Albano (2022), we consider, also, the effectiveness of a blended learning approach that might engage students on both the cognitive and metacognitive level (Schoenfeld,1983): proposing activities that invite students to solve problems as well to discuss possible solutions and their theoretical proofs; and activities that invite students to reflect on their own learning trajectory. According to the definition of blended learning as a combination of face-to-face with distance delivery systems (Osguthorpe & Grahama, 2003): asynchronous/synchronous, face-to-face/distance, paper and pencil/digital, students can be engaged in their personal study time/space inside and outside the classroom. For this reason, we decided to implement a blended learning environment offering students tools for supporting them in the way of thinking and reflecting on the necessary change of mathematics' view, described above. This means providing students with tasks, feedback on the tasks, but also requiring them to reflect on their performances, in particular on the reasons why they failed.

Thus, our research goal is to analyse whether and which features of the implemented blended learning environment can foster the development and the awareness of interrelation among the formal, the algorithmic and the intuitive aspects.

DIDACTICAL SCENARIO

We set a blended learning environment consisting of traditional face-to-face lectures and a dedicated course space on the e-learning platform Moodle. The Moodle course has been populated by the teacher with didactical material and resources. The didactical material consists of videos showing how to prove some main theorems or how to proceed for solving some typical tasks, books, screenshots of the digital boards used during the lectures, worked-out exercises, slides. The resources consisted of tasks, quizzes, FAQ forum, peer workshops for reviewing macro-sections of course content, personal wiki. The use of the e-learning platform allows students to have all the course materials available, to use a familiar environment, and to access their own entire educational history. The teacher also has a global view of each student's history. According to the theoretical framework, we designed the following didactical sequence: a) the students are required to carry out a quiz; b) then the teacher engineers a collective discussion, providing formative feedback on possible errors occurred, making constant references to the processes and mathematical theoretical tools underlying the resolution and choices made, supporting the differentiation of the discussion layers by means of different colours on the whiteboard (see Fig.1); c) the students are required to come back to their performance and to reflect on what did not work and why, writing their reflections on their personal wiki.

The learning portfolio

The student's personal copy of the course represents his/her portfolio, intended as a collection of the student's work such as quizzes performed, task's resolutions submitted, and so on, and also access data to didactical material. In addition, the student is required to use the personal wiki as a learning portfolio, containing his/her written reflections on the subject of learning and on how he/she is learning (Klenowski, Askew, & Carnell, 2006). The teacher introduced the learning portfolio as a document where the student can record what he/she has learnt and can reflect on his/her own learning path, through reporting on results achieved, difficulties encountered, remedies used to overcome difficulties. Therefore, the portfolio focuses on metacognitive aspects of the student as well as cognitive. The students were let free to choose the structure of their learning portfolios. It was made explicit the importance of a clear structure that could help the students come back to their learning trajectory. For each topic, the following questions were suggested in order to guide the drafting/writing of the portfolio: Did I do the quizzes and the proposed homework? How did I solve them? What mistakes did I make? Did I expect them? Why? What did I do to remedy them? Did the remedy used worked? How do you know that it worked or that it did not? What did you understand that you did not before? If you did not do any of the assignments, explain why?

Principles of design of the quizzes

The quiz, mainly containing multiple choice questions, is not procedural because, unlike the traditional tasks, it does not ask you to identify anything. The questions are all based on relationships between concepts: they investigate the intuitive part (procedural concept, intuitive meaning) and the formal one (for a detailed example see Albano, 2022). We chose to set all the questions of the quiz as multiple choice, also in case of just one correct answer, and also to include the "None of the other options", in order to force the students to avoid shortcuts. The discussion following the carrying out of the quiz concerns the truth or otherwise of each item and aims at putting back all the above elements together and the links between the procedural, the conceptual and the formal facets are made.

The questions can be grouped into three categories:

1. Questions about the mathematical notion

The items refer to the mathematical notion, bringing into play the following components: formal definition, procedure, intuitive meaning.

2. Questions with "because"

The items to be selected for these questions are composed according to the propositional structure 'A because B' corresponds to the logical structure 'B => A'. The implications are constructed so that the student can answer the truth of each proposition through a procedure but there is no procedure that tells him/her the truth about the implication. These types of questions involving relations between propositions cannot be reduced to procedural processes. This latter usually requires the students to construct an argument (chain of logical deductions) based on theoretical results (definitions, properties, theorems). There must be significant work to develop the formal component and also its symbolic expression.

3. Questions theoretical-relational with 'if … then'

In these questions the reference is the formal setting: the items to be selected as "True" or "False", are composed according to the propositional structure 'If A then B' corresponding to the logical structure 'A => B'. Differing from the questions with 'because', the student cannot answer the truth of the proposition A (premise) and again but there is no procedure that tells him/her the truth about the implication. The students need the intuitive component rich enough to give them the right example to understand whether each item is true or not.

METHODOLOGY

The context of this study refers to a Linear Algebra course for freshmen Information Engineering students. The course provides the students with synchronous online lectures (7 hours per week), tutoring sessions (2 hours per week), didactical material and resources available by means of the e-learning platform Moodle. The teaching style of the teacher and the tutor is based on the engagement of students in collective discussion starting from a task. The aim is to bring out the various components (procedural and conceptual) and aspects (intuitive, algorithmic, formal) that students put into play and to build together the interrelationships among them. The access to the didactic material and to the resources was not mandatory, but highly suggested, more and more in case of use during the lectures.

The data has been automatically stored in the platform. In this paper we focus on a qualitative analysis of 150 collected learning portfolios.

FINDINGS

The objective of the teaching intervention is articulated on two levels: cognitive and metacognitive:

- cognitive level: fostering the transition from theoretical to procedural aspects, building a relationship between concepts and procedures, i.e.; according to our theoretical framework, constructing and developing the interrelation between the formal, the algorithmic and the intuitive aspects.

- metacognitive level: fostering awareness of the relationship between procedural knowledge, understood as knowledge of algorithms/procedures, and conceptual knowledge, understood in both intuitive and formal dimensions.

To this aim we analysed the students’ portfolios looking at those excerpts which highlight: students’ reference to connections between theory and procedures; awareness of such connection at least in progress.

In this section we consider and discuss some excerpts drawn from students’ portfolios. The analysis allows us to identify some main topics: one is explicitly referred to the relationship between concepts and procedures, the other makes evident the agents (humans or resources) which are pivotal to the desired students’ moving and awareness.

- **relationship between the formal and the algorithmic dimensions**

The use of the portfolio asks students to reflect on their own difficulties, how and when they tried to cope with them. As expected, these reflections seem to develop students' awareness of the distance between the perceived understanding of the new concepts, specifically of their formalisation into a Theory, and the ability to respond to the tasks posed.

In such a direction goes the S1who realises that she makes a lot of mistakes when solving the exercises, but also that she needed a different approach:

> S1: Although I thought the theoretical lectures were very clear, in doing the exercises I did a lot mistakes. Reviewing the proposed solutions, I realised that my studying was not sufficient for solving the exercises to be done: not surprisingly, although I did a few things well, I realised that I did not understand how to approach to the exercise, missing proofs and correct solutions. Now I realise what I need to do to improve, and what not to do to avoid making the same mistakes again.

Reflecting on errors in the exercises enabled her to reinforce the formal aspect, leading her to relate the procedure to solve the problem to the theory, i.e., relating the algorithmic to the formal aspect:

> S1: The exercises in particular, were very helpful in understanding and reinforcing the theoretical concepts

Analysing students writings, we can observe a recurrent shift from the procedures to the theory and vice versa: a movement to and for studying the theory and solving exercises, as following excerpts clearly show:

> S2: I observed that although I thought I understood what was explained in class about theoretical concepts, with regard to exercises I did not perform well. I therefore reviewed what I had studied, then I tried to overcome the gaps by trying additional exercises.

S5: I was always missing something during the exercise. This was because the theory was not clear for me but after I deepened the theory, I was able to understand everything.

The proposed activities triggered the movement between theory and procedures, contributing to the development of the relationship between the intuitive, the algorithmic and the formal component. It seems possible to interpret S5's expression "I was able to understand everything" as evidence that a good balance between the three components has been achieved.

- **the role of teacher/tutor as mediator**

When the tasks accomplished individually are discussed in the classroom, the key role of the teacher/tutor appears. In the students' writings we find a clear and recurrent reference to the significance of these collective sessions and in particular of the interventions of the teacher. In the following, some examples of what the students wrote.

S1: Very useful was the classroom practice with the tutor, which helped [me] a lot to remove doubts and uncertainties about the explanations

S7: I encountered difficulties in doing some exercises, or rather in understanding some of the tracks but I was able to overcome them with the help of tutoring and classroom correction.

S8: I tried again to do the exercises after the correction in class and I was able to do them successfully

S9: Subsequently with corrections in class, I noticed that the approach to the exercise that takes advantage of mathematical definitions is quicker and more accurate.

Most of the students seem to be aware of the efficacy of collective discussions, few of them noticed specifically the need of moving to a 'new' approach in solving the exercises; as S9 writes: taking "advantage of mathematical definitions …".

According to the general aim of the didactical intervention, the tutor acted purposefully as a mediator to foster students' development of a 'new' relationship between the formal and the algorithmic component of their mathematical knowledge.

- **teacher's mediator means**

In order to best fulfil its function as a mediator, the teacher employs appropriate digital facilities: blackboards, slides, textbooks, video. The following excerpts show how the use of specific means proved to be particularly useful.

S4: As for the practice part, I did not encounter any problems except for in the early exercises, where I found just a few problems that I was able to solve as soon as I read again and focus on the slides' content at home.

S5: As far as the matrices, I did not have great difficulty. This is also thanks to the uploaded videos that give a big hand to solve doubts.

S6: The use of the recommended book was a great help.

S1: Very helpful were the blackboards made on the spot and the exercises done in the classroom.

The ‘blackboards’ mentioned by S1, actually refers to the teacher’s habit of using a whiteboard during the collective discussion and sharing afterwards copies of the screens. In Fig. 1 we show an example of a shared whiteboard.

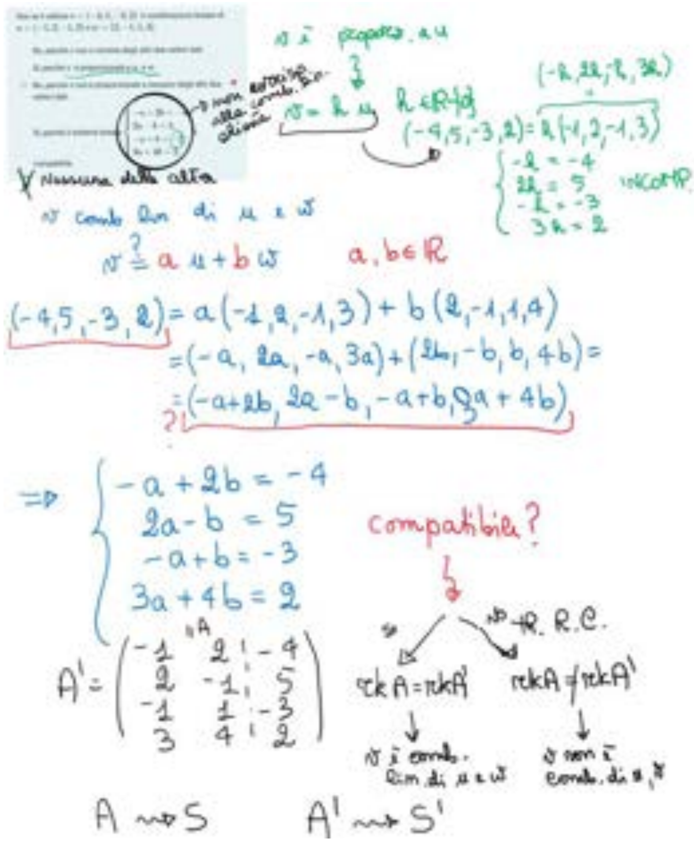

Fig.1: Collective discussion through a shared whiteboard.

The figure shows statically the trace of the flow of a collective discussion that develops over time: the comments and answers given are synthesized by the teacher in specific writings, verbal, symbolic and diagrammatic, with the additional expressiveness of the colors. It is possible to recognize the teacher’s intention to lead students to relate the algorithms used in solving linear systems and the corresponding theory, and in doing that develop the relationship between the algorithmic and the formal aspect.

In the discussion, the concept of “compatible” is elaborated both procedurally (it is related to the control by the algorithms for calculating ranks) and formally (it is related to the definition of linear combination). It can be seen that the dialectic between the three aspects develops around the notion of "compatible": the intuitive (the concept of compatible), the algorithmic (the calculation of matrices’ ranks) and formal (the definition of linear combination and the Rouché-Capelli Theorem). In this respect the screenshot available to the students can provide an effective means for students to evoke the lived discussion. They can relive, through the colors red and blue, the dynamics between the three aspects - formal, algorithmic and intuitive - as developed in the discussion. We could hypothesise that the dynamics of the discussions in which the teacher constantly relates the three aspects, may have been internalized by the students to such an extent that it can reappear in small group discussions, explaining their effectiveness. Difficulties in approaching exercises can be overcome by sharing them with classmates, ass highlighted in the following excerpts from the diaries:

S3: We started the matrices and I encountered several difficulties that I shared with my colleagues. I had the possibility of sharing with them the solutions compared with the latter, and I understood a little better.

S5: The doubts I had about the matrices I solved thanks to the help of my colleagues.

S11: The exercises were not easy right away, but by going over the theory thoroughly and working in groups, I was able to overcome the initial difficulties.

CONCLUSIONS

When students arrive at the university, they tend to approach any task trying to mobilise "the good procedure", that is the procedure taught for that circumstance. Such an attitude is the consequence of a traditional approach to mathematics which sees a predominance of a procedural approach based on memorising specific procedures, often condensed into a formula. Such a 'predominance', usually contrasted with the 'conceptual approach', can be usefully interpreted as a disequilibrium between the formal, the algorithmic and the intuitive aspects whose combination, according to Fischbein, constitute the essence of mathematics as a human activity. Specifically, the introduction of these aspects allows to better describe the theoretical perspective requested in the University courses. The analysis presented above shows how the activities carried out could orient students to establish a relation between the procedural and the conceptual component, intended as a good balance between intuitive, algorithmic and formal aspects. Moreover, we assumed that the activities proposed articulated both at the cognitive level, but mainly at the metacognitive level through the proposal of tasks that require the learners to make personal reflections explicit in written form through the use of the portfolio, could foster their awareness of the role of the formal component. We claim that the development of the awareness of the role of the formal aspect of the concepts learned, could have strengthened students' theoretical perspective as an advanced intellectual tool to be used in the future when solving professional problems.

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Paradigms of linear algebra

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This study is part of the ECOS-ANID C22H03 project, which involves four universities in Chile and France: UV, UPLA, PUCV (Chile) and UPCité (France). The aim of this project is to characterise and analyse the personal mathematical work of future teachers of mathematics in linear algebra at the beginning of university. It also aims to develop open tasks with automatic feedback on digital media in the field of linear algebra. We present here the constitution of the theoretical framework, more particularly the paradigms, and the analysis and implementation of a task that encourages students'work in different paradigms of linear algebra.

Keywords: paradigm, linear algebra, mathematical work, task, automatic feedback

INTRODUCTION

The theory of Mathematical Work Spaces (MWS) takes into account epistemological and cognitive aspects of work by crossing them with semiotic, discursive and instrumental dimensions for the specific study of mathematical work in an educational environment (Kuzniak et al., 2022). The notion of paradigm is central to the theory. It is used to characterise the MWS of a mathematical domain. To identify paradigms, we consider the historical, epistemological and curricular aspects of the mathematical domain in question. Paradigms help the researcher to analyse mathematical work and ensure that work processes are consistent.

Paradigms have already been worked on for geometry, analysis and probability. For each domain, we have identified three paradigms (P1, P2, P3) with an idea of horizontal and vertical mathematisation in three stages (Treffers, 1978). Horizontal mathematisation refers mainly to the formalisation and unification of objects and procedures from paradigm 1 within paradigm 2, while vertical mathematisation refers to the generalisation of objects and procedures within paradigm 3.

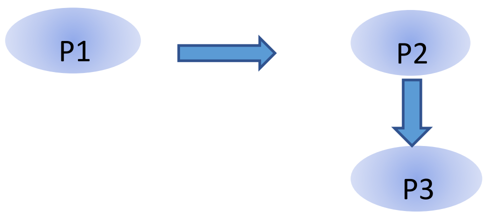

Figure 1: Schema of paradigms within horizontal and vertical mathematisation

In this paper, we focus on linear algebra (LA). We performed an historical, epistemological and curricular review to highlight the three paradigms we have identified. We experiment a task with some students with an online assessment platform. The task aims to favour the student's flexibility between paradigms.

HISTORICAL, EPISTEMOLOGICAL AND CURICULUM APPROACH

For this study, we based ourselves essentially on the work of Dorier (1997). To simplify we distinguish 3 periods in order to align with the 3 paradigms we will define later.

First period: linear systems and geometry

Systems of linear numerical equations played an important role in providing initial problems and concepts. Techniques for solving systems of linear equations are initially based on eliminations or substitutions (Eastern and Western civilisations). In 1750, Euler composed linear equations as sums of two equations, which was the first step towards interpreting them as mathematical objects that could be operated on. But determinants, along with minors and Cramer's method (1850), are the predominant method for solving them. For instance, in 1879, Frobénius defined dependence, linear independence and the notion of rank with minors. The algebraisation of geometry, with Descartes (1637) and Fermat' s methods (1643), brings geometry and algebra closer together. But this rapprochement is not achieved directly between points and *n*-uples, which could be interpreted as a definite advance towards AL, but rather between geometric curves and algebraic equations. Although algebraic methods demonstrate the simplicity of linear geometry, they are still insufficient for the emergence of a new theory.

Second period: unified theory of linear algebra for systems and geometry

The theory is initiated with the work of Grassmann (1844), who developed a theory with results equivalent to the vector spaces theory, such as the properties of operations, linear combinations and dimension. But his work was discarded by the mathematician until his death. In the 2nd half of the 19th century, a kind of unified theory of finite-dimensional linear algebra emerged that could be considered as a theory of $\mathbf{R}^n$ spaces. This theory, which takes into account the initially implicit duality between systems of linear equations and their solution with *n*-uples, see the development of the first concepts of AL, in particular linear dependence and rank. However, the linearity of $\mathbf{R}^n$ in itself is not primitive; the studies of linear systems are predominant. Determinants and matrices are still amalgamated, because matrices have to be considered as algebraic objects in order to go further in the conceptualisation. Hamilton's theory of quaternions played an important role in the development of what will become matrix algebra.

Third period: toward a formal theory

At the end of the 19th century (and for 40 years), two different approaches are taken: the axiomatisation of AL and the infinite dimension. Peano (1888) proposes the first axiomatisation of AL, referring to Grassmann, with an implicit reference to dimension (if we have a base with *n* elements, can we find a smaller generating family?). Pincherle (1901) studies problems in analysis and infinite (countable) dimension. He introduces the notions of rank of families of vectors, vector subspaces, hyperplanes and changes of basis, and he justifies that the vector space of numerical sequences is of infinite dimension. But his axiomatic theory seemed complex enough to deal with problems involving analytic functions. Burali-Forti and Marcolongo (1912) defines an intrinsic calculus with coordinate-free points and vectors. Weyl (1918) shows that the study of linear systems led to the same axiomatisation as geometry. Dorier says that the desire for axiomatisation is a posture that is not limited to a few applications. Finally, the structures of algebra, including AL, emerge from several contributions in algebra.

A few points on the curricular approach

Traces of these three eras can be found in the curricula of the four universities of the project, with two different introductions to linear algebra, only the first being closer to the historical genesis:

- Linear systems, pivot, geometry in dimensions 2 and 3 THEN the vector space structure with matrices and linear applications, reductions of endomorphisms (PUCV and UPC);
- Matrices and applications to systems, without geometry in dimensions 2 and 3, PLUS vector space structure, linear applications and back to matrices, reductions of endomorphisms (UV and UPLA).

We therefore define three paradigms of linear algebra, based on the historical genesis and consistent with study programmes in universities.

PARADIGMS OF LINEAR ALGEBRA

Roughly speaking, AL1 corresponds to the first period of AL, AL2 to the unified theory of $\mathbf{R}^n$ at the end of the 19th century (second period) and AL3 to the axiomatisation of the 20th century.

AL1: Paradigm 1

This paradigm proposes algorithmic work on linear systems and on the geometry of the plane and space (dimensions 2 and 3) with coordinates. The notion of linearity is central but algorithmisation is favoured (work is mostly on instrumental dimension) with calculations without reference to the algebraic structures. The objects are treated with few links between them. In particular, the difference between affine space and vector space is not observed: it is the same structure (it is not distinguished from the Euclidean structure). Algorithmic processes serve to validate the work (discursive dimension of the work). One of the main features of working in AL1 is that there is a direct, or usual, representation of the objects we are working with: objects of plane and spatial geometry as well as algebraic (linear) equations (semiotic dimension of work). Matrices can appear as a semiotic rewriting of systems, which limits the number of signs used by focusing on the row-column position. The systems are solved using Gauss's pivot method, with operations on the rows or columns, or Cramer's method with the calculation of the determinant.

AL2: Paradigm 2

This paradigm proposes working in $\mathbf{R}^n$ with a reinterpretation of linear systems in terms of matrix products, based on matrix algebra. Matrix algebra appears but algorithmisation is always present (instrumental dimension of work). Some algebraic structures are made explicit (in particular those of matrices and $\mathbf{R}^n$) and objects are treated with greater connections between them, with reference to the usual and implicit structures of $\mathbf{R}^n$ (discursive dimension of work takes a greater part). Systems of linear equations are reinterpreted using the product of matrices. Planes and lines are interpreted as special cases of vector subspaces, as are $\mathbf{R}^2$ and $\mathbf{R}^3$. In particular, we are no longer limited to dimension 3 and we go beyond the obstacle of dimension 4. The notion of linear application between $\mathbf{R}^n$ spaces is central (especially for validation), with the notions of rank, kernel and image, and we work explicitly with the bases of $\mathbf{R}^n$, mainly the canonical base, but also with other bases. There is a possibility for change of base and conjugation, in particular for the reduction of matrices (diagonalization, trigonalization) using eigenvectors and eigenvalues (possibly with a use of $\mathbf{C}$) with algorithmic work.

AL3: Paradigm 3

This paradigm proposes work based on the algebraic structure of vector spaces, in finite dimension or not (discursive dimension of work). It is characterised by the fact that the algebraic structures are now explicit. They serve for validation of the work and there is a strong relationship between the mathematical objects involved. As a result, there is a greater capacity for abstraction and generalisation of the basic notions of linear algebra. In this paradigm, the notions of dimension and basis of a vector space are explicit and important. The work is done with different vector spaces, without being limited either to $\mathbf{R}^n$ or to finite dimension. The semiotic treatments of these objects are essentially algebraic (no more link with geometry) and their representation depends on the space considered. A vector can now designate a matrix, a function, a sequence or a formal series, among other things. The objects are abstract. Their generality goes far beyond the framework of $\mathbf{R}^n$ spaces.

A TASK TO PROMOTE THE FLEXIBILITY BETWEEN PARADIGMS

This project aims to study the mathematical work of trainee teachers when they interact with an online assessment platform that performs automatic feedback (Gaona and Menares 2021, Gaona et al. 2022). Tasks plays a central role in this work. We distinguish open-ended tasks as ones that allow for different solutions.

Starting from our analysis of research into linear algebra, we particularly note the tasks proposed by Taylor et al. (2008) (Figure 2) which allow the transformations of a figure in the plane with matrices to be worked on from different points of view. The task can be approached using a trial-and-error strategy or with a system of linear equations (mostly in paradigm AL1) or taking into account the matrix as a linear application (paradigm AL2).

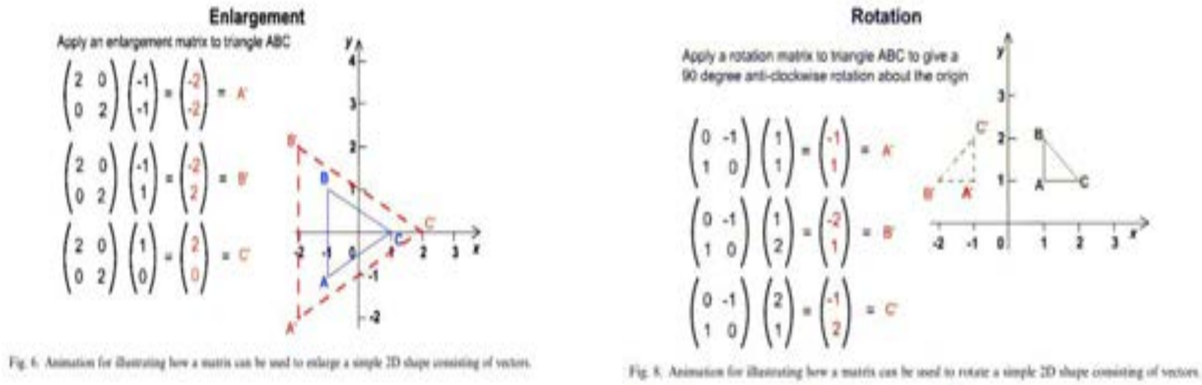

Figure 2. Tasks extracted from the article by Taylor and al. (2008)

In the same way, the tasks that were designed in the project were designed to allow for a flexibility between the different paradigms. Two of these tasks adapted from the one Taylor and al. are shown in Figure 3 (task 1 and 2).

> Task 1: Find a 2×2 matrix A such that when multiplied by the vectors joining the origin with the vertices of the triangle, it becomes a triangle which is in the second quadrant.
>
> Task 2: Find a 2×2 matrix A such that when multiplied by the vectors joining the origin with the vertices of the triangle, it becomes a segment which is in the second quadrant.

In these tasks, the vertices of the given triangle are random with integer coordinates and could be in different quadrants. For the tasks 1 and 2 (Figure 3) they appear in the first quadrant and the answer (triangle or segment) is awaited in the second quadrant. There are other versions of these tasks where a triangle is asked to be moved to a specific quadrant, with non-orthonormal axes or with non-congruent triangles. Due to space limitations, only the version in figure 3 (two initial tasks 1 and 2) will be analysed in the paper. Depending on the answer given by the student, the system provides specific feedback. If the answer is correct, the feedback indicates that the given matrix when multiplied by each of the vectors that are the vertices of the given triangle, results in a triangle in the second quadrant. If the answer is incorrect, the feedback aims to make the student reflect on the resulting triangle and determine the error (Figure 3).

For the tasks 1 and 2, before receiving feedback, we consider four possible strategies: E1) Find the matrix by trial and error, E2) Fix three points in the chosen quadrant and assemble a system of four variables and six equations, E3) Fix two vectors in the chosen quadrant and assemble a system of equations, and E4) Deliver a matrix from knowledge about matrices as operators.

The E1 strategy of trial and error seems to us to be quite expected given the characteristics of the platform. The work is mostly with the semiotic dimension and the AL1 paradigm.

To approach the task with strategy E2, students visually determine the approximate coordinates of the starting point and assemble a system of six equations and four variables, arbitrarily defining the resulting vectors (semiotic dimension of the work). We place mostly this strategy in the AL1 category as it is an algorithmic work (instrumental dimension of the work) on a system of equations without any discursive work involving linear knowledge. However, the system is likely to be inconsistent and its resolution can foster the work in AL2.

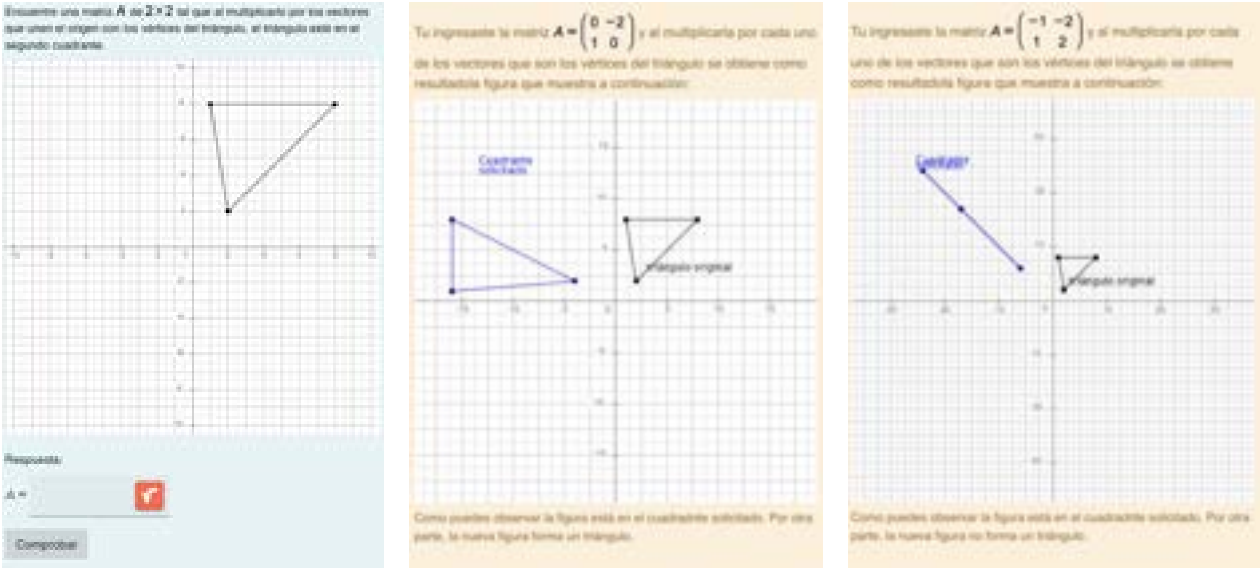

Figure 3. Task 1 constructed in the online assessment system, with the statement (left), with automatic feedback for a correct answer (centre), and a feedback for an incorrect answer (right).

If they choose strategy E3, that is, if they choose only two arbitrary vectors, they build a system of four variables and four equations and use the fact that the third vector is a linear combination of the two previous ones. Then the work has a greater discursive dimension with theoretical knowledge of linear algebra. The work, in this case, is mostly located in the AL2 paradigm as it works with vectors and the idea of linear dependence. The task 2 is specifically designed to foster this shift of paradigm.

To approach the task with the E4 strategy, students are expected to use transformations (central and axial symmetries, projections, rotations) with an interpretation of the situation in affine plane geometry and the production of a matrix. This approach can be mostly with visualisation (semiotic work): matrices can be seen as a game of changing the signs of some coordinates showing work in AL1. It can, however, play a role in understanding the linear algebra of quarter-turns and favour a shift of paradigm. The ambiguity in AL1 between affine geometry and the vector plane is unlikely to be resolved by students. However, the matrix work is specific to AL2, so the strategy E4 is classified between AL1 and AL2.

In all these strategies, the use of technology can be present, not only classical software such as Geogebra (among others) but also those with generative AI such as ChatGPT (among others). In the latter case, it is interesting to think about the potential dialogue that can be produced between the machines and the students. In our methodology the dialogues between students and ChatGPT are recorded, together with screen recording (unfortunately without audio due to a technical problem), in order to understand students' work and identify the paradigms at stake, especially if they choose the strategy E4.

OBSERVATION OF TWO GROUPS OF STUDENTS

Group 1

The session observed lasts 1 hour. The two students of group 1 initially work unsuccessfully on one task 1 and one task 2. They ask ChatGPT questions such as "how can you multiply a vector with a 2×2 matrix?" "How do you transform a vector into a matrix?" ChatGPT suggested various off-topic answers and essentially gave the formula for multiplying a 2×2 matrix by a vector in the plane.

At 30 min, the students return to a task 1. The proposed triangle is (ABC) where A(1,2); B(3,5) and C(6,4). Mouse movements are observed to drag point A to point (-1,2), then to drag B to point (-3,5). The students return to chatGPT to view the multiplication formula again. This is followed by a long moment on their paper (Figure 4).

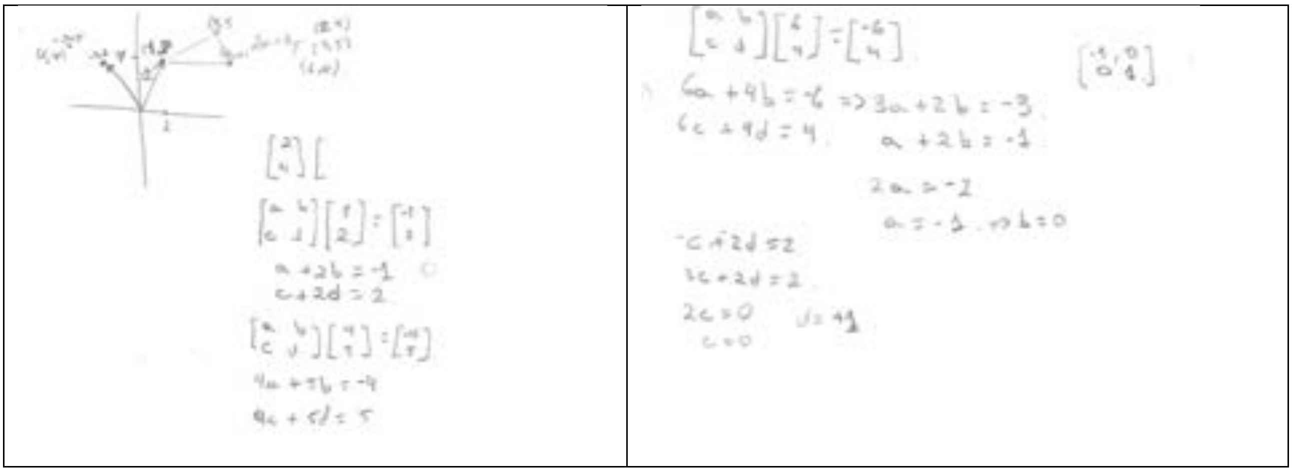

Figure 4. Extract of students' production of group 1 on one task 1.

The students write the 3 systems of 2 equations, using the E2 strategy, with the same variables *a*,*b*,*c*,*d* as those proposed in ChatGPT's answers. They determine *a* and *b* from the third system and then *c* and *d* using the equation $c+2d=2$ from the first system. This is a method of elimination/substitution that is functional here. This allows them to find good values for *a*,*b*,*c*,*d*. Since they are looking for the symmetry matrix with respect to the line $x=0$ without realising it, and since they have given each point A, B and C its correct image by this symmetry, the system is compatible and the imperfect solution they develop still allows them to find a good matrix. The error in the coordinates of B is transparent. At 39'15, they validate their task 1. At this point, they spent a long time looking at the feedback (2 minutes), whereas previously they had not read the feedback during the unsuccessful trials. When they tried one task 1 again, they immediately proposed the same matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which tends to show that they had developed decontextualised knowledge at this moment.

Students also try task 2. They no longer misread the coordinates by transferring them to their rough draft. There is work for about ten minutes which led to the identification of a generic matrix. They write "*to form general, to find triangles in the 2nd quadrant we have* $\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$*", underlining twice the matrix*", which marked the start of work in AL2. This does not allow them to reach a conclusion.

Finally, they essentially worked in the AL1 paradigm on the E2 strategy. They only started working after an unsuccessful attempt at each of the 2 tasks (first 16 minutes) and after questioning ChatGPT several times. The questions were formulated in technical terms and the ChatGPT responses were at the same level. However these responses helped students to convert matrix product into a system. Working in the AL1 paradigm on the first task enabled success, but only the Sem-Ins dimensions of work was used. Solving the system of 6 equations with 4 unknowns was not a problem insofar as the students constructed a compatible system. There was no need to discuss the system, which ultimately confirmed that the students were working in the AL1 paradigm. Working several time on the same task with the software nevertheless enabled the genericity of the matrix found to be identified but it doesn't permit to develop a discursive work for task 2. The paradigm shift required for task 2 (they had to move towards E3 or E4 strategies) remained too difficult for them, without any right answers for task 2.

Group 2

This group of 3 students quickly explored tasks 1 and 2 through trial and errors. For example, they enter 2 for matrix A in task 1. The system interprets this as 2Id and the feedback therefore focuses on the wrong quadrant. On task 2 they enter $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. The result is a segment, but in the same quadrant. This is followed by a question in Google "2×2 matrix to form triangle" and then they click on "how to develop a 2×2 matrix". After a long moment without any change on the screen, the students retry a question in Google (copying the statement) with an unsuccessful attempt to use ChatGPT. But this leads nowhere.

After a while (17'30), at the start of one task 1, the students enter the matrix $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, which gives a correct answer. So they move on to task 2 (18'30). They quickly enter the matrix$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which gives the right quadrant, but not a segment. It is clear that at this point they have understood that this matrix send the triangles to the right quadrant. We also see, at a moment, that the statement for task 1 is copied into ChatGPT. The answer suggests the same matrix and ChatGPT explains the change of quadrant. This should reinforce students' answers while adding a discursive dimension of their work.

Students do several series. For tasks 1, they always use the same matrix, which works. For tasks 2, on the other hand, they try for instance -2, interpreted as -2Id. This gives neither a segment nor the right quadrant. They go to Google with the question " through a matrix, how to find a segment?" but the answers are unconvincing. At one point there were 6'30 of work and the students tries the matrix $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$ which gives a segment, but not in the right quadrant. There was clearly a discursive dimension that took place during these 6'30, as evidenced by the written productions (Figure 5). In particular, we can see matrices with a row or column of 0s that give segments, as well as the question on the links between obtaining a segment and the determinant of the matrix.

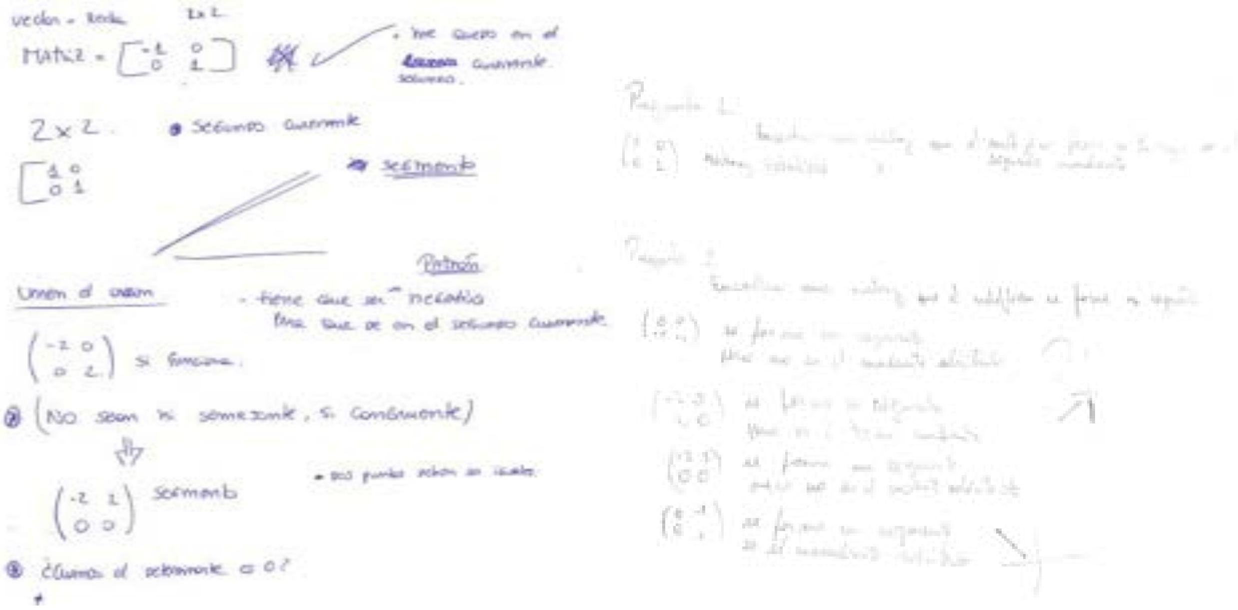

Figure 5. Extract of students' production of group 2 on one task 2.

This group tended to use the E4 strategy, even though it could also be seen some use of the E1 strategy, with trials that were perhaps not much discussed ("just to see"?). Students found two matrices that transform the triangles into the second quadrant. These matrices are supported by the response from ChatGPT, which gives the indication that the abscissa must change sign. Students also answered task 2 correctly with non-invertible matrices consisting of a row or column of zero and a more general question about the determinant. It seems that they had a discursive work, with a search for the genericity of matrices, which enabled them to work in the AL2 paradigm, quickly taking matrices to be an object in their own right which transforms the points of the plane (we can however note the confusion between a number and a 2×2 matrix, which was quickly regulated).

CONCLUSION

The paradigms make it possible to analyse the syllabuses and question their coherence. On the one hand, in the light of the historical genesis: is it suitable to introduce linear algebra using the AL2 paradigm when history shows the long genesis during the 19th century, initially based on work close to the AL1 paradigm? Secondly, the coherence between the paradigms present in the syllabus and what is expected in the examinations can be interrogated, with paradigm changes that are not made explicit. This contributes to our understanding of education systems.

Of course, and this is the continuation of the project, it is important to analyse the students' work on tasks and refine the analyses in terms of paradigms with the ETM theory. This contribution included an exploration of two pedagogical tasks designed to promote a flexibility and smooth transition between paradigms AL1 and AL2, highlighting the importance of task design in the process of teaching and learning linear algebra.

In parallel, the project has turned to the use of information and communication technologies, in particular online assessment platforms, to study the mathematical work of trainee teachers and the role of technologies. The triangle task, presented as an example, illustrates how students can approach problems using different strategies, highlighting the diversity of possible paradigmatic approaches together with the influence of ChatGPT. However, we show, in this study, the difficulty in working in AL2 paradigm, which seems an important step before working in AL3.

NOTES

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An Investigation of Features of Linear Algebra Peer Instruction Questions using Kelly's Repertory Grid Technique

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In this study we consider the views of a group of expert instructors on a bank of Peer Instruction questions used to promote active learning in an introductory linear algebra course. This bank was developed and refined over a ten-year period, and therefore can be considered to contain successful Peer Instruction tasks. Using Kelly's Repertory Grid technique, the experts identified 20 salient features of questions in this bank. These features were classified into two categories: those that involve the format of the question, and those that concern the mathematical thinking required of the student. Our findings will enable further work to determine which features of the questions bear most strongly on the quality of in-class discussion.

Keywords: teaching and learning of linear algebra, digital and other resources in university mathematics education, teacher expertise, task design.

INTRODUCTION

Peer Instruction (PI) is a well-established pedagogical technique which seeks to increase active learning in the classroom by means of voting. PI involves posing a question and allowing students to think and vote individually. Then, without the results being shown, the students are encouraged to discuss with each other and revote before the instructor explains the solution. Research has found that PI leads to improved conceptual reasoning and quantitative problem solving compared to traditional lecture-based teaching (Crouch & Mazur, 2001; Vickrey et al., 2015).

Effective implementation of PI involves access to appropriate software or hardware infrastructure for classroom voting, instructor training, and careful question design. In Physics Education research, Mazur and colleagues have advocated for the use of a style of multiple-choice questions which they named ConcepTests (Crouch et al., 2007). They suggest that these questions should have particular features, such as a focus on a single concept which the instructor believes is central to a common student difficulty. The questions should require students to think and not just compute an answer, and should include a plausible incorrect answer (related to a common misconception). ConcepTests have been widely used in Physics Education, and while some principles developed there may be transferable, there remains a lack of research into the characteristics of effective questions for PI in university mathematics.

In this paper, we report on features of questions identified by expert instructors in a PI question bank used in an introductory linear algebra course. This paper's primary contribution is a report on the Repertory Grid Technique, a method which is being used for the first time in university mathematics instruction with our study.

The PI question bank that we studied was developed for the course Introduction to Linear Algebra (ILA), a first-year mathematics course at a large, research-intensive university in Scotland, UK. It is a compulsory course for students on mathematics and computer science degree programmes, and an option for students on many other degree programmes. The cohort size has ranged from 300-900 students over the past decade. The course operates using a flipped classroom design, where students are required to read sections of the textbook in advance of the lectures, and most of the class time is spent on PI questions (for more details of the course design, see Docherty, 2023). A question bank of around 150 PI questions was developed by a team of lecturers involved with the course, and iteratively refined over the past decade based on the instructors' reflections on the success or otherwise of each question after class. However, the questions have never undergone a formal analysis and this is what we seek to do in this study.

The research questions guiding our study were therefore: (1) What features of mathematics Peer Instruction questions do experts identify? (2) How consistently can those features be recognised in questions?

LITERATURE REVIEW

Some previous mathematics education research has investigated the features of questions used for classroom voting in undergraduate mathematics, whether using Mazur's Peer Instruction approach or not. Particularly relevant for our study is the work of Cline et al. (2013), who developed and evaluated a collection of 311 multiple-choice questions for linear algebra. Their evaluation focused on the percentage of students voting for each answer option, "to identify the questions most likely to produce diverse votes and thus significant discussions" (Cline et al., 2013, p. 3). The authors go on to offer some comments about the features that these questions appear to have in common (e.g., that they go beyond requiring a specific calculation), however they note that in their experience, it is "difficult to anticipate which questions would engage the class in active discussion" (Cline et al., 2013, p. 12) since questions with apparently similar features can perform quite differently. For our study, we seek to develop a rigorous characterisation of the features of in-class voting questions *before* considering how the questions were answered by students, to give this claim a solid empirical grounding.

Our study is focused on features of questions which were noticed by a group of experienced university-level mathematics instructors. Drawing on the experience of expert practitioners has a rich history in mathematics education. For example, Lai, Weber and Mejia Ramos (2012) worked with a group of mathematicians to characterise the features of a good pedagogical proof. Our study makes use of the Repertory Grid Method (described in more detail in the next section), which has been used successfully in previous research to make explicit the tacit knowledge of experts. For example, Suto and Nadas (2009) asked examiners to identify features of mathematics and physics tasks which cause examination questions to be marked less accurately than others. Similarly, Holmes, He and Meadows (2017) interviewed experienced teachers to identify dimensions on which problem-solving tasks varied.

THEORETICAL FRAMEWORK

In order to elicit the tacit knowledge in this study we chose to use the Repertory Grid Technique, which was developed as a tool in Personal Construct Theory (Kelly, 1955). This theory asserts that each of us makes sense of events around us by creating an implicit framework of personal constructs. Kelly describes constructs as follows:

> They are reference axes, upon which one may project events in an effort to make some sense out of what is going on. ... A construct is the basic contrast between two groups. When it is imposed it serves both to distinguish between its elements and to group them. (Kelly, 2017, p. 12).

Constructs can usually be thought of as continuous spectra with two extremes (called *poles*). When thinking about a mathematics course we might have constructs such as: introductory–advanced; large group–small group, et cetera. Kelly (1955) posited that people's decisions and actions arise from a complex system of interconnected constructs. Kelly's Repertory Grid technique is a way of exploring the structure of a person's implicit personal construct framework, and of making tacit knowledge explicit.

The Repertory Grid Technique

The Repertory Grid Technique can be used in an interview setting to gain information about a person's tacit constructs related to a particular subject, and, in particular, to make these constructs visible and explicit. Usually participants are asked to consider sets of objects (called *elements*) taken three at a time, and to identify ways in which two are similar and the other is different. In our study, the objects were the ILA question bank items, and the participants worked in small groups during the interview. Rozenszajn, Zer Kavod and Machluf (2021) detail a protocol for conducting such interviews. They advise beginning with an introductory section where an overview of the method is given along with assurances that there are no 'right' or 'wrong' answers. The topic of the study should be stated in the form of a question that allows the participants' experience and related beliefs to be elicited. Our motivating question was: *What makes a "good" Peer Instruction question?*

During the interview, the participants are asked to select three elements at a time and to look for features that are common to two of the selected elements but not to the third. The participants are asked to write down names which describe the extremes of these features; these names are called the *poles* of the construct involved. This process is repeated as often as necessary to elicit as many different constructs as possible. The constructs and the poles should emerge from the participants' views and beliefs, and are not evaluated or critiqued by the interviewer during the interview. By the end of this stage of the interview, the participants should have produced a list of constructs, each of which is described by two poles. The final stage of the interview involves the participants rating the elements on each of the construct scales. This can be done by asking participants to rate the elements from 1 to 5 on each construct scale, where the award of a score of 1 means that the participant strongly agrees that the element represents the left pole of the construct and a score of 5 means that they strongly agree that it represents the right pole. Thus, at the end of the interview the participants will have created a set of constructs (along with its poles) and have rated a subset of the elements on scales derived from these constructs.

The Repertory Grid technique is particularly useful when exploring the opinions of experts on complex topics. Unlike other methods (such as using surveys), this method does not force participants to use terms chosen by others but allows them to create their own concepts and language to describe them. The method has been used in various fields but has only recently been employed in mathematics education research. One such study, for example, concerned primary mathematics teachers' beliefs and conceptions on argumentation (Klöpping & Kuzle, 2019). The technique has been employed to study features of high-stakes national examinations in the UK; Suto and Nadas (2009) interviewed two senior examiners to elicit features of questions that affect the accuracy of marking. Particularly relevant for our study is the work of Holmes et al. (2017), who employed the Repertory Grid technique with a group of five experienced teachers analysing exam questions. The teachers worked together through a structured process, to develop a shared list of features. Our study takes a similar approach which is detailed below.

METHOD

Description of the participants

We invited all 16 lecturers who had taught on the ILA course in the past 10 years, as well as other colleagues who either had experience of teaching linear algebra or of using the electronic voting system (TopHat) in other first-year mathematics classes. Five lecturers took part in the workshop. Table 1 contains information on the level of experience of the participants as they described it.

Experience	**E1**	**E2**	**E3**	**E4**	**E5**
How many years experience do you have of teaching mathematics at university level?	15+	5	6	15	17
How many times have you taught an introductory Linear Algebra course?	5	1	2	3	1
How many times have you taught a course using the method of Peer Instruction?	3	1	4	0	3
How many times have you taught a course using TopHat?	4	1	4	0	3
How many times have you taught a course using the ILA TopHat question bank?	3	1	1	0	0

Table 1: Prior experience of the five expert participants

The Elements

We identified 132 multiple-choice questions from the ILA question bank. These were shuffled to form 44 triples, with each triple printed on an A3 sheet. An example of one of these is shown in Figure 1.

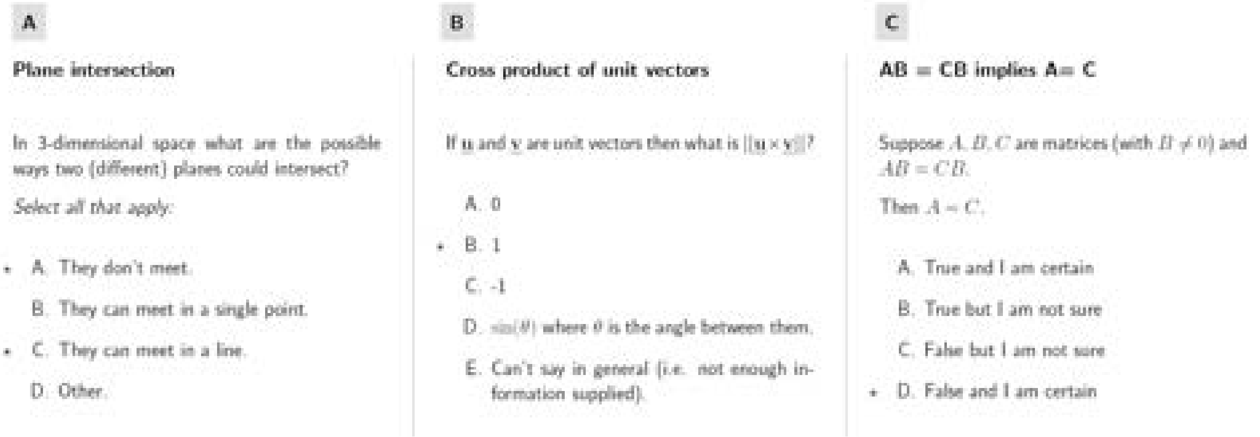

Figure 1: An example of one of the triples of questions as presented to participants (correct answers are indicated by stars).

The Workshop Procedure

The workshop began with an introduction to the Repertory Grid technique as well as the statement of the motivating question: *What makes a "good" Peer Instruction question?* It then proceeded in three phases. In the first phase of the workshop (90 minutes), the five lecturers were allocated to a group of two (E1 and E2) and a group of three. The set of 44 triples were divided between the two groups. The groups reviewed triples of questions and generated a list of features/constructs without the researchers' input. Participants were asked to avoid generating constructs related to specific mathematical topics or to difficulty but were otherwise free to identify features. To begin with, the groups took 5-10 minutes to discuss each triple in depth and to identify similarities and differences. Halfway through the first phase, the group of two had generated 7 feature dimensions, while the group of three had generated 15. Both groups were invited to review the remaining triples with less in-depth discussion, to see if any further features could be identified in the remainder of the first phase. The participants then had a short break. In the second phase of the workshop (70 minutes), the whole group discussed the separate lists of constructs from the first phase and consolidated these into a single list of agreed constructs along with the associated poles. This process was facilitated by the researchers, but care was taken to avoid influencing the outcome of the discussion. In the third and final phase, the participants were asked to complete an online survey to rate a subset of the ILA questions against the agreed list of constructs. We had intended to include this phase in the workshop but ran out of time, so we asked the participants to complete the survey later. The survey included a selection of 10 questions from the question bank, and each question was scored on a scale of 1 to 5 for each construct. Three of the expert participants and all three researchers provided ratings.

RESULTS

The group of experts formulated 20 constructs in all; these give some insight into what the experts noticed about the ILA questions. We have grouped the constructs into two categories, based on whether the focus is on aspects of the *Format* of the question (shown in Table 2) or on the *Mathematical Thinking* required to answer the question (shown in Table 3). The third author suggested this grouping after all the scoring was completed. All three authors independently classified the constructs; we had complete agreement in all but three cases (constructs 10, 11, 17) which we resolved through discussion.

Construct	Pole-Low	Pole-High	Mean Rating	Mean SD of Rating
14	Options look very similar	Options look different	2.33	1.01
7	Question/options dominated by numbers	Question/options dominated by text	2.92	0.81
12	Many options presented	Few options presented	2.95	0.71
8	Question is short	Question is long	1.75	0.69
18	Question asks students to select a single option	Question asks students to identify all correct options	1.90	0.24
13	“it depends” is present as an option	“it depends” is not present as an option	4.72	0.18
3	Options include measure of confidence	Options are answers only	5.00	0.00
19	Some options are correct but not optimally so[1]	Options are either correct or incorrect	5.00	0.00

Table 2: The constructs elicited in the workshop that focus on the format of items

The constructs are described by their poles in Tables 2 and 3, along with summary statistics from the rating of 10 questions on these constructs (by three experts and three researchers). The mean rating shows whether the 10 questions tended to be nearer the low (1) or high (5) end of the scale. The mean standard deviation shows the extent to which the constructs could be scored consistently: for each of the 10 questions we computed the standard deviation of the ratings, then took the mean of those standard deviations (in line with the approach taken by Holmes et al., 2017). The constructs with very low mean standard deviations are the ones that the raters found easiest to agree on; for example, construct 3 is concerned with whether the answer options include a measure of confidence (e.g., “yes, and I am sure”), and the Mean SD of 0 reflects the fact that raters had perfect agreement on this construct. The judgements about

[1] This construct arose from items where more than one answer was correct but one was more complete than the others.

constructs at the beginning of the tables require a greater degree of subjectivity; for example, whether an answer is easily verifiable or not (Table 3, construct 11), where raters gave a wide range of scores to most questions.

Construct	Pole-Low	Pole-High	Mean Rating	Mean SD of Rating
16	Tempting but wrong answer as an option, e.g. based on misconception	No obvious misconceptions underlying the wrong options	3.02	1.21
11	Answer is easily verified once known	Answer cannot be easily verified	2.57	1.18
1	Multiple ways to approach	Only one way to approach	3.80	1.17
20	Requires generating an argument or answer	Requires checking or verifying something that is given	3.45	1.13
15	Question has a natural visual interpretation	No natural visual interpretation	3.03	1.06
2	Requires understanding and connecting multiple concepts	No need to make connections between concepts	3.47	1.06
5	Solution requires only memory	Solution requires processing	3.03	1.00
6	Solution involves abstract reasoning	Solution involves computation	2.60	0.94
10	Solution comes from working out a single answer	Solution comes from checking the provided options	3.35	0.82
4	Requires multiple steps of reasoning	Single step of reasoning (at most)	4.03	0.80
17	Question can be solved without using all the given information	Solution requires all information in the question	3.95	0.78
9	Requires analysing/critiquing a given argument	Does not require analysis/critique of a given argument	4.73	0.61

Table 3: The constructs elicited in the workshop that focus on mathematical thinking

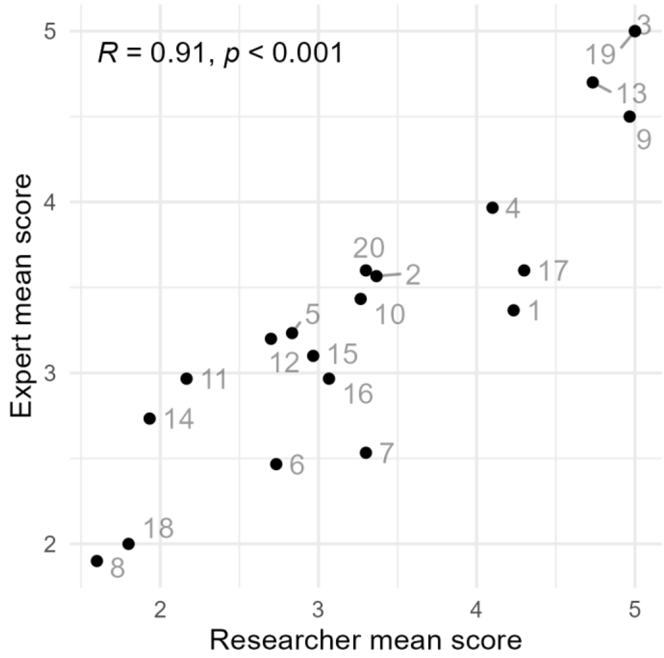

Figure 2: Mean scores on the 20 constructs assigned by Experts and Researchers

The rating statistics shown in Tables 2 and 3 are based on the combined set of ratings from the three experts and the three researchers. We were interested in the level of agreement between these two subgroups, since our plans for future research are based on producing scores for all the ILA questions – a task that would be too much to ask of our expert participants. For each construct, we computed the mean score given by the experts and by the researchers. Figure 2 shows that there was a high level of agreement, with Pearson's product-moment correlation of 0.91 (95% CI [0.78, 0.96]).

DISCUSSION

We used Kelly's Repertory Grid technique to elicit 20 constructs that experts use to distinguish between in-class voting mathematics tasks. We found that our scoring of a sample of 10 tasks using these constructs agreed closely with the scores assigned by experts, giving confidence that we have developed a shared understanding of their meaning. The resulting set of constructs includes many that are essentially binary (e.g., 13: "it depends" is/is not present), although this is also true of many of the constructs from Holmes et al. (2017). We found that the constructs could be divided into two categories, based on whether their focus is on the *Mathematical Thinking* required to answer the question, or on its *Format*. We noticed that the constructs with the highest mean standard deviation of their ratings were in the Mathematical Thinking category. This makes sense as many of the constructs in the Format category require less interpretation (such as whether students are asked to select a single correct option or not).

The constructs developed by our group of experts include features of ConcepTests identified by Crouch et al. (2007). In particular, many of the constructs in the Mathematical Thinking category relate to the importance of engaging students in reasoning and not just computation (e.g., Construct 6), and we note that Construct 16 concerns the inclusion of an answer option targeting a common misconception as recommended by Crouch et al. (2007). However, our list of constructs highlights further features, such as having many different solution methods (Construct 1), multiple steps in the solution method (Construct 4) and multiple underlying concepts (Construct 2).

We think this method is potentially useful for other research on undergraduate mathematics education that seeks to draw on expert perspectives. We acknowledge that a limitation of the method is the way that the resulting constructs depend on the set of participants and the features they happen to attend to during the time available. We sought to mitigate this in our study by inviting participants with a broad range of experience, and by allowing sufficient time for the whole-group discussion in the second phase to consolidate ideas across the groups. While we had planned to complete all three phases during a 3-hour workshop, we adjusted our plans to allow the discussion to continue by making the third phase into a follow-up survey. This modification is another limitation of our study and we would recommend that researchers planning to use this method be aware that the discussion may take longer than anticipated. Another adaptation of the method that may be helpful in future research is to return to a set of example questions during the second phase, asking participants how those examples would be rated on each of the proposed scales to make the distinctions more concrete. We found this often happened informally, with participants referring to examples of questions to demonstrate their points, but this could have been facilitated by having a core set of example questions that all participants could see.

During the workshop, we observed a noticeable difference in the conversations in the two groups (although these were resolved during the whole-group discussion phase). By chance, the group of two were both researchers in abstract algebra and their discussion tended to focus on the mathematical content, while the group of three (two teaching focused-lecturers and an applied mathematician) more often commented on the surface features of the question presentation or how students might approach them. In each case, the activity seemed to prompt thoughtful discussion between participants, and we are intrigued by the potential for activities like this to stimulate exchange of expertise from more senior to more junior colleagues.

In future work, we plan to rate all of the ILA questions using this set of constructs, and compare the scores with student voting data for each of the questions. We hope to identify which of the task features are likely to lead to fruitful discussions in classes, and thus provide guidance to task designers.

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Students’ understanding of the scalar product at the entry to university – Comparison of desirable and actual associations

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The scalar product plays an important role in linear algebra courses at university. For being able to build upon their prior knowledge, it is important to explore students’ understanding of the concept at the entry to such courses. For this, we first describe – based on the literature – associations concerning the scalar product that contribute to an understanding of the concept, and which might therefore be desirable to acquire in a first pre-university course on elementary vector arithmetic, e.g., at upper secondary school. Thereafter, we present a study exploring the associations beginning undergraduates enrolled in a mathematics teacher program had. This study shows that many of them still lacked important associations that contribute to an understanding of the scalar product. Therefore, these should not be taken for granted at university.

Keywords: scalar product, teaching and learning of linear algebra, teacher education, concept image.

INTRODUCTION AND EMBEDDING OF THE RESEARCH

The scalar product (or dot product) has a high relevance for several groups of university students, especially those enrolled in STEM-subjects. But it also plays an important role for future teachers, as basic vector algebra is one of the major topics mathematics teachers are required to teach at upper secondary level in many countries – for example to prepare students for STEM study programs. For being able to build upon students’ previous knowledge in linear algebra courses at university adequately, it is important to investigate which understanding of the concept beginning undergraduates have.

The literature is still rather sparse on students’ understanding of the scalar product, despite its relevance. An early study was conducted by Knight (1995). He developed a test – *the vector knowledge test* – on students’ skills related to vectors and the operations addition, scalar product, and vector product. He then administered this test to 286 beginning physics students at a university in the US. Of the 213 students who had studied vectors at school before, 53 claimed to be able to evaluate scalar products. However, only 40% of these were able to do so algebraically, and even only 12% of them were able to do so geometrically using angles.

Barniol and Zavala (2014) later developed a multiple-choice test that did not only focus on skills but also on understanding vectors and their operations, named TUV (*test of understanding vectors*). Regarding the scalar product, the TUV did not only require calculations of such, but also contained an item asking students to describe the result of the scalar product of two vectors geometrically. Barniol and Zavala administered the TUV to 423 students at a Mexican university who had finished a calculus-based physics course. Most of these knew that they could calculate the scalar product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ with the formula $ABcos(\theta)$ (78%). However, much fewer could interpret the result geometrically as the product of the magnitudes of the first vector and the orthogonal projection of the second vector onto the first (33%), which is important for being able to interpret the scalar product in contexts such as physics. Instead, 27% thought that the scalar product of $\boldsymbol{A}$ and $\boldsymbol{B}$ gives the magnitude of a *vector between* $\boldsymbol{A}$ *and* $\boldsymbol{B}$, and 20% even considered it as such a vector itself. This suggests that even if students can determine scalar products, they are not necessarily able to give these a meaning using orthogonal projections. Similar findings were also obtained by Rakkapao et al. (2016). Zavala & Barniol (2013) furthermore discovered that many first-year physics students cannot interpret the scalar product in physical contexts.

In addition, Craig & Cloete (2015) found, by administering a 31-item multiple choice test on vectors with 8 items involving the scalar product to second-year engineering students at a South African university, that students also have problems to apply the concept in geometric tasks. Among the test items, the ones requiring the calculation of angles or orthogonal projections turned out to be the most difficult items of the test.

Overall, the studies above with physics or engineering students indicate that beginning undergraduates are often able to calculate scalar products, but have problems to interpret values of the scalar product in extra-mathematical or geometric contexts.

This paper now extends the abovementioned research on students' understanding of the scalar product in two ways:

1. It theoretically describes associations related to the scalar product that contribute to an understanding of the concept.
2. It presents an empirical study investigating beginning undergraduates' actual understanding of the scalar product for a new student group for whom the concept is relevant: prospective mathematics teachers for upper secondary level.

Point 2 is also our major research question: Which understanding of the scalar product do beginning undergraduates (enrolled in a mathematics teacher program) have?

THEORETICAL FRAMEWORK

Even if mathematical concepts at university are usually introduced via a precise formal concept definition, a student's understanding of a concept does rather emerge from her/his experiences with it. Tall and Vinner (1981) introduced the construct of *concept image* for this, which describes "the total cognitive structure that is associated with the concept, which includes all the mental pictures, associated properties and processes" (p. 152). It is built up over years based on experiences with the concept.

Selden and Selden (2008) highlighted several components that are part of one's concept image: examples, non-examples, facts, properties, relationships, and visualizations. Furthermore, it may contain a personal reconstruction of the definition. A special kind of association that has been highlighted by the German mathematics education community is a so-called *Grundvorstellung* (GV) – sometimes also translated as basic mental model or basic idea. Greefrath et al. (2016) defined a *Grundvorstellung* of a concept as “a conceptual interpretation that gives it meaning” (p. 101). The following features of *Grundvorstellungen* (GVs) are described in the literature: they constitute meaning to a concept by establishing connections to familiar knowledge/experiences, by generating a mental presentation, and by linking it to real-life situations (Vom Hofe & Blum, 2016). Hence, GVs can be considered as special associations in a students’ concept image that fulfill these features. For more advanced concepts, however, links to “real-life situations” might not always exist. In this case, GVs should especially link to ways of using the concept that illustrate its inner-mathematical significance.

The usage of the construct *Grundvorstellungen* in mathematics education research is of a dual nature. On the one hand, it is used as a prescriptive notion to describe associations that should be part of a valid concept image. But it is also used as a descriptive notion for describing adequate associations that individuals use for making sense of a mathematical concept (Greefrath et al, p. 102).

Frohn (2020) has proposed four important *Grundvorstellungen* of the scalar product that upper secondary students should acquire, and which might therefore be desirable for beginning undergraduates (translated into English by the authors):

1. *Generalized product:* The scalar product is an operation that fulfills certain algebraic properties like a product, such as the commutative or the distributive law.
2. *Orthogonality indicator:* The scalar product is a number that shows whether vectors are orthogonal or not.
3. *Angle indicator:* The scalar product indicates the size of the angle between two vectors. Especially the sign shows whether this angle is acute, right, or obtuse.
4. *Orthogonal projection*: The scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ gives the (signed) product of the magnitudes of $\boldsymbol{a}_{\parallel}$ and $\boldsymbol{b}$ with $\boldsymbol{a}_{\parallel}$ being the orthogonal projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

The first *Grundvorstellung* (GV) is essential for grasping the algebraic nature of the concept, the second and third are important for grasping its main meaning in geometry, and the GV *orthogonal projection* is significant for contextual interpretations of the scalar product in physics, for instance, in the equation $W = \boldsymbol{F} \cdot \boldsymbol{s}$. Although these four GVs are different in their nature (1. describes what the scalar product is while 2.-4. are interpretations of its numerical value), all of them can help to constitute a meaning to the concept.

For an appreciation of the scalar product, students should furthermore associate “applications” of the concept that illustrate its usefulness (Senate Administration for Education Berlin, 2014). These include its usage in extra-mathematical contexts such as physics, but also for solving geometric problems such as determining normal vectors and distances, or for proving statements from elementary geometry.

In the following empirical study, we explored the concept images of the scalar product beginning undergraduates enrolled in a program for upper secondary mathematics teachers had. We especially investigated whether their concept images contained the associations just described, which are of high relevance for them because they should convey these themselves to pupils later.

METHODS OF THE STUDY

Participants and data collection

The study took place at the beginning of a first-semester linear algebra course for future teachers at a large university in Germany. Its participants will later teach mathematics and a second subject chosen freely up to the end of secondary level. They were surveyed about their concept image of the scalar product in their first tutorials.

In the first part of the session, the first author, who will be tutor of one tutorial class later, introduced the construct of *concept image*. To illustrate possible constituents of a concept image, he collected with the students different aspects that could be part of a concept image for the derivative concept on the board, e.g., its geometric interpretation as tangent slope, its interpretation as a rate of change, or the differentiation rules. He wrote these into boxes surrounding the concept definition. Meanwhile, he emphasized the categories that are typically part of one's concept image, such as visualizations, properties involving the concept, or applications. The created visualization was similar to a concept map (Novak, 2010). However, its aim was not to visualize links between different concepts, but to collect propositions that involve the concept of interest.

In the second part of the session, the students were asked to create such a visualization of their concept image of the scalar product by themselves. An example of such a visualization can be seen in Figure 1. Furthermore, we asked them to state a definition of the concept. Finally, we gave them a short survey with some biographical questions. The students had 45 minutes to create the visualization individually, and to answer the survey. 99 submitted both, and agreed that their data may be used for this study.

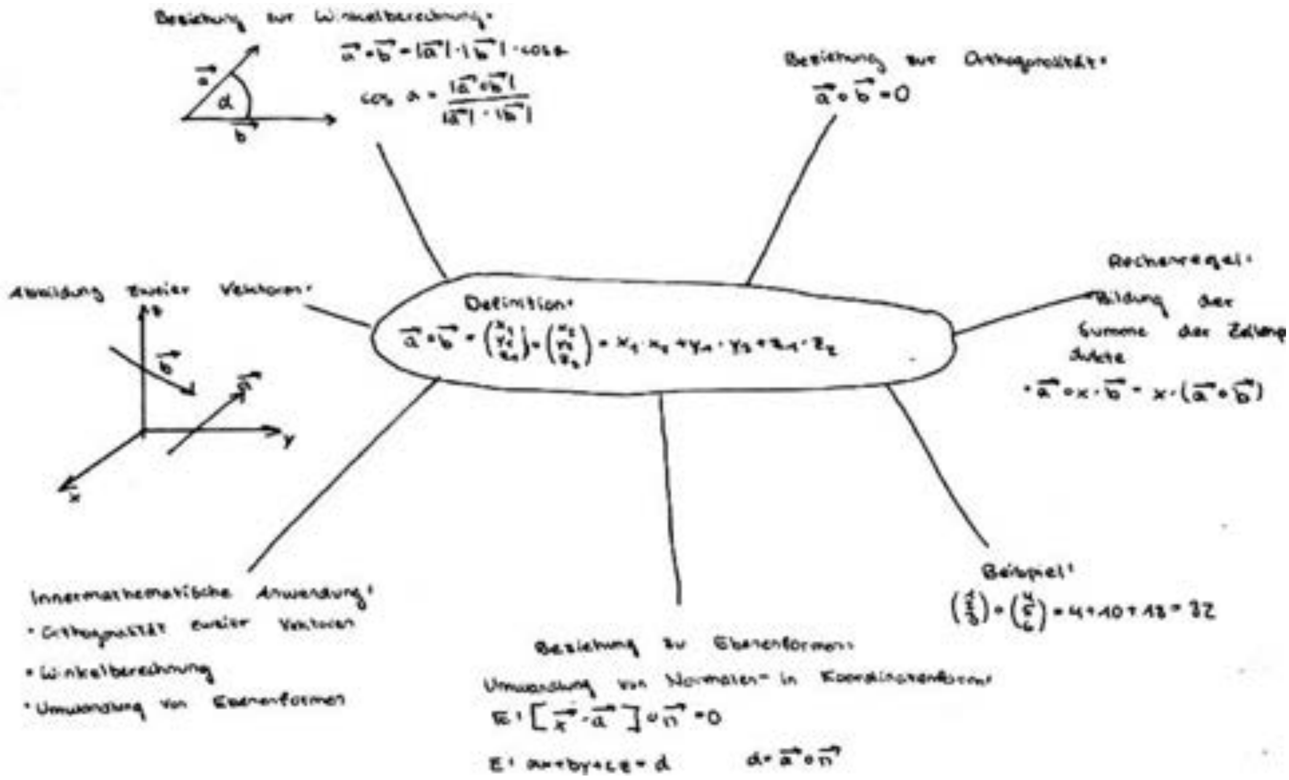


Figure 1: Example of a visualization of a student's concept image of the scalar product

Data analysis

Analysis of the stated concept definitions: In a first step, the first author classified whether the students stated 1) an arithmetic definition as a sum of products, 2) the geometric definition via $|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos(\gamma)$, 3) an algebraic definition as a bilinear form, or 4) another definition. In a second step, he categorized the responses from the category "other definition" inductively on the basis of the data. Then the second author coded the students' stated definitions with the category system developed. Finally, both authors compared their codes and resolved disagreements in a discussion.

Analysis of the visualizations of the concept images: We also aimed at categorizing students' visualizations of their concept images – according to the aspects contained in them. As this was more complex, we proceeded in multiple steps.

1. Before coding, we developed "superordinate categories" based on our theoretical framework describing associations related to the scalar product that contribute to an understanding of the concept. These comprised the four Grundvorstellungen *generalized product, orthogonality indicator, angle indicator, orthogonal projection,* the arithmetic formula for its calculation, and inner- and extra-mathematical applications. In addition, we chose the additional superordinate category "Relationships to other concepts", as such are also an important component of one's concept image according to Selden & Selden (2008).
2. The first author then started to code the data with these superordinate categories. However, since these were rather general, he refined them during the coding with subcategories representing what the students actually wrote. In the case of applications, for instance, these subcategories were specific applications/purposes of the scalar product that the students mentioned explicitly.
3. Afterwards, the second author coded 20 cases with this refined system of subcategories to check whether it was appropriate. Then the two authors discussed "borderline cases" among these, and finalized the coding instructions.
4. Finally, both authors coded the whole data independently using these coding instructions, compared their codes, and resolved disagreements in a discussion.

The method of analysis aimed especially at finding out which associations of the scalar product many students have (or not), rather than identifying individual holistic concept images.

RESULTS OF THE STUDY

Of the 99 survey participants, 25 stated that they had heard about the scalar product in a previous course at university. Since we wanted to find out about beginning undergraduates' understanding of the concept, we will exclude these from now on. Of the remaining 74 students, 68 stated that they had learned about the scalar product at school.

Results regarding the stated concept definitions

The concept definitions the students stated can be seen in Figure 2.

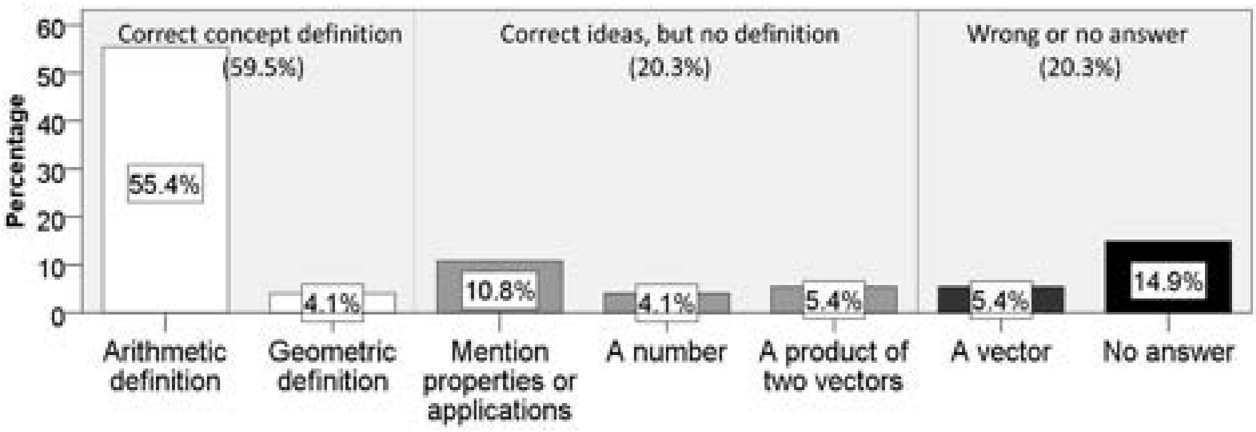


Figure 2: The definitions stated by our beginning undergraduates ($N = 74$)

It shows that the majority stated the arithmetic definition as a sum of products, while only few stated the geometric definition via $|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos(\gamma)$. 10.8% did not state an explicit definition, but mentioned that the scalar product is a number that is used to find out about certain geometric properties of vectors, e.g., the angle between them, or about their position in space. 9.5% just stated that the scalar product yields a real number or that it is an operation or product of two vectors. Only 5.4% gave a really wrong answer by defining the scalar product as a vector, e.g., with components $x_i \cdot y_i$ in the columns. Only one participant mixed it with the vector product in this task.

Overall, most of our participants associated a correct definition of the scalar product. However, only few mentioned the geometric definition that relates closer to three of the four Grundvorstellungen (*orthogonality indicator*, *angle indicator*, *orthogonal projection*) and to its usage in the natural sciences such as physics.

Results regarding the visualizations of the students' concept images

Table 1 shows the categories and all subcategories that occurred in our participants' visualizations of their concept images, and illustrates these with examples. The enumerations in the category descriptions point out the different subcategories found in the data.

Category	Description and subcategories	Examples
Arithmetic formula	The student mentioned the arithmetic formula of the concept 1) as a definition or 2) in an example with numbers.	"$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_3 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$"
Generalized product	The student mentioned 1) at least one of the algebraic laws of the scalar product, 2) that it is a product or a multiplication of vectors, or 3) that it yields a number.	Rechenregeln $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ $\lambda \cdot (\vec{a} \cdot \vec{b}) = (\lambda \cdot \vec{a}) \cdot \vec{b}$ "Operation of two vectors"

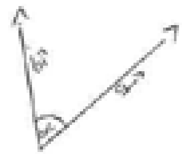
Orthogonality indicator	The student mentioned 1) the orthogonality criterion involving the scalar product (maybe within an example) or 2) that the scalar product is used to check for orthogonality.	“If two vectors $\vec{a} \cdot \vec{b}$ have a right angle, then their scalar product $=$ 0.” “Inner-mathematical application: Check whether two vectors are orthogonal.”
Angle indicator	The student mentioned 1) the geometric definition of the scalar product with the cosine, 2) the angle formular involving the scalar product, 3) that the scalar product is used to determine angles, or 4) drew a picture of two vectors and the angle in between or referred to this angle verbally.	“$\vec{a} \cdot \vec{b} = \lvert\vec{a}\rvert \cdot \lvert\vec{b}\rvert \cdot \cos(\gamma)$” „$\cos(\gamma) = \frac{\vec{a} \cdot \vec{b}}{\lvert\vec{a}\rvert \cdot \lvert\vec{b}\rvert}$“ „Determination of the angle of two vectors” 
Orthogonal projection	The student mentioned verbally or illustrated with a picture an orthogonal projection.	
Other inner-mathematical application	The student mentioned that the scalar product is used for 1) calculations involving normal vectors (including distances), 2) calculating areas or volumes, 3) proving geometric statements, 4) or mentioned an unspecific phrase indicating an inner-mathematical application	“Transformation of the normal form into the coordinate form: $E: [\vec{x} - \vec{a}] \circ \vec{n} = 0$ $E: ax + by + cz = d$” „$A_\Delta = \frac{1}{2}\sqrt{\vec{a}^2 \circ \vec{b}^2 - (\vec{a} \circ \vec{b})^2}$ “ „Inner-mathematical appl.: positional relationships”
Extra-mathematical application except angles	The student mentioned an extra-mathematical application *aside from angles* or a context in which the scalar product can be used.	“W=$\vec{F} \cdot \vec{s}$” “Vectors as flight routes” “Engineering”
Relationship to other concepts	The student mentioned a correct relationship to the absolute value of a vector or the vector product.	“$\vec{a} \cdot \vec{a} = \lvert\vec{a}\rvert^2$” “$\vec{a} \cdot (\vec{a} \times \vec{b})$=0”

Table 1: Categories and subcategories found in the students’ visualizations of their concept images

Figure 3 then shows the proportion of students who referred to the different (superordinate) categories from Table 1 in the visualizations of their concept images.

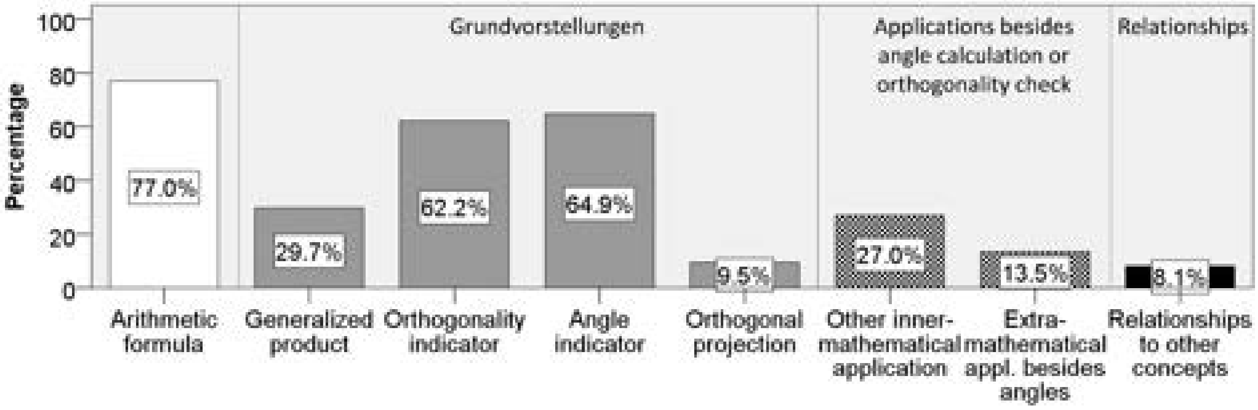


Figure 3: Aspects that were present in our beginning students' concept images ($N = 74$)

It first shows that 77% of our participants knew the arithmetic formula for the calculation of the scalar product. Furthermore, it indicates, on the positive, that about two thirds of them possessed the Grundvorstellungen (GVs) *orthogonality indicator* and *angle indicator*, because the students either stated corresponding formulas or that the scalar product can be used to check for orthogonality or to determine angles.

On the other hand, much fewer students demonstrated ideas that are related to the GV *generalized product* in the visualizations of their concept images. Of these, only 16.2% mentioned at least one of the algebraic laws the scalar product fulfills, which are essential for this GV. The others simply stated that the scalar product is a product/ multiplication of two vectors or that it yields a number. Furthermore, only a few students showed in their visualizations facets of the GV *orthogonal projection* – mostly with a vague picture indicating an orthogonal projection as in Table 1.

Finally, our data suggest that our participants did not have many applications besides angle calculation in mind. Concerning the extra-mathematical applications, only one student referred to a specific one: its usage to determine work via $W = \boldsymbol{F} \cdot \boldsymbol{s}$. The others stated only unspecific phrases or keywords such as "engineering" (see Table 1). But also concerning inner-mathematical applications, our data indicate – via an analysis of the corresponding subcategories – that our participants' associations were often limited to angle calculations and a usage of the scalar product to determine normal vectors and distances. In particular, only one student mentioned that the scalar product can be used to prove properties in elementary geometry.

SUMMARY AND DISCUSSION

Overall, we provided theoretically grounded associations that contribute to an understanding of the scalar product, and which might be desirable for students to have at the beginning of a linear algebra course at university (after a first course in elementary vector arithmetic). Then we compared these with associations mathematics teacher students brought in when entering university.

Our analysis first showed that the majority of our participants had several relevant associations concerning the scalar product in mind (see Figure 3): the arithmetic concept definition, the GV *orthogonality indicator*, and the GV *angle indicator* (including mere statements that it is used for checking orthogonality or determining angles). On the other hand, only a minority put forward associations related to the GV *generalized product* or to the GV *orthogonal projection*. Hence, these two GVs of the scalar product cannot be taken for granted at the beginning of a linear algebra course at university. Of course, the students' visualizations can only provide snapshots of students' understanding of the scalar product. For deeper results, in-depth questions or interviews focusing on the individual GVs would have to be designed.

Our results coincide with the cited literature in several respects. As in Barniol & Zavala (2014) or Craig & Cloete (2015), many of our beginning undergraduates knew how to calculate scalar products, but lacked of some associations that are relevant for an understanding of the concept and its usage in contexts, e.g., its geometric interpretation relying on orthogonal projections. However, our data also indicate a novel finding that is especially relevant for linear algebra courses at university: Beginning undergraduates often do not bring in associations about the arithmetic laws the standard scalar product fulfils, which are important for connecting their prior knowledge about the standard scalar product with the abstract concept defined as a bilinear form that is usually taught in linear algebra courses at university. Hence, the GVs *generalized product* and *orthogonal projection* should be emphasized in such courses.

A possibility to foster the GV of *orthogonal projection* could be to introduce the scalar product with a physical example: pulling an object a certain distance into a certain direction. The work required equals the product of the distance and the magnitude of the force acting in direction of the distance, which is just the orthogonal projection of the force vector onto the distance vector. This idea then leads to the geometric definition of the scalar product, but emphasizes first the idea of orthogonal projection. A possibility to foster the GV *generalized product* could be to cover also at university the standard scalar product first, prove its algebraic properties, and compare similarities and differences to the ordinary product of real numbers – before introducing the scalar product as a bilinear form. Donevska (2015) also proposed a dynamical geometry environment to explore the algebraic properties of the (standard) scalar product geometrically, which might help to connect the geometric definition and orthogonal projections to the algebraic properties. Besides this, teacher students should be also exposed to problems that illustrate the utility of the scalar product in geometry, e.g. its usefulness to solve elementary geometric problems or to prove geometric theorems, because they later have to be able to convey such "geometric applications" to pupils themselves (Senate Administration for Education Berlin, 2014).

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The skateboard drawing of a linear system of equations.

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In this communication, we report advances of an innovative project in which we investigate the construction of the linear combination concept in relation with the linear system of equations. Our observations indicate that this relationship serves as the foundation for constructing several other concepts in the course, such as Span, Linear Independence, and Basis. We designed activities based on APOS theory to promote the construction of that relation. Two groups of students were interviewed, in one, students were enrolled in a linear algebra course using conventional teaching, the other group worked on a model and activities designed with a genetic decomposition. Results contribute to literature by focusing on the construction of the linear combination representation of systems of equations.

Keywords: teaching and learning of specific topics in university mathematics, teaching and learning of linear and abstract algebra, APOS theory, linear combinations, magic carpet problem.

INTRODUCTION

Research on didactic strategies for the teaching of linear algebra began in the late 1980s and early 1990s. They were prompted by findings about students' need to develop a richer understanding of Linear Algebra. Researchers such as Harel (1989) and Hillel and Sierpinska (1993) expressed their concerns regarding students encountering difficulties to understand abstract linear algebra concepts. Stewart et al. (2019) provided an overview of the current state of research in the field of linear algebra learning and teaching. They synthesize key themes, questions, results, and perspectives highlighted in the papers within that issue, along with a selection of those published between 2008 and 2017.

They observed that research on systems of equations is scarce (e.g Sandoval and Possani (2016); Oktaç (2018); Possani et al. (2010); Trigueros, (2018)). The concept of linear combination appears secondary in the text, despite being the foundation for understanding concepts such as spanning set, linear independence, and linear dependence. In a study conducted by Harel (2017), an instructional experiment involving in-service teachers was described. The experiment focused on equivalent systems of equations, highlighting the conceptual challenges inherent in the manipulations of systems that preserve equivalence. Larson and Zandieh (2013) devised a framework to understand student reasoning, identifying three key interpretations of the matrix equation Ax = b, where A is an m×n matrix, x is in R^n, and b is in R^m. Specifically, Ax = b can be constructed as a system of equations, a linear combination of column vectors, or as a transformation from R^n to R^m. Zandieh and Andrews-Larson (2019) extended this framework to the interpretation of augmented matrices. They observed algebraic and geometric contexts as two arenas in which an individual may engage with each interpretation. Considering these interpretations of the system of linear equations, it is noteworthy that we have not found specific studies dedicated to constructing the connection between system of equations and linear combinations. As we mentioned before, we did not find specific studies on the construction of linear combinations either, but some of the studies related to the Span set take it into consideration. Kú et al. (2011) presented a genetic decomposition using APOS Theory, to analyse how students construct spanning set and span concepts in Linear Algebra, together with the analysis of empirical data coming from interviews to students to validate it.

Wawro et al. (2012) introduced a didactic sequence called The Magic Carpet Ride, designed for an introductory linear algebra course and aiming to facilitate students' reinterpretation of the concept of span. This model has served as the foundation for designing activities using APOS theory and has been widely employed by professors at a private university in México. The activities used are based on a genetic decomposition and related to a similar model to the magic carpet model, the skateboard model. In this paper we present the results of a study on students' understanding of linear combination and its relation to linear systems of equations in the context of the skateboards problem using APOS theory as a theoretical framework.

THEORETICAL FRAMEWORK

APOS Theory is a cognitive theory based on Piaget epistemology interested in understanding how students construct mathematical knowledge. The structures of APOS Theory are Actions, Processes, Objects, and Schemas together with the associated reflective abstraction mechanisms that enable the transition between the different structures while students learn. An Action refers to the external transformation of a mathematical Object, carried out step by step according to explicit instructions. Through repetition and reflection, an Action can be interiorized into a Process. A Process is an internal construction, replicating the same Action(s) without external stimulus and allowing anticipation of results without explicit execution. Processes are coordinated with others to form new Processes and can be reversed. Applying Actions to a Process may raise awareness of the Process as a complete transformation, encapsulating it into a cognitive Object. Once an Object is constructed, it can be de-encapsulated back into the Process it originated from as needed. Students may autonomously apply constructed Objects to various problem situations, performing new Actions. A Schema encompasses Actions, Processes, Objects, and other already constructed Schemas, interrelated by general principles to create a coherent framework usable in solving diverse mathematical problems. Actions on a Schema may lead to its thematization into an Object. Contrary to a linear progression, APOS theory recognizes a dialectical progression with partial developments, transitions, and returns between conceptions. Applying APOS theory to describe students' mental constructs requires the formulation of a genetic decomposition (GD) – a model describing the specific mental constructions a typical student might construct in understanding mathematical concepts and their interrelations. The GD is a predictive model proposed by researchers, subject to experimental testing. After using it in an experimental situation, it can be rejected, refined, or validated by experimental data.

While modeling is not explicitly integrated into the APOS theoretical framework, its incorporation in the classroom aligns with APOS structures (Figueroa et al. 2018). When students encounter a modeling problem, they coordinate the mathematical schemas developed through their learning experiences. Through Actions and Processes on Objects within these schemas, as well as the coordination of Processes, a mathematical model emerges. The model is encapsulated into an Object, and new Actions and Processes can be done on it. The model we use in this study is the skateboards model. As students work on the model, the teacher can introduce activities designed with a genetic decomposition to foster its development together with new related mathematical knowledge. In the skateboard model, based on the Magic Carpet problem (Wawro et al. 2012), students work with a given set of skateboards traveling in different directions per unit of time. Skateboard's movements can be represented mathematically as vectors. While moving with the given skateboards, different points in a space can be reached. Reachable points are linear combination of the vectors associated with skateboards. Skateboards travel in a given space R^n.

Research questions.

Does skateboards model help in the construction of the relationship between linear combination and systems of equations?

Does constructing the linear combination representation (LC) of a system of equations facilitate the construction of linear combination?

METHODOLOGY

Context of research.

We started our work on this problem by interviewing 10 volunteer students (Group 1) who had just finished a lectured based introductory linear algebra course at a Mexican university.

The interviews were analyzed with Ku et al. (2008) published a genetic decomposition (GD) for the basis concept. This analysis showed that students had difficulties to determine the relation between systems of equations and linear combination. As this construction was not considered in Ku's GD, we decided to refine it as follows:

> Given an $\mathbf{R^n}$ space, a specific set $\mathbf{S}$ of vectors in $\mathbf{R^n}$ and specific scalars in $\mathbf{R}$, students need to perform Actions on the vectors and scalars. These Actions consist of performing scalar multiplications and sums of vectors (linear combination) to obtain a new vector in $\mathbf{R^n}$. Interiorization of these Actions into the Process of constructing a new vector $\mathbf{w}$ which is an element of $\mathbf{R^n}$ space, that is, into the construction of particular linear combination as a Process. This Process is coordinated with the systems of equations Process into a new Process where

each solution of the system is a vector with entries consisting of the scalars that are to be multiplied with the vectors in **S** to obtain was a linear combination of **S**. This Process is reversed so that given a system of equations, the linear combination **w** corresponds to the liner system of equation's constants vector, and the vectors in set **S** are the columns of the augmented matrix representing the system. This last Process is encapsulated into an Object and Actions can be performed to determine if this system of equations is consistent, which means that **w** can be written as a linear combination of S.

We used this resulting GD to design new activities that were implemented in another basic Linear Algebra course at the same university. The professor used activities based APOS theory, including those designed with the refined GD into her class of 22 students.

These activities were introduced when students had already covered the system of linear equations where they engaged in activities focusing on the geometric representation of linear equations, solutions of a linear equation and of linear systems with two and three variables. They also had worked on the construction of the normal form of a line in R^2, and the normal form of a plane in R^3. They interpreted linear systems' solutions as the intersection of geometric representation defined by the solutions of each equation. Problems were crafted for students to generalize to systems of equations with m equations and n variables. The extended matrix and the Gauss-Jordan method were employed to find the set of solutions for a general system of linear equations. They worked on activities to construct the solutions of each linear equation as a set of points in R^n and the system's solutions as the intersection of solutions from each equation which was named as the "row drawing" of the system of equations.

The skateboard model was introduced together with activities designed with the new GD. The professor followed the ACE Teaching Cycle (Arnon et al., 2014 Chapter 5) composed by Activities worked collaboratively in small groups, Classroom discussion where the previous work is presented and discussed by the whole group and the teacher. In this phase, the teacher may steer the discussion by posing questions or offering definitions necessary for the continuation of activities. The final component comprises homework Exercises, featuring standard problems designed to foster students' understanding and the construction of mathematical concepts.

The students worked in groups of two or three people. They were asked to submit their work individually so that they could draw their own conclusions. Once the course was completed, interviews were conducted with 10 students from various levels of development in the whole course (Group 2). The interview questions were the same as those used with students from the previous semester.

Description of the activities of Group 2.

We describe some of the activities designed to construct the coordination of the processes related to each concept. We start the activities by giving the student a set of two skateboards in R^2. The first one traveling in one time from the origin to (1,1) and an the second traveling in one time from the origin to (1,-2). With these skateboards, they must solve the problem of reaching a point (-2, 7). As documented, some of them use drawings, while others propose a system of equations. Different strategies used are discussed as a group. Then, a new skateboard is added to the original set of skateboards: the (1,1) and (1, -2) along with (5,6), and the task is reaching a new point (10, -11). They are asked to formulate the vector equation of a linear combination to record the journey. Then, they are asked to formulate a system of equations with these new skateboards to solve reaching (10, -11), thus doing actions to interiorize the relation between the system of equations and finding a path to (10, -11). They are asked to describe the meaning of the variables in the system of equations they have formulated and the augmented matrix of the system. Once the system of equations is formulated, they are asked to express the vector (10, -11) as a linear combination of the system. Given that the system has infinite solutions, students search ways to find a solution. This requires the interiorization of actions into the solution set of a system of equations Process. Students struggle with this problem, but we found that group discussion enables them to continue with the activity. Then, they are asked to find a path where the skateboard (1,1) is scaled by -2 and another path where the skateboard (1, -2) is scaled by 3. Additionally, they are requested to draw the trajectories and the corresponding vector expressions. They are prompted to reflect on how many paths exist if the system of equations has a free variable and what happens if there is no solution.

To encourage the construction of the reverse Process, students are asked to express a 2x3 augmented matrix as a skateboards problem. At this point, most of them show the interiorization of the meaning of the matrix columns and the constant vector. Some students even draw skateboards as the columns, as shown in Figure 1.

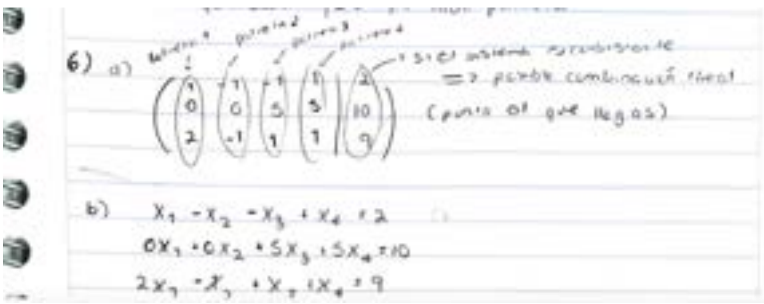

Figure 1: the skateboard drawing of a linear system.

The first activity of the next class asks to draw the trajectory -1/2(1,0,0) + 3(0,1,0) - 2(-2,1,0). This activity intends to foster the interiorization of the Process for linear combination in R^3. The task proves to be quite challenging for most students, not so much due to understanding what they want to do but rather because of the difficulty of drawing in R^3, as they referred. A class discussion is introduced to discuss the formal definition of a linear combination through the interpretation of each part of the definition as skateboards and scalars to travel with.

An augmented matrix of 4x3 is introduced. A list of vectors in R^4 is provided, and students are asked to verify if these vectors are linear combinations of the columns of the matrix. This is the moment when the linear combination of the system of equations Process is coordinated with the Gauss-Jordan method Process. Most students calculate the solution set each time, although a few realize that to answer the question, it is enough for the system to be consistent. A group discussion is introduced here for students to discuss this fact.

Finally, activities are introduced to foster the interiorization of the processes related to Gaussian elimination method and the interpretation of its results in terms of linear combinations.

The interviews.

Interviews included three key questions that we deemed crucial to assess their understanding of the linear combination concept and its relation to the corresponding system of equations.

Question 1. Can you explain what a linear combination is?

Question 2. Can you see a way to solve a linear combination problem with this augmented matrix $\begin{pmatrix} 1 & 2 & 3 & 3 \\ -1 & -1 & 5 & 3 \\ 2 & -1 & 7 & 3 \end{pmatrix}$? Explain your answer.

Question 3. What should you do if you want to determine if vector (1,7,3) is a linear combination of vectors (2,-5,3), (3,1,1) and (1,7,-2)?

The interviewer asked more questions as the students responded, thereby obtaining a richer set of qualitative data.

RESULTS AND DISCUSION

In the first question, most students in group one started with an algebraic description in terms of actions. Overall, they all agreed that a linear combination involved adding and scaling vectors. Two of the students who answered correctly but they confused which were vectors and which were numbers in their definition. In contrast, students in group 2 began with a drawing in R^2 or with a set X of two vectors and the given expression for two vectors as an action, but they could perform the same actions with different sets of vectors when they were asked for more explanation.

Students who used an algebraic definition in their response were asked to illustrate it with an example. Most students in Group 2 drew a trajectory illustrating the linear combination and the final point. Almost every student in Group 1 drew a pair of vectors on the plane and the resulting linear combination vector. A few of them provided evidence of the coordination between geometric (triangle method) and algebraic addition Processes. Among those who did, they constructed a parallelogram with the original vectors without considering the possibility of scaling them. After some interviewer questions, one of them finally drew a trajectory as a linear combination of three vectors, referencing a high school physics course. Apparently during the interview, he constructed a linear combination as a Process. This accounts for the advantages of using geometry in Group 2; students in Group 1 mentioned not having worked with geometric examples for linear combination.

In question two, the differences between the two groups were substantial. Group 2 students immediately referred to the vector of constant terms as the linear combination of the matrix columns showing the coordination between the linear combination Process and the linear system of equation Process. Although a couple of them who used memorized procedures, couldn't explain why, showing the construction of Actions. The interviewer inquired about the meaning of the system's solutions. One student mentioned that the vector of constant terms was the result of the system. This student couldn't dissociate the vector of constant terms and the solutions of the system, but when questioned, it became clear that he constructed the connection between the system of equations and the augmented matrix as an Action. All the other students in Group 2 were able to complete the task, explaining the meaning of the solutions of the system. When interviewed, one of these students demonstrated the interiorization of all the Processes.

Student:	It's as if each of the solutions were instructions to scale the columns and reach this vector... Yes, I believe that if the system has no solution, then I can't reach it, and if it does, I'll be able to find paths to do so.
Tutor:	What do you mean by instructions?
Student:	Well… if I take the skateboard (1, -1,2), and the solution is 3, then I travel with it three times.
Tutor:	And if the solution is -2, can you travel -2 units of time?
Student:	no, of course not, but I flip it and travel with it for two periods.
Tutor:	Is there a difference if the system has a unique solution or an infinity of solutions?
Student:	Oh yes, but that doesn't matter if we don't consider the linear independence part.
Tutor:	Does anything change in the expression you wrote below if there is one solution or many?
Student:	Not much, well, I don't know... the combination doesn't... change
Tutor:	An for ex..
Student:	Maybe you'll have infinite ways to arrive to the point.

For the students of Group 1, Question 2 was much more difficult. Five of them could not answer anything. One of the students explained that the vectors involved in the linear combination were the rows of the augmented matrix. Then he mentioned that if a row was a linear combination of the other two then the vectors were linearly independent. Three students tried to change the representation: one used Ax=b and the rest wrote it as a system of equations. Only one of the students who wrote the system of equations performed a sum of all the equations and said “ now it looks more like a linear combination”. Another Group 1 student could remember that there was an augmented matrix in the procedure of checking if a vector was a linear combination of others, and from this idea she was to perform the actions involved in forming the corresponding linear combination. However, she couldn’t interpret the solution of the system as the vector of scalars for the linear combination. Four students of Group 1 mentioned the word linear independence without being aware of its meaning.

Question three was the typical problem asking if a given vector is a linear combination of a given set of vectors. Almost all students of the two groups could perform this task by writing a linear system of equations and solving it. But, half of students in Group 1 could not interpret the solutions of the system. There were, surprisingly, only two students in Group 1 that could notice that the idea in question three could be used to answering Question 2. As the interview was coming to an end, a student from Group 2 added:

Student: I could also view the problem as an intersection of planes

Tutor: Do you believe that linear combinations are intersections of planes?

Student: No, they are very different things, but at the core, it's the same thing.

This suggest that the student constructed relations between the row representation of the linear system of equations and the (LC) representation in his system of equations Schema.

The results of the interviews show that it is not straightforward to construct the Process that identifies the columns of an augmented matrix with the vectors in X, the vector of constant terms with the linear combination, and a solution as a vector of scalars that multiplies the vectors in the linear combination expression.

The students in Group 1 were more formal in general, and precise in writing the definition of a linear combination. Some even specified that scalars belong to a field. However, they demonstrated that constructing the Process of building a path of vectors in R^2 and R^3 to describe a linear combination is not automatic. It requires coordination between the Process of algebraic addition and scaling with the geometric addition and scaling of vectors Process. This is why the skateboard model has been so useful for us, as it helps students build this coordination, which they use as a reference for the abstraction of linear combination. It is a tool for them to apply Actions on the model and then generalize those actions to vectors with more dimensions.

It was demonstrated that most of the students from the first group did not write a vector as a linear combination as a Process. They follow a series of mechanical instructions to solve the problem. They don’t even realize the relation between Questions 2 and 3. Therefore, they did not construct the linear combination as a Process when vectors are in R^n since they had not constructed the LC representation of a system of equations.

CONCLUSIONS

The skateboard model was found to be useful in the construction of the relationship between linear combination and system of equations. Students in Group 2 constantly referred to the vocabulary developed by de model in their answers. But we think that the difference in the answers between Group 1 and Group 2 was related to the work with the Activities designed using the proposed GD. These activities promoted students' reflection on the relationship between the model and the abstract concepts.

We consider our activities useful for students to make sense of geometrical interpretation of both linear combination and the interpretation of solutions of the system of equations. Also, they help students make sense of the matrix columns and constant vector of the augmented matrix. The vocabulary acquired through the skateboard model enables students to have an internal language that is highly recommended for the interpretations of abstract concepts. The link between the geometry of the linear combination in R^2 and R^3 with the algebra involved, may allow students to generalize ideas to higher dimensions and promote the interiorization of both, the linear system, and the linear combination conceptions. On the other hand, designing activities with the APOS theory allows for a detailed analysis of the constructions needed to coordinate the Process of linear combination to the system of equations Process, promoting the development with understanding of a method to verify that a vector is a linear combination of other vectors. The inverse Process of constructing a linear combination from a system of equations is fundamental to understand relations between concepts, for example, linear dependence with multiple solutions. This is only one example among others in an introductory Linear Algebra course where students need to understand this Process. Therefore, the construction of the LC is necessary in the construction of the linear combination Object.

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Understanding eigentheory: The modes of thinking in students' works

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In this study, we explore the understanding of eigenvectors and eigenvalues among university students by analysing their written responses to a task and conducting subsequent interviews. We make use of Tall and Vinner's notion of the concept image and Sierpinska's modes of thinking to assess their comprehension. Our findings show that the majority of students is able to engage with multiple modes of thinking when explaining the concepts of eigenvectors and eigenvalues. However, some students experience difficulties facing different modes. These insights are expected to inform our future work in designing tasks to further support students' learning of eigentheory.

Keywords: linear algebra, eigentheory, modes of thinking, concept image.

INTRODUCTION AND THEORY

Understanding linear algebra is an essential part of undergraduate mathematics education. To characterise the complex phenomenon of understanding, we make use of Tall and Vinner's notions of the concept image and, more specifically, the concept definition image. According to Vinner (2002), "To understand […] means to have a concept image" (p. 69). The *concept image*, as defined by Tall and Vinner (1981), refers to the mental representation or internalisation of a mathematical concept that individuals develop through their experiences and interactions with mathematical ideas. Hence, for this study, we consider the concept image as referring to a part of an individual's subjective understanding of a concept. Closely linked to the concept image is the *concept definition*, which is a verbal explanation precisely characterising the mathematical object. Tall and Vinner (1981) further describe the *concept definition image*, as the part of the concept image generated by the concept definition (Figure 1).

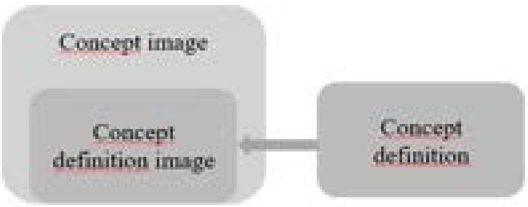


Figure 1: Diagram depicting the relationship between concept image, concept definition and concept definition image (adapted from Viholainen, 2008).

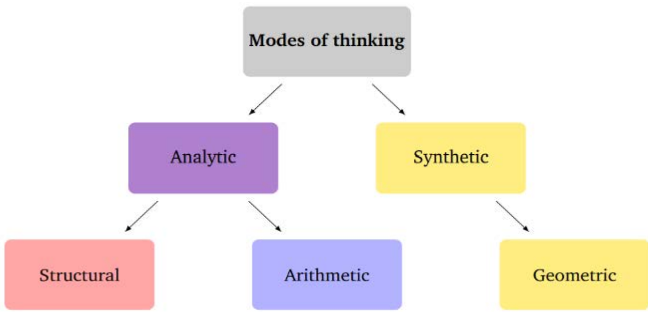


Figure 2: The hierarchic structure of the modes of thinking (based on Sierpinska, 2000).

From our perspective, the *modes of thinking* conceptualised by Sierpinska (2000) describe a part of students' understanding specific to linear algebra. In this study, we apply the modes of thinking as a theoretical lens to characterise certain parts of students' concept images of eigenvectors and eigenvalues, most of which align with the concept definition image. Sierpinska (2000) identifies three modes of thinking in linear algebra, namely the synthetic-geometric, analytic-arithmetic, and analytic-structural. These categories have a two-level hierarchy (see Figure 2). The first level distinguishes between synthetic and analytic modes of thinking, while the second level further separates them into geometric, arithmetic, and structural modes. Sierpinska (2000) describes the difference between *synthetic* and *analytic* modes of thinking as follows:

> [I]n the synthetic mode the objects are, in a sense, given directly to the mind which then tries to describe them, while, in the analytic mode they are given indirectly: In fact, they are only constructed by the definition of the properties of their elements. (p. 233)

In the *analytic-arithmetic mode*, an object is defined by the formula enabling its computation. Hence, the arithmetic mode concerns n-tuples of specific numbers satisfying equations or inequalities (p. 235). The *analytic-structural mode*, on the other hand, is more general, concerning the characteristic properties defining the mathematical objects. Finally, the *synthetic-geometric mode* employs the vocabulary of geometric figures, such as points, lines and planes (p. 234). This mode is concerned with the geometric characteristics of the objects and their visual representations. In a study by Gol Tabaghi and Sinclair (2013), the modes of thinking were used to analyse students' reasoning. Their analysis led to the conceptualisation of an additional mode of thinking, the *dynamic-synthetic-geometric mode*, emerging from students' emphasis on the dynamic aspects of eigenvectors.

According to Sierpinska (2000), each of the three modes of thinking correspond to a specific "system of representation" (p. 234), which we interpret in the sense of Duval (2006). Sierpinska (2000) highlights a shared characteristic between the synthetic-geometric and analytic-structural modes of thinking, namely the independence of a coordinate system. However, the analytic-structural mode is characterised by schematic representations, illustrating the inherent properties and the abstract relations between objects, whereas the synthetic-geometric mode portrays the mathematical objects in a more concrete manner.

METHOD

In this study, we employ the modes of thinking and concept image to elucidate specific facets of students' understanding of eigentheory. It is our perspective that eigentheory, the field of linear algebra concerning eigenvectors, eigenvalues, and eigenspaces, can prove to be particularly challenging for students to grasp. In their 2011 study, Thomas and Stewart observed that while many students demonstrated proficiency in performing the arithmetic computations related to eigenvectors and eigenvalues, they encountered difficulties in associating these calculations with their geometric interpretations. Moreover, within the context of eigentheory, students must adeptly navigate several key concepts simultaneously, including linear transformation, vector space, and span (Wawro et al., 2018, p. 275).

This paper is based on the master's project of the first author (Lyse-Olsen, 2023), aiming to explore aspects of students' understanding of eigenvectors and eigenvalues through the following research question:

> *What parts of students' concept images can be described through the modes of thinking identified in their characterisations of eigenvectors and eigenvalues?*

The presented study took place in a first linear algebra course of a Norwegian university during the fall of 2022, attended by ca. 700 students from various engineering study programs, most of which were in their second year of study and aged in their early 20s. The instructional approach of the course employed a flipped classroom style, with interactive lectures, exercise classes and exercise lectures. To prepare for these activities, students were expected to watch a set of short lecture videos and read the course materials. The following definition was used to introduce eigenvalues and eigenvectors in the written materials:

> *Let $T: V \to V$ be a linear transformation. A scalar λ is called an eigenvalue of T if there exists a vector $\vec{v} \neq \vec{0}$ in V such that $T(\vec{v}) = \lambda\vec{v}$. The vector $\vec{v}$ is called an eigenvector of T corresponding to the eigenvalue λ. When T is given as an $n \times n$ matrix A, λ is called an eigenvalue of A and $\vec{v}$ is an eigenvector of A corresponding to the eigenvalue λ.* (NTNU, n.d.)

The preparational material also emphasised geometric aspects of eigenvectors and eigenvalues, including visual representations of two-dimensional eigenvectors. As all activities and preparational material was voluntary, it is likely that different students were exposed to different modes of thinking prior to this study, depending on their decisions to engage with or omit certain preparations or activities. To come to know about students' concept images and modes of thinking, four tasks were designed and implemented. In this paper, we will only present the results of the analysis of one of these tasks (see Figure

Task 9: Explain in your own words:

a) What is an eigenvector?
b) What is an eigenvalue?

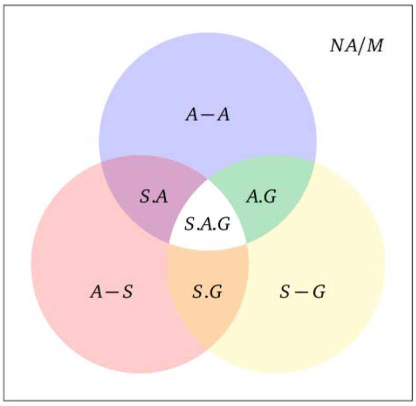
You may use sketches to illustrate.

Figure 3: The task given to the students (translated from Norwegian).

3), where students were asked to explain eigenvectors and eigenvalues in their own terms and encouraged to produce an illustrating sketch. The task was designed with an open phrasing to challenge students with a different task than they were used to and to prompt reflection upon the concepts, thereby allowing our exploration of their understanding. The results are based on the written answers of 170 students and additional semi-structured interviews with five individual students lasting up to 45 minutes each. The interviews were conducted and audio-recoded by the first author of this paper, further exploring their reasoning in completing the tasks and their overall experiences with the course, approximately 3-5 weeks after submitting their homework. The students were selected based upon a preliminary analysis of the students' written homework and the selection aimed to include students exhibiting different modes of thinking and various concept images.

Data analysis: The students' written answers were analysed in a two-step coding process using the qualitative data analysis software NVivo. The first level codes were based on the words or short phrases used by the students, aiming to capture the essence of their ideas. In the second level of coding, we created codes corresponding to the modes of thinking and categorised the first level codes accordingly. It should be noted that the answers to parts a) and b) were analysed and coded together due to the interconnectedness between the concepts of eigenvectors and eigenvalues. The transcriptions from the audio-recordings of the interviews were employed as supportive data in this study.

To effectively characterise students' modes of thinking, we must first establish what the modes entail in the field of eigentheory, as Sierpinska (2000) does not specify her framework for this particular context. First, it is our perspective that an analytic-structural mode of thinking about eigenvectors and eigenvalues would involve a general description true for all eigenvectors, not just particular examples. Thus, this mode of thinking could present as characterising eigenvectors as preserving their span when imaged by a linear transformation or providing a schematic sketch illustrating these properties. An analytic-arithmetic mode of thinking, on the other hand, could manifest as characterising eigenvectors and eigenvalues as the solutions of the equations allowing their computation, $(A - \lambda I)\vec{x} = \vec{0}$ or



Abbreviation	Explanation
$A-S$	Analytic-structural
$A-A$	Analytic-arithmetic
$S-G$	Synthetic-geometric
$S.A$	Structural-arithmetic
$A.G$	Structural-geometric
$A.G$	Arithmetic-geometric
$S.A.G$	Structural-arithmetic-geometric
NA/M	Not answered/ Miscellaneous

Figure 4: Overview of modes of thinking exhibited in the students' answers to Task 9.

$\det(A - \lambda I) = 0$, respectively. Finally, a synthetic-geometric mode of thinking about eigenvectors and eigenvalues may involve geometric descriptions of the representations of particular examples, such as characterising eigenvectors as remaining on the same line or preserving their direction under a linear transformation (or matrix multiplication). From our understanding, descriptions in the synthetic-geometric mode describe certain characteristic properties of *some* eigenvectors and eigenvalues, but they do not define them.

As we shall see, we quickly came to realise that many of the students' answers aligned with multiple modes of thinking. To capture the nuances of answers incorporating multiple modes of thinking or falling between modes, we created four intersectional modes of thinking, which we call the *structural-arithmetic*, *structural-geometric, arithmetic-geometric* and *structural-arithmetic geometric* modes of thinking (see Figure 4). These modes will be elaborated in the upcoming section.

RESULTS AND ANALYSIS

In this section, we first present examples of students' work aligning with the original modes of thinking described by Sierpinska (2000), and later present examples of answers that extend beyond the initial categorisation.

Analytic-structural mode of thinking: The following work of one student (see Figure 5), who we shall call Alex, was identified as an example of a structural representation. The sketch shows two Vs connected by an arrow labelled T, which we interpreted as symbolising a linear transformation within a vector space denoted V. Below, the linear transformation T is depicted as operating on a vector $\vec{x}$ and mapping it to $\lambda(\vec{x})$. Hence, we understand this as a schematic sketch of the relations between the mathematical object, depicting how a *general* linear transformation T acts upon a corresponding eigenvector $\vec{x}$ by scaling it with a factor of λ, the eigenvalue. Other answers, describing eigenvectors and eigenvalues in relation to the notion of span or the image of linear transformation were also interpreted as exhibiting elements of an analytic-structural mode of thinking.

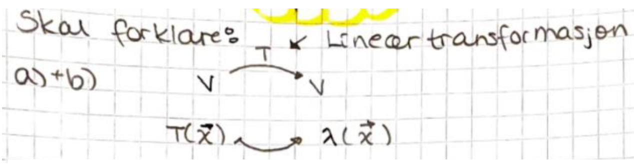


Figure 5: Alex's sketch.

Analytic-arithmetic mode of thinking: Some students explained the concept of eigenvalue according to the procedure for computing them, namely by computing the roots of the characteristic polynomial. For instance, a student, Tyler, stated that: "Eigenvalue or characteristic value is a solution of the characteristic equation $\det(A - \lambda I) = 0$". In describing an eigenvalue as the solution of an equation, the answer was deemed as aligning with an analytic-arithmetic mode of thinking.

Synthetic-geometric mode of thinking: Several students gave geometric descriptions of eigenvectors and eigenvalues using words and phrases like “stretching”, “shrinking”, “changing length” or “preserving direction”. For example, Sam stated that “[The] Eigenvalue is how much the vector is stretched.”. Hereby, Sam gave a visual description of a particular type of eigenvalues, namely real eigenvalues with an absolute value greater than 1, suggesting a synthetic-geometric mode of thinking. In fact, several other students gave similar descriptions of eigenvalues being factors of stretching or eigenvectors being stretched, thus excluding the possibilities of shrinking or preserving the length. Notably, the definition presented to students (see Method) does not imply such restrictions. In this study, none of the 170 participating students gave a comprehensive description of all possible effects on eigenvectors by linear transformations or matrix multiplications, including stretching, shrinking, rotating by 180 degrees, preserving its length or direction, or combinations thereof.

While the previous example of Sam concerned a change in *length*, other students focused on the effect of the linear transformation on the eigenvector’s *direction*. For instance, Robin explained an eigenvector as “[A] vector which does not change direction”. However, characterising eigenvectors as always preserving their direction is inaccurate as real eigenvectors can reverse direction with negative eigenvalues, and eigenvectors with complex eigenvalues can undergo both scaling and rotation. Thus, Robin’s answer is a geometric interpretation of *specific* examples of eigenvectors, which we consider evidence of a synthetic-geometric mode of thinking.

Robin was selected for an interview to further explore their concept image. When asked to explain the concept of eigenvectors and eigenvalues in the interview, Robin gave a verbal rephrasing of the eigenequation (the following is our own translation from Norwegian):

> Robin: [long pause] Yes, that, if you have a... a matrix, then you can… And you multiply it with the eigenvalue, then you will have the same as if you multiply the eigenvector… an eigenvector with the eigenvalue. Is actually the only thing I know about that […]. So, basically, an eigenvalue is a value which you can multiply by the matrix and a vector and obtain the same result.

Thus, while the written task evoked one part of Robin’s concept definition image associated with a synthetic-geometric mode, the interview appeared to trigger another. In describing eigenvectors and eigenvalues as fulfilling an equation, specifically the faulty equality $A\lambda = \vec{x}\lambda$, Robin’s oral response aligns with an arithmetic mode of thinking. To support Robin in linking the different modes of thinking expressed in their written and oral response and restore their error, Robin was reminded of their written description:

Interviewer: It seems to me that you know another thing because you wrote that.... In [Task 9] a) you wrote: “A vector which does not change direction”?

Robin: Oh… Yes… Uhm… [long pause]. I don’t really know what I meant by that. If I… Maybe I meant, if I… That I multiplied by a number and then… No… Now I’m not quite sure what that means.

While Robin’s written and oral characterisations do not entirely contradict each other (scaling a vector $\vec{x}$ by a number λ can preserve its direction), Robin’s apparent confusion could suggest they *perceive* their answers as conflicting. Nevertheless, when prompted to recall their written explanation of eigen*values* affecting the length of the eigenvector, Robin appeared to align their oral and written answer:

Interviewer: In [Task 9] b) you wrote: “The scalar which determines the length of the eigenvector”.

Robin: That makes a bit more sense. Maybe if you take a vector and you multiply it by a number, it would change length.

Hence, Robin seems able to visually interpret scalar multiplication as a means of altering the length of a vector (in this case, an eigenvector), thereby navigating between the synthetic-geometric mode and the more arithmetic thinking of scalar multiplication. Thus, Robin’s interview illustrates a development of their concept image.

Intersectional modes of thinking: It is noteworthy that a substantial proportion of the students participating in this study demonstrated engagement with multiple modes of thinking in their written answers to Task 9. In particular, we identified an overwhelming majority of 123 answers expressing the symbolic eigenequation of the linear transformation ($T(\vec{x}) = \lambda\vec{x}$) or matrix ($A\vec{x} = \lambda\vec{x}$), or a verbal rephrasing of it. For instance, Jordan gave the following explanation of eigenvectors and eigenvalues: “If one has a matrix A and a vector $\vec{x}$, the product will give a number λ multiplied by $\vec{x}$. Then λ will be an eigenvalue and $\vec{x}$ an eigenvector: $A\vec{x} = \lambda\vec{x}$.”. It is our perspective that characterising eigenvectors as fulfilling the eigenequation aligns with both an analytic-structural mode of thinking (by expressing a defining property of eigenvectors and eigenvalues), as well as an analytic-arithmetic mode (in stating an equation allowing the computation of eigenvectors and eigenvalues). To capture these nuances, we introduced the intersectional mode *structural-arithmetic*, and categorised such answers accordingly. Nevertheless, Jordan’s answer is brief and therefore provides little other information regarding their concept images of eigenvectors and eigenvalues. For instance, there is no mention of the relation between eigenvectors and eigenvalues to other key concepts like linear transformation or span, and there is no sketch or geometric interpretation of eigenvectors. Other answers incorporating elements from analytic-structural, analytic-arithmetic and/or synthetic-geometric modes of thinking were classified as structural-geometric, arithmetic-geometric, and structural-arithmetic-geometric. However, due to space limitations, we are unable to elaborate and exemplify all categories here.

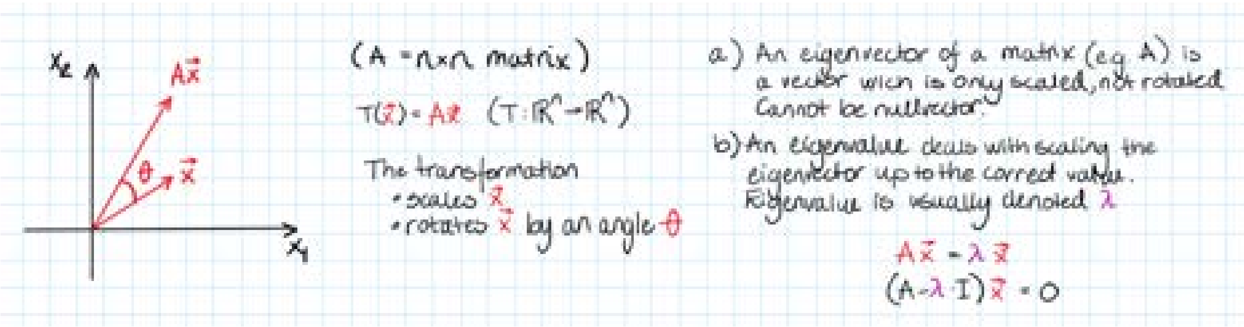


Figure 6: Excerpt (translated from Norwegian) of Riley's written answer.

Figure 6 shows a particularly interesting example of the work of student Riley, who incorporated all three modes of thinking in their answers. Our interpretation of Riley's work is that they first explain and illustrate how a general linear transformation may act upon any vector $\vec{x}$, and in Task 9 a) the student specifies that the vectors which are "only scaled" are eigenvectors, so long as they are not the nullvector. By relating eigenvectors to the notion of linear transformation, the answer can be characterised as incorporating an analytic-structural mode of thinking. By further stating that eigenvectors are scaled, not rotated, Riley provides a geometric description of typical examples of eigenvectors, aligning with a synthetic-geometric mode of thinking. Moreover, in providing the equation $(A - \lambda I)\vec{x} = \vec{0}$, the answer could indicate an analytic-arithmetic mode of thinking as well. Consequently, our analysis suggests this student possesses a concept image encompassing multiple characteristics of eigenvectors, suggesting a rich understanding of them.

Summary: Initially, based on Sierpinska's framework, our expectation was that most students would adhere to one mode of thinking in their responses. However, it became evident that the majority of the students incorporated more than one mode of thinking in their answers, implying a richer concept image. As shown in Figure 7, only few answers were characterised as belonging to one of the original categories described by Sierpinska (2000), while combining two modes of thinking was much more prevalent (98 students). Interestingly, as many as 54 students gave answers which encompassed all three modes of thinking. In Figure 7, the number 9 outside the Venn diagram represent the written answers that could not be classified into either

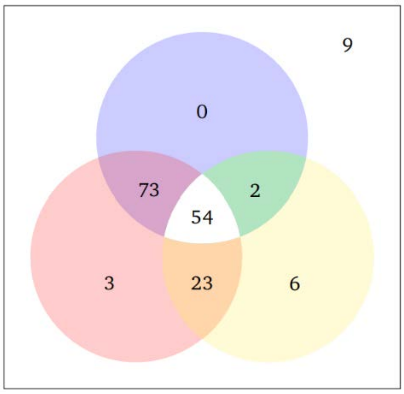


Figure 7: Quantitative overview with the observed modes of thinking.

mode of thinking (task unanswered: 4 students; aligns with none of the modes: 5 students).

DISCUSSION AND CONCLUSION

Through our analysis, we have seen the participants effectively connecting eigenvectors and eigenvalues with linear algebra concepts such as span, linear transformations, and vector spaces – a skill emphasized by Harel (1997) as key to understanding linear algebra. Diverging from the findings of Thomas and Stewart (2011), our study revealed that many students demonstrated an ability to describe eigenvectors and eigenvalues in terms of their geometric properties. Interestingly, we have seen that the majority of students engaged with multiple modes of thinking or gave answers falling between Sierpinska's (2000) categories in their characterisations of eigenvectors and eigenvalues. In doing so, we consider that the students demonstrated rich concept images and an ability to navigate multiple modes of thinking. This prompted us to introduce intersectional modes of thinking, thus refining Sierpinska's (2000) framework in the context of eigentheory, similar to the approach taken by Gol Tabaghi and Sinclair (2013). Our analysis further aligns with Sierpinska's (2000) observation that there is no empirical basis for a general preference among students for one mode of thinking over another. Instead, we propose that certain types of tasks may trigger specific modes of thinking in students. While most participants expressed multiple modes of thinking in their answers to the presented task, the majority exclusively employed the analytic-structural mode in their answers to subsequent tasks not covered in this paper (see Lyse-Olsen, 2023).

In our quest to describe students' understanding of eigenvectors and eigenvalues, it is important to acknowledge that as students express their ideas in sentences (whether written or oral) and illustrations, there is a potential for meaning to be lost, transformed, or even added to their utterances. While our methods do not allow us to identify *all aspects* of students' understanding, our theoretical lenses have enabled us to describe *certain aspects* of their concept images (closely aligning with the concept definition image) of eigenvectors and eigenvalues and their modes of thinking. Moreover, it is important to recognise that absence of specific expressions of knowledge does not necessarily imply that students lack awareness of them. For instance, students' concept images may contain several different ways in which a linear transformation (or matrix) can transform eigenvectors, even though not all these ways are explicit in their answers. We expect these findings to inform our future work, which aims to develop tasks that effectively address students' challenges and foster a deeper understanding of eigentheory.

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Students’ preferred resources for learning mathematics in an online linear algebra course

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This study investigates which resources students prefer in a fully digital learning environment and how different preferences relate to students’ personal and psychological characteristics. We draw on questionnaire data from 78 students of an online linear algebra course, indicating that students prefer the course's resources over resources from other sources. Additionally, profile analyses identify four profiles that mainly differ in students’ perceived usefulness of a full-class tutorial and videos and books from different internet sources. First-semester students rather relied on such other resources and perceived interaction with others as less useful than other students. Such findings contribute to understanding students’ preferences and to developing group-specific support measures.

Keywords: Digital and other resources in university mathematics education; Teachers’ and students’ practices at university level; Teaching and learning of linear and abstract algebra.

FULLY DIGITAL LEARNING OF MATHEMATICS AT UNIVERSITY

Due to the COVID-19 pandemic, university teaching had to be implemented into fully digital learning environments. In university mathematics, up to this time, completely non-digital and purist chalk-talk was predominant (Artemeva & Fox, 2011). Thus, this shift to fully digital learning environments may be considered a fundamental change. Additionally, as students considered peer learning as one of the most important tools to master the transition from school to university in mathematics in traditional learning environments (Göller, 2021; Liebendörfer, 2018), peer collaborations had to be transferred and implemented into fully digital solutions.

In this paper, we investigate which resources students preferred in such a fully digital learning environment and how different preferences were connected to students’ personal and psychological characteristics. To do so, we first present theoretical and empirical considerations on resources in university mathematics education and factors associated with students’ learning processes on which our empirical study is grounded.

THEORETICAL BACKGROUND

Resources in university mathematics education

Resources (not only in mathematics education) comprise all (also digital) materials that are used or developed by students and lecturers for learning and teaching mathematics (Pepin & Gueudet, 2018). Beneath such materials, students can also draw on other persons such as lecturers, tutors, or peers as well as different kinds of teaching and support opportunities (e.g., lectures, tutorials, learning centers) to influence their learning processes (Anastasakis et al., 2017; Göller, 2021).

In traditional tertiary mathematics learning environments, students most often use and prefer resources provided by the lecturers of their respective courses (Anastasakis et al., 2017; Inglis et al., 2011; Maclaren, 2018). Consequently, students' use of resources depends on the learning environment (Gueudet & Pepin, 2018; Kock & Pepin, 2018), may change during the study (Stadler et al., 2013), and is often oriented towards exam-related goals (Anastasakis et al., 2017). Additionally, peers and other persons are an important resource for students' learning (Göller, 2021; Liebendörfer, 2018).

For a fully digital learning environment of a linear algebra course at a German university, Kempen & Liebendörfer (2021) found that traditional aspects of mathematics teaching, such as lecture notes or attending (online) lectures and tutorials were still rated as particularly useful, while traditional literature such as textbooks were rarely considered useful. Despite the obstacles given by the online environment, communication with peers was rated as the most useful resource (Kempen & Liebendörfer, 2021). In addition to such similarities, profile analyses showed that some students comparatively preferred external digital resources such as videos, webpages, etc. ("digitals"). In contrast, another group preferred traditional resources such as live lectures and tutorials ("traditionalists"), and a third group rated the usefulness of all resources as comparatively high ("all resource users").

Factors associated with students' learning processes at university

From a self-regulated learning perspective, students' preferences and use of different recourses are guided by their *self-regulation* which describes efforts to initiate and direct the pursuit of their (learning) goals by planning, monitoring, evaluating, and adapting their cognition, behavior, motivation, and affect (Greene et al., 2023). Thereby, students' mathematics *self-efficacy*, i.e., their belief in being able to realize strategies that lead to learning the mathematics contents, is theoretically and empirically strongly connected to students' learning approaches and performance (Sun et al., 2018). More recent theories highlight the importance of social aspects for self-regulated learning (Greene et al., 2023). *Social relatedness*, e.g., which involves feeling close, connected, and belonging to others at university, as well as caring for and feeling cared for by them (Longo et al., 2016; Ryan & Deci, 2020), has empirically shown to be strongly associated with self-regulated learning processes (Zhou et al., 2021). In traditional mathematics courses, students describe social relatedness as a key to mastering the transition from school to university (Göller, 2021; Liebendörfer, 2018) and it also seems of high importance in fully digital contexts (Kempen & Liebendörfer, 2021).

On the other hand, studies also have shown that students with different personal characteristics participate differently in learning mathematics at university. For example, female students learn more frequently with peers and use more organization strategies (Johns, 2020; Liebendörfer et al., 2020), and generally, students' learning strategies are related to their respective study program and their performance (Credé & Phillips, 2011; Inglis et al., 2011; Liebendörfer et al., 2020). This means that students' personal characteristics such as gender and study program should be considered when analyzing their preferences and use of resources or their self-regulated learning processes in general.

Research Questions

To validate and further explore the findings of Kempen & Liebendörfer (2021), based on one specific implementation and relatively small sample size, the present study investigates which resources students prefer in a fully digital learning environment and how possibly different preferences relate to students' self-regulation, self-efficacy, social relatedness, and personal characteristics. More concretely, we aim to answer the following three research questions:

> RQ 1: Which resources do students identify as being useful for their learning in a digital linear algebra course?
>
> RQ 2: Which different student profiles regarding the usefulness of different resources can be identified?
>
> RQ 3: How are these profiles related to students' self-regulation, self-efficacy, and social relatedness, as well as to their personal characteristics (gender, age, high school grade point average, study program, study semester)?

METHODS

In the winter term 2020/21, which in Germany lasted from October to February, the students of a first year 'Linear Algebra 1' course, held entirely digitally due to the COVID-19 pandemic, were asked to complete an online questionnaire at two points in time (one in December, one in February). The questionnaire contained questions regarding the usefulness of different resources, coordinated with the materials and meetings provided by the lecturer and the tutors of the course. Students had to rate on a 6-point Likert scale how useful they perceived a) the lecture videos (which were provided by the lecturer), b) the lecture notes, c) attending the Zoom lecture, d) attending the small group tutorials, e) attending the full-class tutorials, f) tutors, g) peers, h) other videos, i) books, or j) other internet resources for their learning in the linear algebra course. Additionally, the questionnaire covered personal characteristics of the students (gender, age, high-school grade point average, first semester (yes/no), study program) as well as their mathematics self-efficacy (4 items, Cronbach's $\alpha = .90$, Hochmuth et al., 2018; originally from Ramm et al., 2006), social relatedness (relatedness satisfaction, 3 items, Cronbach's $\alpha = .83$, Longo et al., 2016), and self-regulation (4 items, Cronbach's $\alpha = .89$, Kempen & Liebendörfer, 2021).

For the analysis, we only used data from students who provided complete information on the questions about the usefulness of the resources. In total, data from 53 students of time point 1 (December, middle of the semester, 27 female, 14 math majors, 27 preservice teachers) and 25 students of time point 2 (February, end of the semester, 11 female, 4 math majors, 15 preservice teachers) were analyzed.

To answer research question 1, we provide some descriptive statistics for the usefulness of the resources. For research question 2, we conducted a latent profile analysis with the different resources as variables using the R-package "mclust" (Scrucca et al., 2023). According to the Bayesian information criterion (BIC), a four-profile solution fits the data best. Regarding research question 3, we first conducted ANOVAs with the assignment to these four profiles as grouping variable and the other variables as dependent variables. Additionally, "mclust" provides the predicted probabilities of each observation (student) to be classified in one of the profiles, which allows for correlation analyses of these profile probabilities with the other variables.

RESULTS

Results for RQ 1: Usefulness of resources

Table 1 provides descriptive data as well as the results of the ANOVAs. Regarding research question 1, the first three columns (total sample) indicate that overall, students perceived the resources of the course as rather useful for their learning of linear algebra. Especially lecture notes, small group tutorials, and peers were rated as very useful resources. Resources from other sources (other videos, books, and other internet resources) were rated as comparatively less useful.

Results for RQ 2: Profiles

The profile analysis provides a four-profile solution that fits the data best (according to the Bayesian information criterion). Descriptive statistics for these four profiles are given in Table 1. Additionally, Figure 1 provides a visualization of the means of the different resources for the different profiles. These four profiles can be characterized as follows:

Profile 1: Lecture video is characterized by a comparatively lower perceived usefulness of almost all regarded resources. Only the usefulness of the lecture videos was rated slightly higher by students in Profile 1 than by the total sample (in mean). Students in Profile 1 are also characterized by not finding the full-class tutorial useful at all.

Profile 2: Interaction comprises students who perceived resources that enable interaction with others, such as Zoom lectures, small- and full-class tutorials, and communications with tutors and peers, as particularly useful for their learning of linear algebra. The provided lecture videos, as well as resources from other sources (other videos, books, other internet resources), were perceived as less useful, both in interpersonal and intrapersonal comparison.

Profile 3: All resources comprises students who perceived all indicated resources as useful. The means of all resources were higher for students in Profile 3 than in the total sample.

Profile 4: Other resources is characterized by a comparatively higher perceived usefulness of resources from other sources (other videos, books, other internet resources). Although students of Profile 4 also rated lecture notes, small group tutorials, and peers relatively high, the mean values here scarcely differ from the mean values of the total sample. The means of the perceived usefulness of lecture videos, Zoom lectures, full-class tutorials, and tutors are for students of Profile 4 lower than in the total sample.

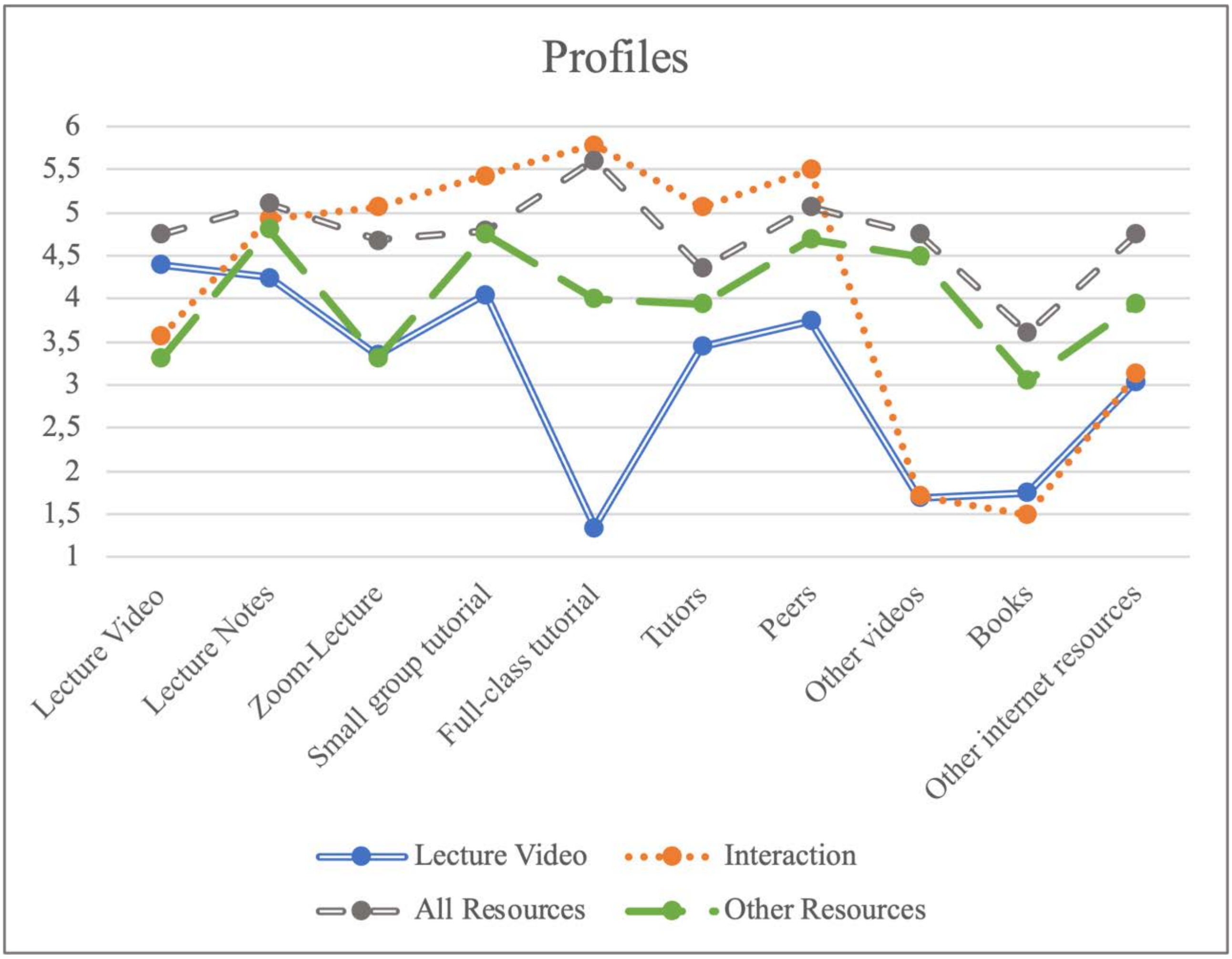

Figure 1. Visualization of the means of the four profiles.

Results for RQ 3: Profiles in relation to personal and psychological characteristics

The right-hand columns of Table 1 show the results of the ANOVAs. The most significant effect is found for the full-class tutorial (students of Profile 1 here differ massively from students of the other profiles) followed by the resources from other sources (other videos, books, other internet resources) and the Zoom lecture. The effect of the four profiles on the other considered variables is not significant except for self-efficacy where the effect is significant but small.

The correlations given in Table 2 provide an additional perspective on the four profiles: Students with higher self-efficacy and lower self-regulation tend to be in Profile 1 (lecture video), and students with higher self-regulation tend to be in Profile 3 (all resources). Female students are rather in Profile 1 (lecture video). First-semester students are more likely in Profile 4 (other resources) and less likely in Profile 2 (interaction). Preservice teachers are unlikely to be in Profile 1 (lecture video), while students of other study programs are rather in Profile 1 (lecture video) and rather not in Profile 3 (all resources). Age, high school grade point average (HSGPA), social relatedness, the math major study program as well as the time point do not correlate significantly with the four profiles.

	Total sample (n = 78)		P1 Lecture video (n = 20)		P2 Interaction (n = 14)		P3 All resources (n = 28)		P4 Other resources (n = 16)			
Resources	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*F*	η_p^2
Lecture video	4.15	1.67	4.40	1.76	3.57	1.74	4.75	1.00	3.31	2.02	3.59*	0.13
Lecture notes	4.79	1.27	4.25	1.55	4.93	1.27	5.11	0.92	4.81	1.33	1.90	0.07
Zoom lecture	4.13	1.69	3.35	1.93	5.07	1.07	4.68	1.09	3.31	1.96	6.16*	0.20
Small group tutorial	4.71	1.19	4.05	1.54	5.43	0.88	4.79	0.65	4.75	1.18	4.32*	0.15
Full-class tutorial	4.22	1.90	1.35	0.59	5.79	0.43	5.61	0.50	4.00	0.73	265*	0.92
Tutors	4.17	1.39	3.45	1.76	5.07	0.61	4.36	0.91	3.94	1.61	4.65*	0.16
Peers	4.73	1.62	3.75	1.86	5.50	1.02	5.07	1.27	4.69	1.78	4.45*	0.15
Other videos	3.37	1.84	1.70	0.98	1.71	0.73	4.75	1.00	4.50	1.63	44.9*	0.65
Books	2.64	1.51	1.75	1.02	1.50	0.65	3.61	1.13	3.06	1.88	13.8*	0.36
Other internet resources	3.86	1.57	3.05	1.54	3.14	1.29	4.75	1.11	3.94	1.77	7.15*	0.23
Age	20.0	2.99	20.2	1.22	20.3	1.22	20.1	4.40	19.7	1.68	0.12	0.01
HSGPA	1.73	0.53	1.63	0.40	1.52	0.59	1.81	0.56	1.82	0.56	1.01	0.05
Self-efficacy	3.75	1.26	4.43	1.19	3.83	0.92	3.49	1.12	3.35	1.57	2.89*	0.11
Social relatedness	3.52	1.36	3.44	1.11	3.44	1.56	3.51	1.47	3.75	1.38	0.14	0.01
Self-regulation	4.26	1.28	3.72	1.34	4.30	1.17	4.67	1.04	4.05	1.52	2.22	0.09

Table 1. Descriptive statistics and results of the ANOVA for the total sample and the four profiles (lecture video, interaction, all resources, and other resources). High school grade point average (HSGPA) ranges from 1 (best) to 4 (poorest). $^*p < .05$.

DISCUSSION

Investigating students' preferred resources for learning mathematics in an online linear algebra course, our results on RQ 1 show that students perceived the resources of the course as rather useful than resources from other sources (other videos, books, other internet resources). These results confirm previous studies showing that students prefer resources closely connected to the course (Anastasakis et al., 2017; Inglis et al., 2011; Kempen & Liebendörfer, 2021; Maclaren, 2018).

	1.	2.	3.	4.	5.	6.	7.	8.	9.
1. P1 Lecture video	---								
2. P2 Interaction	-.30**	---							
3. P3 All resources	-.44***	-.34**	---						
4. P4 Other resources	-.28*	-.26*	-.38***	---					
5. Age	.03	.04	.01	-.08	---				
6. HSGPA	-.10	-.14	.13	.07	.25*	---			
7. Self-efficacy	.30*	.04	-.17	-.16	-.05	-.15	---		
8. Social relatedness	-.03	-.01	-.02	.06	-.11	-.27*	.10	---	
9. Self-regulation	-.24*	.01	.25*	-.06	.04	-.15	.20	.04	---
10. Female	-.31**	.02	.23	.05	-.27*	.02	-.33**	-.03	.07
11. First semester	-.18	-.33***	.18	.27*	-.15	.07	-.26*	.23	-.02
12. Math major	.11	.04	-.01	-.15	-.11	.02	.12	-.23	-.27*
13. Teacher	-.39***	.09	.18	.12	.00	.03	-.37**	.22	.18
14. Another program	.40***	-.18	-.25*	.02	.13	-.06	.37**	-.03	.09
15. Time point	.10	-.08	-.09	.07	-.04	-.09	.05	.11	-.00

Table 2. Pearson correlations. For variables 10 to 15 we report point biserial correlations. High school grade point average (HSGPA) ranges from 1 (best) to 4 (poorest). $^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$.

However, the profile analyses (RQ 2) showed that students differ in their preferences. While the huge differences in the perceived usefulness of the full-class tutorials are probably due to the specific characteristics of the learning environment investigated in the present study, the differences in the preferences regarding the resources from other sources (other videos, books, other internet resources) seem to be more systematic. Kempen and Liebendörfer (2021) even identified a profile that preferred these other resources over the course resources. Such a profile has similarities with Profile 4 (other resources) but was not found in the present study in this distinctiveness.

Regarding research question 3, no significant correlation was found between high school grade average, age, or time point and the four profiles. This indicates that such profiles seem to capture preferences that are rather not due to achievement, and which might be relatively stable over time. A correlation between social relatedness and a preference for interaction (Profile 2) could perhaps have been expected but was not found. On the other hand, correlations of the profiles with self-efficacy, self-regulation, gender, study semester, and study program were found. Perhaps students with higher self-efficacy assume that they do not need many resources for their learning (Profile 1). In this vein, using fewer resources could require less self-regulation (Profile 1), while using all resources would require more self-regulation (Profile 3). Interestingly, first-semester students rather relied on other resources (Profile 4) and rather less on interaction (Profile 2) than other students, which might indicate that such profiles also trace back to different experiences in different learning environments. These findings are similar to those of Kempen and Liebendörfer (2021), where "digitals" were likely to be first-semester students.

By-products of Table 2 are that female students have better HSGPAs and nevertheless report lower self-efficacy (first-semester students and preservice teachers also report lower self-efficacy, no grade difference), while math major students report lower self-regulation. Such findings replicate the results of other studies (Else-Quest et al., 2010; Zander et al., 2020) and give rise to considering group-specific support measures.

Limitations and outlook

When interpreting the results, the specific characteristics of the present study, especially the specific implementation of the different resources and the limited number of participants should be considered. Missing data were treated here by including only complete cases regarding the different resources. With multiple imputations, the number of cases could be increased. This might be realized in future analyses of the data.

Although we found no significant correlation between the time point (middle of the semester, end of the semester) and the assignment to the four profiles regarding the total sample, there might be students who adapt their preferences for specific resources in the course of their study (cf. Stadler et al., 2013). The present study did not analyze such individual trajectories. However, longitudinal studies that examine the development of preferences for certain resources in relation to other variables would be desirable.

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Raison d'être of mathematical works and epistemological responsibility in inquiry: the case of the Fibonacci sequence under various moduli

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This study explores raison d'être of mathematical works and epistemological responsibility of students in university mathematics education, analysing the case of inquiry into the Fibonacci sequence under various moduli. Employing the anthropological theory of the didactic, it examines how the students construct the milieu through the inquiry. As a result, the mathematical works were learned in a manner different from the commonly disseminated teaching style. However, while the students were given some responsibility for constructing the milieu, the teacher often determined the necessary mathematical works, indicating challenges in fully delegating epistemological responsibility to students. The findings suggest the possibilities for inquiry-based approaches into university mathematics education.

Keywords: Teachers' and students' practices at university level, transition to, across and from university mathematics, anthropological theory of the didactic, didactic paradigm, milieu construction.

INTRODUCTION: THEORETICAL BACKGROUND

Research on university mathematics education has paid significant attention to the problem of transition, namely, discontinuity between secondary-level and tertiary-level mathematics, particularly in the context of teacher education (e.g., Gueudet et al., 2016). Our study also addresses this issue based on the *anthropological theory of the didactic* (ATD). The reason for adopting ATD is that, as will be detailed below, it helps to identify the research problem in university mathematics education related to the transition, and provides approaches to address the issue. Over the past decade, research within ATD has used the concept of the *didactic paradigm* to clarify the problematic (e.g., Bosch et al., 2018). These studies have referred to two typical didactic paradigms: the *paradigm of visiting works*, which has been widely disseminated, and its counterpart, the *paradigm of questioning the world*.

The core issue noted with the paradigm of visiting works are the disappearance of the *raison d'être* of the objects (to be) learned and the reduction of the *epistemological responsibility* of students. In this paradigm, a curriculum is composed of a sequence of mathematical works constrained by various factors. Students learn such works as if visiting monuments, treating them as valuable in themselves. As a result, these works have lost their function within the original body of knowledge. ATD refers to this as the disappearance of the *raison d'être* of the works (cf. Chevallard, 2015). Besides this, students are considered to be taking on epistemological responsibility when they are deciding, on their own initiative, how to construct the necessary environment to solve problems or answer questions. In the traditional paradigm, students merely follow teacher's guidance in visiting works, and thus have little epistemological responsibility.

The paradigm of questioning the world has been proposed as an alternative. In this paradigm, learning principally occurs in *inquiry*. ATD defines inquiry in a broad sense, as a process in which students, with teachers' support, generate their own answers to an initial question. These questions and answers involve not just a single simple question and its corresponding answer, but rather a chain of derivative questions and partial answers. In an inquiry process, students may study works from books or the Internet (called *media* in ATD), or obtain data through experiments and simulations. ATD refers to a set of derivative questions, partial answers, works, and data as (didactic) *milieu*. Students continue to construct their milieu and, at the same time, interact with it to generate the final answer. Such a process is conceptualized as the *study and research path* (SRP). This concept emphasizes the dialectical development of the two activities of study and research in inquiry.

Milieu is a key concept for considering *raisons d'être* of works and the epistemological responsibility of students. When construction of the milieu is undertaken with the purpose of "answering the question," the works learned in the inquiry can be said to have a *raison d'être*. Furthermore, when students are responsible for selecting and deciding how to construct the milieu, such as choosing which media to reference or which existing answers to apply, they are considered to be assuming the epistemological responsibility for the final answer. Therefore, delegating the responsibility for milieu construction to students within inquiry appears to be a promising approach to addressing the issue raised by the paradigm of visiting works.

Research question

Our research targets *book-club-style* mathematics seminars that are typical in Japanese university mathematics education. In this format, a small group of students reads a book together with a teacher. Typically, the book to be read is preselected by the teacher at the time the seminar is designed, and it is customary to read the book sequentially from the first page. Textbooks in mathematics are logically organized in a sequence where many definitions and propositions are introduced because they are needed for proving theorems that appear later. This implies that the students often find the *raison d'être* of these works to be unclear and have low epistemological responsibility towards the constructed knowledge.

This paper discusses a case where a teacher, who had traditionally used this style, redesigned the seminar as SRP to overcome these issues. The SRP was implemented at a Japanese national teacher training university for 2 years. It formed part of a seminar for third-year undergraduate students specializing in mathematics education. This SRP was initially planned within the area of elementary number theory and the topic chosen was the Fibonacci sequence under various moduli. The initial question of the inquiry was formulated as follows: *What mathematical properties hold for the Fibonacci sequence under various moduli?* The aim of our study is to describe and analyse, from the perspective of milieu construction, the *raison d'être* of the works and the students' epistemological responsibilities in the SRP. This serves to demonstrate the potential and possibilities of incorporating inquiry into university mathematics education. From this, the following research question emerges:

> RQ: In the context of a university mathematics seminar focused on the Fibonacci sequence under various moduli, how and for what reasons is a milieu constructed, and how is the responsibility for its construction distributed among members in the SRP?

METHODOLOGY

Didactic engineering

This research adopts the methodology of *didactic engineering* (Artigue, 2020). In the process of didactic engineering, an *a priori analysis* describes how the SRP can develop from the initial question. It is common to represent the development of the SRP using a tree structure known as a *Q-A map* (Winsløw et al., 2013). In addition to this, in this paper, we also consider mathematical works that can become a component of the milieu. A priori analysis allows us to predict the possible development of an SRP prior to its implementation, enabling us to contemplate what occurred and what did not in the actual SRP, as well as its characteristics. Then, *in vivo* analysis is conducted. The development of the implemented SRP is described as the actualized Q-A map. This analysis is conducted continuously over the long-term implementation period of the SRP. A posteriori analysis involves the consideration of the characteristic phenomena. In this study, a posteriori analysis can be seen as equivalent to the "Discussion" section in a general research paper. By comparing the prediction made in the a priori analysis with the reality of the implemented SRP, it is possible to highlight the characteristics of the results. Furthermore, the a posteriori analysis includes the perspective of *ecological analysis* in ATD. This involves examining the factors that led to such characteristics, i.e. the conditions and constraints.

Participants and procedures for data collection and analysis

This paper focuses on a two-year SRP conducted in a mathematics seminar for two third-year undergraduate students at a national teacher training university. At the end of the second year, the students presented about what they had studied to their supervisor and another professor in the department. The analysis covers a total of 30 periods of teaching in the first year, 13 periods in the spring semester and 17 periods in the autumn semester (each period of 90 minutes).

One of the two students aspired to be a primary school teacher, while the other aspired to be a high school mathematics teacher. Before starting this SRP, the students had taken two algebra courses that covered mathematical knowledge such as properties of integers (e.g., Euclidean Algorithm, prime factorization), modular arithmetic (e.g., residue classes,the Chinese Remainder Theorem), and matrices. However, they did not understand them well enough to apply their knowledge to this inquiry. The teacher (the second author) is a mathematics researcher specialising in algebraic combinatorics. In his previous teaching experience, he conducted seminars by book-club-style. This was the first time he conducted a seminar in the form of an inquiry.

In this SRP, the students worked on inquiry basically in pairs. The teacher intervened in their inquiry as needed, for example, by providing questions or presenting the media needed for their inquiry. Moreover, although they worked on inquiry primarily in the seminars, the teacher often provided homework when needed, and they worked on inquiry individually outside of the seminars.

This SRP was initially planned within the area of elementary number theory and the topic chosen was the Fibonacci sequence under various moduli. The initial question Q_0, was "What mathematical properties hold for the Fibonacci sequence under various moduli?" This SRP was conducted over a total of 20 periods, spanning the latter part of the spring semester (3 periods) and the entire autumn semester (17 periods). During the first 10 periods of the spring semester, a preliminary investigation of the mathematical properties of the Fibonacci sequence was conducted.

Data was collected by recording the dialogues of the students and the teacher. And, the first and third authors also observed the seminars as observers and collected memoranda of the seminar observations, photographs of the blackboard, and students' notebooks. Based on these data, we describe how the inquiry was actually addressed.

A PRIORI ANALYSIS: A POSSIBLE DEVELOPMENT OF THE SRP

At the start of the inquiry, a typical question derived from Q_0 could be Q_1: "Is the Fibonacci sequence modulo any integer $n \geq 2$ always periodic?" Concrete data from the Fibonacci sequences modulo some integers obtained through manual calculations, spreadsheet software, or programming will play an important role. It is expected that concepts related to the pigeonhole principle and elementary number theory such as congruences, will be used in response to Q_1. By learning and applying such mathematical works as needed, a positive response can be obtained; that is, the answer A_1: "It is always periodic."

Subsequently, more detailed properties related to periodicity can be explored. For instance, considering the Fibonacci sequence modulo n and denoting the length of its period as $\pi(n)$, a question such as "Is it true that $n \neq 2 \Rightarrow \pi(n)$ is even?" ($Q_{2.1}$) can be formulated. This question can be considered specific example of a broader question, Q_2, which asks about the patterns in the lengths of the periods. Further questions, like "Does $\pi(p)$ have any particular properties or patterns for primes p?" ($Q_{2.2}$), "Is there a relationship between $\pi(p^k)$ and $\pi(p)$?" ($Q_{2.3}$), or "Is there a relationship between $\pi\left(p_1^{e_1}\, p_2^{e_2} \cdots\, p_l^{e_l}\right)$ and $\pi\left(p_i^{e_i}\right)$?" can be derived.

The answers to the questions $Q_{2.1}$ to $Q_{2.4}$ can be summarized as follows. First, $Q_{2.1}$, formulated as a conjecture, is positively resolved using mathematical induction ($A_{2.1}$). $Q_{2.2}$ focuses on the periods of the Fibonacci sequences modulo primes. For this question, properties like "for a prime p in the form $5p \pm 1$, $\pi(p) | p - 1$" or "for a prime p in the form $5p \pm 2$, $\pi(p)|2(p+1)$" are possible answers ($A_{2.2}$). Uncovering these properties likely requires some familiarity with elementary number theory. This is because one must not only focus on primes, but also classify these primes according to the residues modulo 5 and pay attention to the divisibility by the length of the period $\pi(p)$. These properties can be proven using Fermat's Little Theorem and the Law of Quadratic Reciprocity. While tackling $Q_{2.2}$ may appear challenging due to the need for some works in elementary number theory, the inquiry from $Q_{2.2}$ to $A_{2.2}$ becomes plausible considering references to media. Students can engage with some webpages, applying its works to their own milieu, advancing their inquiry while constructing knowledge.

While $Q_{2.3}$ may derive from $Q_{2.2}$, typically, it would be deduced by focusing on periods for prime powers such as $\pi(2^k)$ or $\pi(3^k)$, based on the generated data. It is anticipated that $Q_{2.3}$ will spontaneously be engaged with ease. This is because the students are expected to be familiar with focusing on prime powers. As an answer to this question, $A_{2.3}$, it can be considered that $\pi(p^k) = p^{k-1}\pi(p)$. In fact, this conjecture remains an unsolved problem, and no general proof has been presented yet. However, this does not mean that students cannot tackle it in their inquiry; it is conceivable that they could construct proofs for some specific prime numbers.

Question $Q_{2.4}$, based on the uniqueness of prime factorization, is a natural question to consider in the inquiry, and the exploration of such questions should be encouraged. An answer $A_{2.4}$ to this question could be expressed as follows: $\pi\left(p_1^{e_1}\, p_2^{e_2} \cdots\, p_l^{e_l}\right) = \left[\pi\left(p_1^{e_1}\right), \pi\left(p_2^{e_2}\right), \ldots, \pi\left(p_l^{e_l}\right)\right]$, where the square brackets mean the least common multiple of several numbers. In fact, more generally, for any two natural numbers m and n, $\pi([m,n]) = [\pi(m), \pi(n)]$ holds. This is something that students can discover through interactions with the milieu.

The proof of this property requires works of elementary number theory, such as the Chinese Remainder Theorem. The students under consideration in this paper had previously taken lectures on elementary number theory. Therefore, it is expected that they can advance their inquiry by applying known works as one of the elements within the milieu. However, there may be situations where known works cannot be effectively applied, and in such cases, it is anticipated that some form of media reference, such as learning through Wikipedia's "Chinese Remainder Theorem" article or a textbook on elementary number theory, may occur.

The above is an a priori analysis of the expected inquiry process. When summarized as a Q-A map, it appears as shown in Figure 1. Each question is labelled briefly to indicate its content, and the typical components of the milieu that will be constructed during the inquiry process are also included. In particular, the data obtained as a result of simulations are expected to have an interactive role, contributing to various questions and answers, rather than serving a one-time purpose.

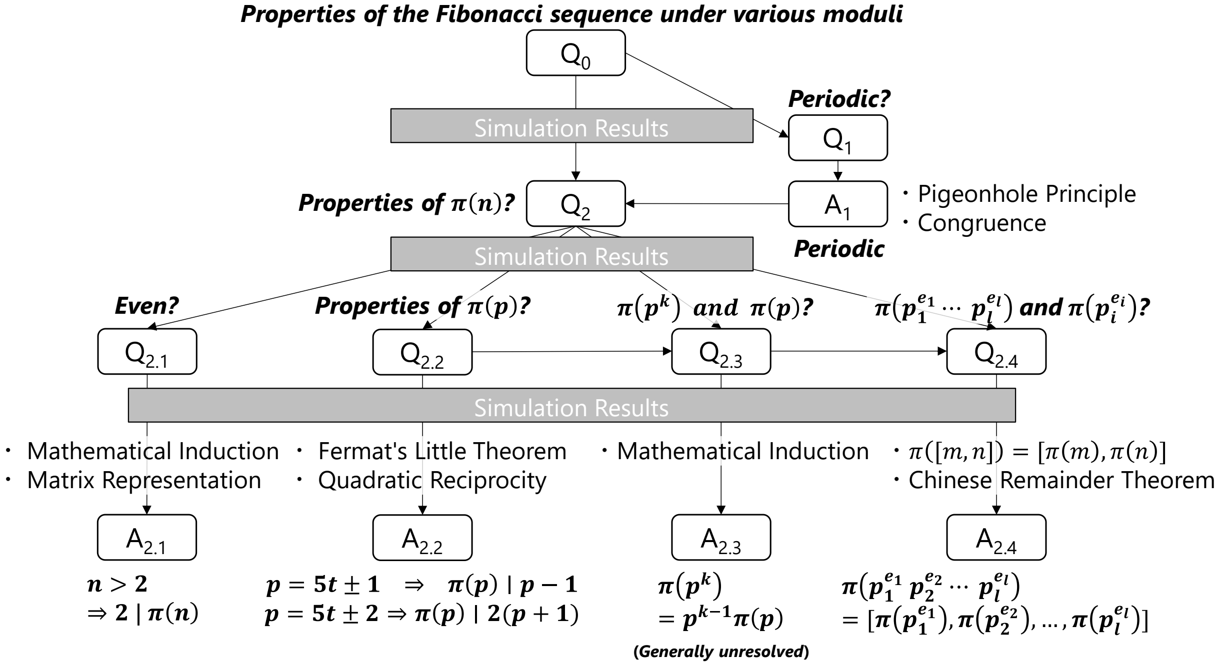

Figure 1: The Q-A map for the possible process of the inquiry

In the following sections, we will describe the realised SRP and analyse its characteristics through a comparison with the results of the a priori analysis.

DESCRIPTION OF THE SRP IMPLEMENTED

Phase 1: Emergence of Questions Related to $Q_{2.4}$ and $Q_{2.1}$

Initially, the teacher introduced the Fibonacci sequence under some moduli and presented the question Q_0: "What mathematical properties hold for the Fibonacci sequence under various moduli?" Over four periods, mathematically formulated conjectures were gradually developed.

The students first manually calculated and recorded the Fibonacci sequences modulo some integers (the left side of Figure 2). They quickly noticed their periodicity. This observation led to the conjecture Q_1, related to periodicity. The teacher engaged the students in discussions about the structure of the recurrence relations and the pigeonhole principle, sharing ideas for their proofs (A_1). The students then attempted to use spreadsheet software (Excel) for calculating the Fibonacci sequences modulo more integers, aiming to gather more data. However, they realized that with large numbers, the expression of the calculation result became insufficient in spreadsheet software. Consequently, the teacher suggested using the computer algebra system Maxima for simulation. Subsequently, they calculated and outputted the lengths of the periods for integers from 2 to approximately 100, and investigated their characteristics. The right side of Figure 2 shows a part of the code and output from the program actually used.

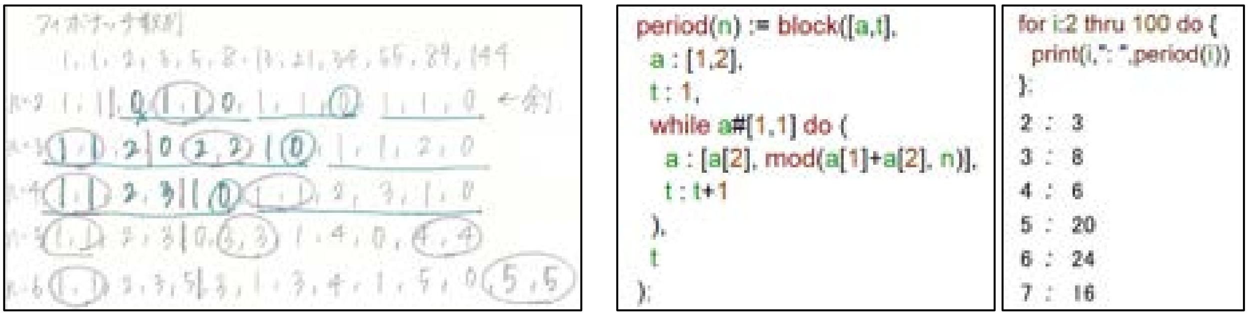

Figure 2: The Fibonacci sequences modulo some integers manually calculated (left side) and the length of the periods calculated by Maxima (right side).

An agreement was reached to explore the following two conjectures, which the students derived from their simulation results: α) $n|m \Rightarrow \pi(n)|\pi(m)$, β) $n \geq 3 \Rightarrow 2|\pi(n)$. Attempts were made to search for related information on the internet, but no relevant Japanese literature was found. Searching with terms like "Fibonacci period modulo" led to Marc Renault's website, and from there, his master's thesis on the Fibonacci sequence under various moduli was discovered and downloaded (Renault, 1996). This thesis contained sections related to the above conjectures. The teacher introduced a theorem related to Conjecture α, formalized as $\pi([m,n]) = [\pi(m), \pi(n)]$. However, it was not introduced as an already established theorem. The teacher made students create some specific examples to help them realize that this general claim seems plausible. Following this, the students took on this claim as a conjecture and set out to prove it. We will refer this conjecture as α′.

At this point, α′and β concerning $Q_{2.4}$ and $Q_{2.1}$ respectively were conjectured. The subsequent inquiry would proceed while reading the master's thesis as the main media.

Phase 2: Inquiry into $Q_{2.4}$ and $Q_{2.1}$

The proof of $\pi([m,n]) = [\pi(m), \pi(n)]$ was constructed while reading and understanding the thesis. In the seminar, the students explained the proof based on the thesis, and the teacher asked questions and provided clarifications. The following fact is notable at this stage: rather than reading the master's thesis from beginning to end sequentially, the students and the teacher often read the thesis retrospectively while searching for mathematical works necessary for the proof of Conjecture α′. The mathematical works, such as other theorems mentioned in the thesis and the Chinese Remainder Theorem, were not pre-learned, but became elements of the milieu as needed. Through this process, the students found that the equation $\pi\left(p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}\right) = \left[\pi\left(p_1^{e_1}\right), \pi\left(p_2^{e_2}\right), \ldots, \pi\left(p_l^{e_l}\right)\right]$ holds. This equation was what we had presented as the answer $A_{2.4}$ to $Q_{2.4}$ in the a priori analysis. That is to say, in the actual SRP, contrary to our anticipated sequence, $Q_{2.4}$ was resolved before other questions (such as $Q_{2.1}$).

During the process of constructing the proof of Conjecture α′, it was discovered that the proposition formulated in Conjecture β, "$n \geq 3 \Rightarrow 2|\pi(n)$", was also proved as a theorem within the same thesis. Following this discovery, the proof of β was constructed.

Phase 3: Inquiry into $Q_{2.3}$

The final phase is $Q_{2.3}$, namely, the inquiry into the equation $\pi(p^k) = p^{k-1}\pi(p)$. As mentioned above, this is an unsolved problem, and a general proof has not yet been discovered. The teacher developed activities by starting with having the students construct specific examples, eventually formalizing the conjecture. Subsequently, a proof was constructed for the case of $p = 2$. Since the proof was included in the same thesis, students followed that proof. It is noteworthy that, even here, the way of tracing back the works necessary for the proof continued. For example, properties of the Fibonacci sequence such as $F_{2n} = F_n(F_{n-1} + F_{n+1})$, $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$, etc., were learned and utilized in the proof for the case of $p = 2$.

DISCUSSION: A POSTERIORI ANALYSIS

Construction of the milieu with *raison d'être* of the mathematical works

One of the characteristics of the SRP is the way in which the milieu was constructed, particularly the order of it, which differs from that of a typical seminar-style mathematics lesson. The SRP may superficially to be similar to a mathematics seminar in a book-club format, because one particular medium—the master's thesis in our case—held a privileged position and the students read it with the support of their teacher. However, in this SRP, the mathematical works were frequently employed in a sequence opposite to that in the traditional style. The students read the master's thesis by going back to the necessary sections and learned the works as needed. A typical example of this characteristic is the properties of the Fibonacci sequence used in Phase 3. These properties could be presented as valuable in their own right. However, in this SRP, they were learned and utilized because they were necessary for the inquiry. This indicates that the mathematical works learned had a clear function in answering the questions of the inquiry, that is, they had the *raison d'être*.

The way in which the milieu was constructed in this case is different from the knowledge development that Barquero et al., (2013) described as an *applicationism* phenomenon. Under the epistemology of applicationism, mathematically fundamental works should be learned before being applied into extra-mathematical situations. Although the authors used this term in the context of applied mathematics, we can expand its scope to encompass all of mathematics. Namely, applicationism can be understood as the view that any work should be learned before being applied in some situations. It can be said that many logically-organized mathematics textbooks are based on this epistemology, and thus, the problematic phenomena caused by applicationism, namely, the dilution of *raison d'être* of mathematical works, can fully occur within inner-mathematical context as well. The above suggests that redesigning university mathematics courses through SRPs will be a promising approach to addressing the issue of transition, even within the context of pure mathematics.

Construction of the milieu under the epistemological responsibility of students

Another characteristic of this SRP is that a part of the responsibility for constructing the milieu was devolved to the students. A typical instance was the teacher asking students at each juncture of the inquiry, "What kind of question would you like to consider next?" The two questions $Q_{2.1}$ and $Q_{2.4}$, formulated in Phase 1, were based on several conjectures proposed by the students. Furthermore, at the end of Phase 2, a similar question was asked, leading to the proof of a theorem on prime powers. The way in which questions appeared and the order in which they were tackled were different from what was shown in the a priori analysis, which is an indication that some responsibility was devolved to the students. On the other hand, there was not much autonomous reference to media by the students. This suggests that another part of the responsibility for constructing the milieu was monopolized by the teacher. Specifically, this refers to the responsibility of determining at certain points in the inquiry which mathematical works might be necessary and/or which might not be useful. The teacher occasionally instructed, "Try reading this part of the thesis", and the students followed these instructions to learn the works or follow the proofs. Then, the students did not seek out additional information that might be useful for the inquiry beyond the section.

This implies that there was little occurrence of *adidactisation* (Chevallard & Strømskag, 2022). When the milieu is described as adidactic for students, it means that the students are *not* trying to figure out the teacher's didactic intentions behind the milieu. As this definition shows, adidactisation is crucial in delegating epistemological responsibility of inquiry to students. The students, in the SRP, largely relied on the teacher to decide where to find the elements of the milieu. In other words, the milieu was constructed through the students' assumption that "it must be necessary for our inquiry because the teacher has presented it at this moment." In this sense, the milieu was not adidactisized for them, Therefore, in terms of media usage, much of the epistemological responsibility lay with the teacher, not the students.

One of the constraints on the adidactisation would be the existing epistemology which the students have spontaneously formed through their mathematics learning under the paradigm of visiting works. Under this paradigm, media to be referenced are predetermined, and furthermore, the timing of their referencing is also planned in advance. Such learning experiences are likely to lead students to the view regarding knowledge construction that "media and milieu are presented by the teacher when needed." To update such an epistemology, long-term new mathematical experiences across both secondary and higher levels of mathematics are necessary.

CONCLUSION

This study examined how the milieu was constructed and how the responsibility for its construction was distributed among members in a university mathematics seminar framed as an SRP. The analysis revealed two key characteristics of the SRP. First, the mathematical works were learned and utilized in a sequence that differed from the typical seminar in a book-club style. This suggests that the SRP can help address the issue of the disappearance of the *raison d'être* of mathematical works, which is a common problem in traditional university mathematics education. Second, while the students were given some responsibility to formulate questions and direct the inquiry, their agency in constructing the milieu was somewhat constrained. This indicates that fully delegating the epistemological responsibility to students is challenging, likely due to their existing epistemological views shaped by past experiences in the paradigm of visiting works. The findings suggest that incorporating inquiry-based approaches like the SRP into university mathematics education holds promise for addressing the transition issue. However, more research is needed to further understand how to foster students' autonomy in constructing the milieu and taking on full epistemological responsibility within such inquiry processes. Longitudinal studies investigating how students' mathematical epistemologies evolve through experiencing diverse inquiry-oriented activities across secondary and tertiary levels would be particularly valuable.

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Students' understanding of spanning set from the APOS Theory

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Students often learn procedurally how to demonstrate whether a given set of vectors is a spanning set or to find their span by manipulating matrices via row-reduction. However, they may need help to really understand these concepts mathematically. We thus implemented an activity designed by the Inquiry-Oriented Linear Algebra project following APOS Theory and analysed the results of this implementation using this same theory. We worked with a group of university students enrolled in a course in linear algebra. Results obtained showed that most students stated related span with the geometry of space, and that when asked about the set being a basis they did not consider the need of the vectors to be linearly independent.

Keywords: Teaching and learning of specific topics in university mathematics. Teaching and learning of linear and abstract algebra, Spanning set, Span, APOS theory.

INTRODUCTION

Linear algebra has become an indispensable university course in engineering and science majors due to its applications in different disciplines (Trigueros & Wawro, 2020). Research has been conducted to study students' obstacles in teaching and learning linear algebra concepts from different theoretical perspectives (Dorier, 2000; Stewart et al., 2018). Stewart and colleagues (2019) surveyed the current state of linear algebra research. Researchers mention that geometry has often been present in teaching and learning linear algebra, some of them claimed that using geometry improved students' learning experience. Others underline that student do better in routine algebra exercises.

Nardi (1997) studied students' conceptual and reasoning difficulties in mathematics during their first year of study. Six students participating in tutorials were encouraged to describe the terms spanning set and span in their own words or, if they wished, to draw a picture and explain it. The researcher identified that the dominant conceptual image of students is that a spanning set represents a basis and that most students confuse the terms spanning set and span, using one instead of the other.

Ku et al. (2008) designed a genetic decomposition of the basic concept. The researchers interviewed six students who took a linear algebra course designed with APOS theory. One of his questions aimed to observe how the student argued about the concept of dimension: when does a given set of vectors form a basis in $\mathbb{R}^2$, $\mathbb{R}^3$ and P_2 (polynomials grade 2)? The researchers observed that students did not consider the fact that the vectors in a spanning set must belong to the vector space or subspace they span. Moreover, it was easier for students to know if a set of vectors forms a basis for a given vector space than to find a basis for a given vector space.

Wawro et al. (2012) designed an instructional sequence called "Magic Carpet Ride (MCR)" to be used by teachers in their classrooms. The researchers used a travel metaphor in $\mathbb{R}^2$, where a person starts from the origin of a Cartesian space and travels to a given location (x, y), which could be changed in another activity, using two different means of transportation, a magic carpet and a hoverboard, that travel each in a specific direction. They also asked participants to determine if there is a place where someone can hide, at a point that cannot be reached using the same two given means of transportation as in the previous problem, and if using the same means of transportation, it is possible to get back home. These materials were designed to help students understand linear combinations, span, and linear in/dependence. According to the authors, students developed an intuitive understanding of the desired concepts using these tasks.

Carcamo et al. (2018) designed a learning environment in the context of looking for secure passwords for students to construct the concepts of spanning set and Spanned space. They observed the work of seven first-year engineering students on this problem. They report that students started by using their prior conceptions of vectors and linear combinations. By scaling a matrix whose rows are vectors in $\mathbb{R}^2$, a team of four students suggested that if the matrix rank is 2, they "can" span all $\mathbb{R}^2$ space. The researchers mention that "[They] indicate to us that they left open the possibility for there to be a set for which the associated matrix has rank 2 but does not span $\mathbb{R}^2$ (p. 211)". Students then observed that two vectors are not enough to conclude whether such a set spans $\mathbb{R}^2$ but that those two vectors should also be linearly independent.

THEORETICAL BACKGROUND

APOS theory aims to study the learning of mathematics and is also used to teach mathematics; this dichotomy of teaching and learning is not separate; it is an intrinsic dialectic of the theory. This theory focuses on students' mental constructs when learning a mathematical concept (Arnon et al., 2014) and comprises four mental structures: Action, Process, Object, and Schema. An individual with a conception of Action carries out operations guided by external algorithms and procedures without demonstrating control over the situation. When an Action is repeated and reflected upon, it can be interiorized into a Process. A Process is recognized by students no longer needing to perform step by step of a procedure as they can recognize the result of applied such Actions. When a Process has been constructed it is possible to go back to the Actions that gave rise to it. Thinking about operations being applied to a Process makes it possible for the individual to thinks of the Process as a whole and encapsulate it in an Object. Once an Object has been constructed, it is possible to perform Actions on it, such as finding or analysing its properties. A Schema is a collection of Actions, Processes, Objects, and other Schemas. When students start studying a topic, they use the Schema they have constructed before to start constructing the new concept. A Schema develops when relations among its component structures are formed.

Taking into account student's difficulties with spanning set, span and basis (Kú et al., 2008; Nardi, 1997), our research question is: What constructions about the concepts of spanning sets and spanned set were evidenced by university students who were introduced to these concepts through the Magic Carpet problem?

METHODOLOGY

APOS theory proposes a research cycle consisting of three elements: theoretical analysis, design and implementation of instruction, and data collection and analysis. These three components are very closely linked. In this theoretical perspective, research begins with a theoretical analysis of the involved concepts to develop a model called genetic decomposition (GD). The GD makes it possible to describe how a student can construct that concept using the structures of the theory (Action, Process, Object, Schema). This analysis leads to the design and application of a teaching strategy, which aims to make sure that students construct the proposed structures in the initial theoretical analysis: Finally, data is collected from students work throughout the application of the APOS didactic strategy (collaborative work on activities, whole class discussion, exercise) by means of questionnaires and/or interviews. Data obtained was transcribed, organized and analysed by each member of the research team. They then compared and negotiated among them until they came to an agreement.

Sierpinska (2004) consider that research reports rarely provide sufficient detail about the task design or the variables involved in their research process, that few studies justify the choice of a task or identify those characteristics that are essential and those that are not relevant to the study. She considers a result, many aspects leading to results obtained remain hidden. When a task is created from scratch or a modified version of another task, task design principles may vary (Watson & Ohtani, 2015). In this research, we use an Inquiry-Oriented Linear Algebra project activity, "Magic Carpet Ride (MCR)," adapting it to the use of APOS Theory through the use of activities the same as Kú et al., genetic decomposition (2008) in a group of university students in a linear algebra course. We aimed to discover how students use the MCR problem to construct the notions of spanning set, span and basis.

This study focused on 19 students (11 enrolled in actuarial science, 5 in mathematics, and 3 in applied mathematics) taking for the first time a first linear algebra course at a Mexican public university in the spring of 2023. The teacher of this course was one of the authors of this paper. Students worked on each activity individually; then, in class, they discussed it in teams of three; during their teamwork, the teacher visited the teams, asked the students questions and provided clarifications where necessary. Each team had to present the work of each of its members and a single collective work to note. The teacher then led a discussion with the whole group and gave the students homework to do at home at the end of the lesson. At the end of the semester, one student from each of the six teams was chosen to be interviewed. We used an instrument proposed in Kú et al., (2008) consisting of seven multi-task questions to conduct semi-structured interviews. Each question was designed to test specific mental constructions. Each interview lasted between 40 and 60 minutes and was audio recorded. Students' work was collected, and all data were coded and analysed independently by the researchers and then negotiated among them. The following are some selected interview questions. Students discussed and wrote their answers during the interview.

2. Ley W be a subspace of $\mathbb{R}^3$consisting in points (x, y, z) such that $x + 3y - 4z = 0$. a) Find a spanning set for W consisting of two vectors; b) Can a set with only one vector span W? c) Can a set of three vectors span W?

3. Let v_1, v_2 and v_3 the vectors in next graph

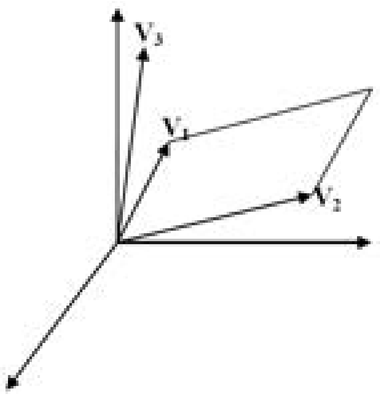

Figure 1. Graphical representation of problem 3.

Is it possible for vector v_3 to be expressed as a linear combination of v_1 and v_2?

4. If you have vectors $\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and the set $H = \left\{\begin{bmatrix}s\\s\\0\end{bmatrix} : s \in \mathbb{R}\right\}$. Then, every vector in H can be written as a linear combination of $\{\vec{v}_1, \vec{v}_2\}$, because

$$\begin{bmatrix}s\\s\\0\end{bmatrix} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Is $\{\vec{v}_1, \vec{v}_2\}$ a spanning set for H?

5. Let $v_1 = (1,0,1)$ and $v_2 = (0,1,1)$. Let S be the set of all linear combinations of v_1 and v_2. Represent S graphically. Find a vector v_3 that is not an element of S.

6. Tell me if each of my statements is true or false and why.

a) Let W be the set of vectors spanned by $\{(1,2,1), (1,0,2)\}$. If $(-1, -6, 1) \in W$, then $\{(1,2,1), (1,0,2), (-1, -6, 1)\}$ spans the same set W.

b) Consider vectors $u, v, w \in \mathbb{R}^2$. If $u = 2v + 4w$, is it possible to be sure that $B=\{v,w\}$ is a basis for $\mathbb{R}^2$?

ANALYSIS OF DATA

We present the results of the analysis of each question. In Problem 2, most students recognized that the equation $x + 3y - 4z = 0$ represents a plane in $\mathbb{R}^3$ passing through the origin. They also did Actions on of a system of linear equations when they checked whether a point satisfies the equality $x + 3y - 4z = 0$.

Student 5: [Writes (1,1,1)]. By substituting the values, I know that x is 1, y is 1, z is 1, and 1+3-4 is zero. The substitution gives zero.

This student does Actions with the given vector to find out if it is on the plane and considers that he has responded to the question asked by doing so. All the other students did Action to find a vector belonging to the plane.

When asked to provide a spanning set for W with two vectors, this student and another student did Actions to provide a vector belonging to the plane and then discussed if they spanned W:

Student 5: [Gives the vectors (1,1,1) and (2,2,2)]. These vectors do not span a plane since they are on the same line.

Student 6: The vectors [Writes (1,1,1) and (3,3,3)] satisfy the given equation. Nevertheless, these two vectors are collinear [He writes (0,1,3/4)]. The two vectors must be linearly independent.

Other two students recognized that two linearly independent vectors can span a plane. In question 2c, three students mentioned that three vectors cannot span the plane, arguing that three vectors span $\mathbb{R}^3$. All of these students showed memorized Actions in their responses. However, two students indicated that more vectors can span a plane since "it depends on what those vectors are because they can span a line, a plane, or $\mathbb{R}^3$". Only one student showed the need to know specific vectors to answer the question. Other students responded:

Student 2: A single vector could not do it because we are in $\mathbb{R}^3$, so we need at least three. Therefore, A single vector can not span W ... To span W, we would need at least two…Could three vectors span W? They can, they could, but ... It only takes two.

Student 3: To span $\mathbb{R}^3$, I need all three vectors. So, with just one vector, we cannot span W. We can span it with two or three vectors, but not just one. The third could be a linear combination of these vectors.

Student 6: A single vector could not span W ... If only one vector exists, it would be like a line on this plane. With three vectors, yes, they would be linearly dependent. One of them would be a linear combination of the other two.

These responses show that students are aware that the spanning set of W can have a third element but that it has to be a linear combination of two vectors belonging to W. Students (S2 and S5) mention that at least two vectors are needed to span W. They demonstrate a spanning set and span as Processes, considering that all linear combinations that span W can be found for this set. Most students considered that two or more vectors can span a plane but did not mention linear dependence.

In problem 3, all six students answered that v_3 cannot be expressed as a linear combination of v_1 and v_2, justifying that v_3 is not in the given plane. As can be seen in the following extracts

Student 3: I think no ... because v_1 and v_2 span a plane, no, because it would have to belong or be in the plane.

Student 4: No, because v_1 and v_2 span a plane unrelated to v_3.

Student 5: No, since graphically, v_3 is not located in the plane spanned by v_1 and v_2.

Most students use the Actions to verify whether the given vectors belongs to the plane. Only student five showed the construction of a Process of span by considering the whole span v_1 and v_2.

In problem 4, most students recognized that the span for v_1 is a line and that the span for v_2 is also a line but was confused by the fact that those lines were different from the line spanned by the vector $\begin{bmatrix} S \\ s \\ 0 \end{bmatrix}$. They knew that both vectors $\{v_1, v_2\}$ span a plane and that the spanned set of H is another line.

Student 4: With two non-collinear vectors, we obtain a plane. The set H is a line. These two [v_1 and v_2] are distinct and span a plane. They cannot span a line.

Student 5: v_1 spans a line, v_2 equals a line, and v_1 and v_2 do not span the same line; they both span a plane. H is a line. It is a spanning set for H because both [v_1 and v_2] can be combined to span the vector H.

Student 6: v_1 and v_2 span H because they can be expressed as a linear combination.

We consider that S4 shows the construction of a Process for span and spanning set. As he shows the possibility of comparing spanning sets, it may be possible that he is in transition to span as an Object. Since S5 and S6 do linear combinations with v_1 and v_2, they realize that they can reach all the points in H with those combinations. However, they do not consider that v_1 and v_2 span $\mathbb{R}^2$, and that their linear combinations include H and many other points, so as v_1 and v_2 are not in H, they are not in the spanning set for H. So, S5 and S6 show they still need to construct this spanning set Process in this problem. Interestingly, in question 3, S5 showed some evidence of a Process construction, while in this question, he contradicts that reasoning by doing only Actions. In their explanations, we consider that they use only Actions in their responses, so they constructed an Action conception of spanning set and span.

Problem 5 is similar to problem 3, except that the student is asked to find a vector v_3 that does not belong to the plane spanned by the set of v_1 and v_2. In this question, we observed whether the students could use a geometric representation of the spanned plane. We also observed whether they were able to coordinate the linear combination and spanning set Processes. Most students could draw the two vectors in three-dimensional space and used the parallelogram method to represent the plane spanned by these vectors. Four students drew the third vector with an out-of-plane orientation, indicating "a vector that is not in this region." we considered they were referring to the spanned space for the set $\{v_1, v_2\}$ (see figure 2).

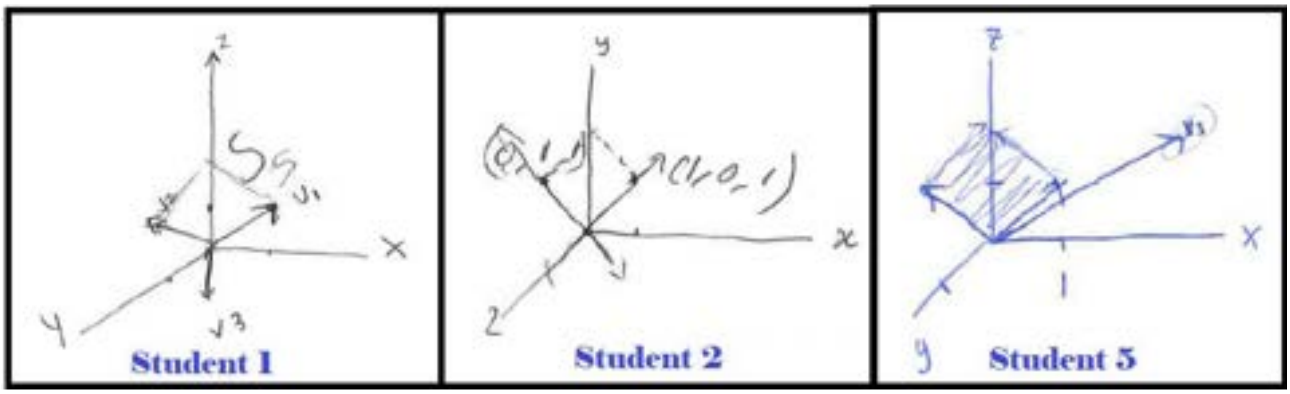

Figure 2. The third vector with an out-of-plane orientation in space.

In problem 6a, students should understand that the number of elements does not determine whether or not they span the given vector space, as shown in the following comments by students:

Student 3: It would be necessary to test if one of the vectors is a linear combination of the other two to see if they span the same set.

Student 4: The statement tells us that this vector [(-1,6,1)] can be represented as a linear combination of these two [(1,2,1) and (1,0,2)]. It would then be a dependent set which spans the same set.

We can observe that student 4 accepts different spanning sets for the same space; he also shows that he has constructed a Process for span by stating that v_3 can be expressed as a linear combination of the set {(1,2,1), (1,0,2)}. The linear combination, linear dependence, and spanned set processes must be coordinated to answer this question. Student 5 responded in a similar way as student 4. The other students did use Actions in this response.

In Problem 6b, we were interested in students' arguments regarding linear combination, spanning set, linear in/dependence and basis of a vector space, and the possible coordination among these Processes. We consider the following excerpts.

Student 3: For them to be basis, they must be linearly independent and span ... we could give them values to demonstrate it.

Student 4: To know if they are basis, I need to check if they span and are linearly independent.

In problem 6b, three students indicated the statement was true; arguing that since v and w can be written as a linear combination and assuming they are linearly independent, we can be sure the set is a basis. Two students mentioned that for the set B to be a basis, one would need to know explicitly the ordered pairs of v and w to verify if they are linearly independent. No student mentioned that v and w can be equal or multiple one of the other, in which case they do not span $\mathbb{R}^2$.

DISCUSSION AND CONCLUSIONS

This research used an Inquiry-Oriented Linear Algebra Project activity called "Magic Carpet Ride (MCR)" to introduce the notions of spanning set and span to students in a first linear algebra course. We use APOS theory to analyse the responses of a group of university students. Interestingly, none of the six students referred to the MCR scenario during our interview to justify or explain their responses as was the case in the original MCR study.

The results of this study show that most students considered that if the vectors in stake are in $\mathbb{R}^3$, a single vector spans a line; two vectors span a plane, and a minimum of three vectors are needed to span $\mathbb{R}^3$. We found that, in spite of the use of genetic decomposition based activities, most students' responses involve the use Actions. That can be related to the genetic decomposition need to be refined to foster students' reflection. Only two students demonstrated some hints of having constructed Process related to spanning set and span.

We observed that for most students it is easier to do Actions on given sets of vectors. The construction of Processes implies reflection on those Actions and the possibility to reverse it when needed. Our results show that only two students constructed spanning set and span as Processes. More activities are needed to foster a deeper understanding of these concepts by using MCR problems. Such activities must be designed with APOS Theory but considering MCR's affordances. This practice may help students to interiorize Actions on MCR into Processes and leave aside their need to memorize facts. More studies are needed to test the last hypothesis.

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Explorando concepciones previas sobre números complejos a través de la resolución de ecuaciones cuadráticas en estudiantes universitarios

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Los números complejos constituyen un concepto que, en México, su estudio se propone en el nivel medio superior, dirigido a estudiantes de entre 15 y 18 años. Posteriormente, su estudio continúa en la universidad. Desde la teoría APOE, es fundamental identificar las concepciones previas que han construido los estudiantes para dar paso a la construcción de nuevas estructuras mentales. En este sentido, a través de la resolución de tareas que demandan resolver ecuaciones cuadráticas, se busca identificar las concepciones sobre los números complejos que han construido estudiantes que ingresan a la universidad. Como resultado, se encontró que los alumnos no han desarrollado estructuras mentales asociadas a los números complejos antes de ingresar a la universidad, a pesar de ser propuestos para su enseñanza.

Palabras clave: números complejos, ecuaciones cuadráticas, alumnos universitarios, concepciones, teoría APOE.

INTRODUCCIÓN

Las investigaciones que se han realizado sobre la enseñanza y aprendizaje de los números complejos desde la educación matemática señalan que estos números son enseñados comúnmente a estudiantes mayores de 15 años (Bagni, 2001; Pardo y Gómez, 2007). Además, según Aznar et al. (2010) la instrucción de estos números se ha limitado a su tratamiento en un registro de representación algebraico -en el sentido de Duval (2006)-, comúnmente en su forma binómica $(a + ib)$. Asimismo, pese a su limitada instrucción se han reportado dificultades para la comprensión de esos números (Randolph y Parraguez, 2019; Bagni, 2001). Esto resalta un problema en la educación, especialmente en las carreras universitarias donde el dominio de los números complejos es esencial. La instrucción centrada exclusivamente en un único registro de representación descuida otros aspectos fundamentales de estos números. Desde el sentido de la teoría APOE, esto ocasiona problemas en la formación de estructuras mentales que enriquecen las concepciones sobre este concepto.

Es comprensible que la enseñanza de los números complejos comience desde su tratamiento en un registro algebraico, ya que, desde una perspectiva histórica de las matemáticas, en el siglo XVI, matemáticos como Cardano promovieron la existencia de los números complejos al resolver ecuaciones de segundo grado con el método de completar el cuadrado. Hay registro de aceptar aplicar la operación de raíz cuadrada a cualquier número que se obtenía con dicho método, dando origen a los números imaginarios (Kline, 1992). Este suceso histórico es común que se tome en cuenta en la enseñanza, tal como lo menciona Bagni (2001).

Una revisión a los planes de estudio y libros de texto del nivel medio superior en México dejó ver que la enseñanza de los números complejos se propone a partir de la solución de ecuaciones cuadráticas (Secretaría de Educación Pública [SEP], 2015). Sin embargo, se ha planteado la hipótesis de que, a pesar de su propuesta, este tema es poco enseñado en el nivel medio superior. Esto lleva a la pregunta ¿cuáles son las concepciones previas sobre los números complejos que han construido los alumnos que ingresan a ingeniería? En consonancia con esta pregunta, se ha planteado como objetivo: caracterizar, desde la teoría APOE, las concepciones de los estudiantes de primer semestre de ingeniería sobre los números complejos, asociadas a la resolución de ecuaciones cuadráticas.

MARCO CONCEPTUAL

Como marco de investigación para el desarrollo de este estudio se asume la teoría APOE (acrónimo de Acciones, Procesos, Objetos y Esquemas). Esta teoría fue propuesta por Dubinsky y sus colaboradores, y es entendida como un modelo que permite caracterizar las concepciones matemáticas de los estudiantes a través de la construcción de ciertas estructuras mentales, las cuales son: las acciones, procesos y objetos que se disponen a través de esquemas. Esas estructuras se forman a partir de la activación de mecanismos para su construcción, los cuales son por ejemplo la interiorización, coordinación, reversión, encapsulación y des-encapsulación (Dubinsky y McDonald, 2001).

Aunque las estructuras mentales poseen una secuencialidad, es decir, van de acciones a procesos, de procesos a objetos y se materializan en esquemas, el aprendizaje de un individuo no sigue necesariamente esa secuencia. Incluso, Oktaç et al. (2021) sugieren que el progreso de un individuo desde una estructura mental a otra no es inmediato, pudiendo haber transiciones entre ellas, por ello es necesario reconocer niveles o momentos de transición. Para el caso de esta investigación se identifican elementos de las estructuras mentales para hacer referencia a esos niveles.

De acuerdo con Arnon et al. (2014), para la construcción de nuevas estructuras mentales de un individuo, se comienza con acciones sobre objetos previamente construidos. Por ello, es fundamental indagar sobre las concepciones que los alumnos han construido sobre los números complejos, en particular a partir de la instrucción para la resolución de ecuaciones cuadráticas. Como lo mencionan García et al. (en prensa), este concepto es fundamental para definir la unidad imaginaria y comprender los números imaginarios, así como establecer relaciones entre este conjunto numérico y el de los números reales. Las concepciones previas sobre las ecuaciones cuadráticas son un punto de partida importante para construir estructuras mentales fundamentales, asociadas a las formas binómica $(a + ib)$ y par ordenado (a, b) de un número complejo, partiendo de un registro algebraico, pero extendiéndose a un registro geométrico, tal como lo describen García et al. (en prensa) en la Figura 1.

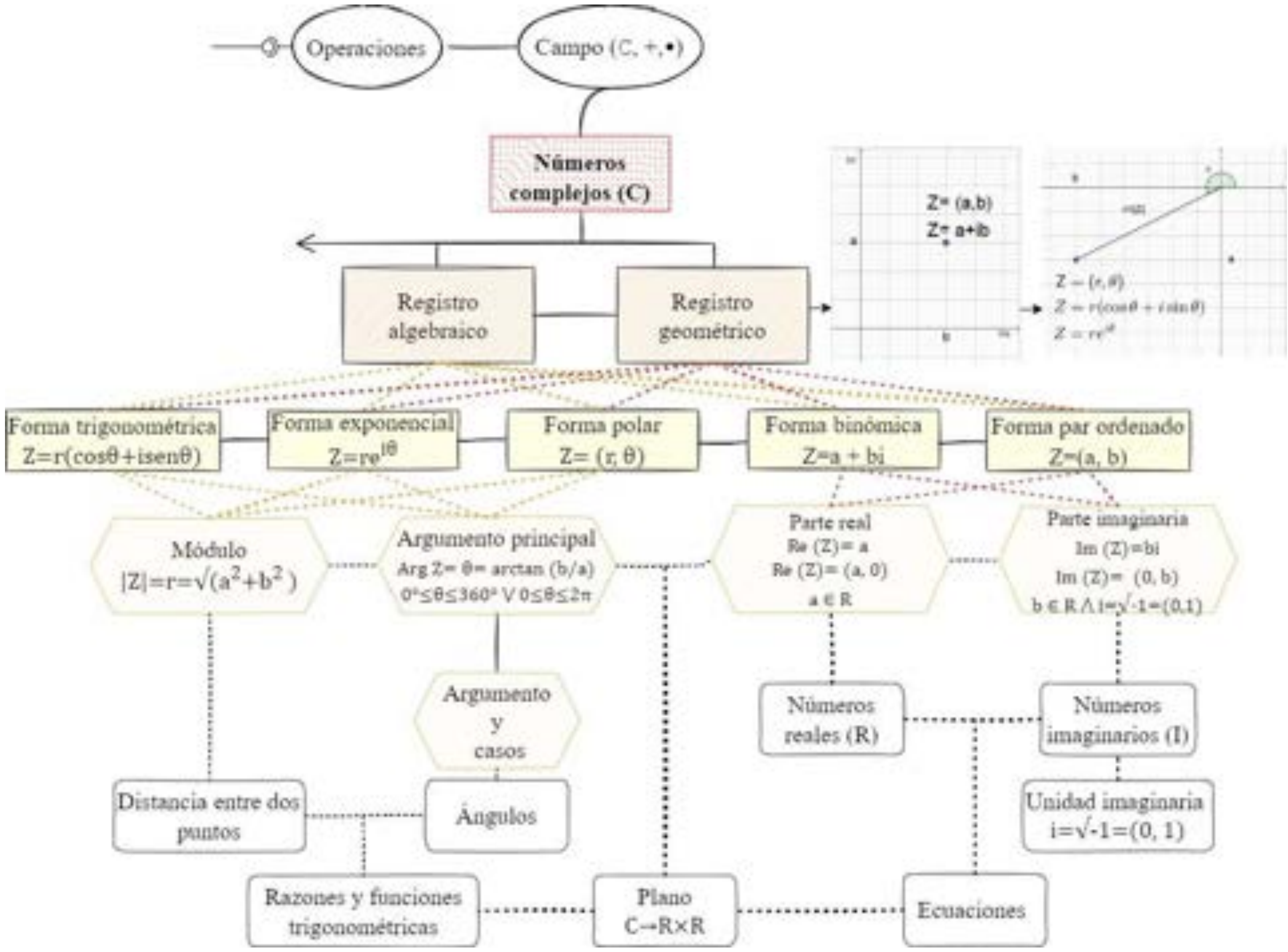

Figura 1: Concepciones previas asociadas a los componentes de los números complejos (tomado de García et al., en prensa).

Así, las estructuras mentales que pudieran identificarse en este estudio están relacionadas con los componentes de los números complejos, como son la parte real y la parte imaginaria a partir de la solución de ecuaciones cuadráticas.

METODOLOGÍA

La teoría APOE contempla un aspecto metodológico, ya que para los estudios que se hacen bajo este marco se propone un ciclo de investigación que consta de tres componentes (Asiala et al., 1997). El primero es el análisis teórico, el cual permite entender la epistemología de un concepto y determinar las estructuras y mecanismos que necesita el estudiante para comprenderlo; al resultado de este análisis se le llama descomposición genética. El segundo componente implica la creación y ejecución de actividades educativas; la descomposición genética actúa como guía, dado que es un modelo que orienta la generación de nuevas estructuras mentales para fortalecer las concepciones de los aprendices. El tercer elemento abarca la recopilación y el análisis de datos, permitiendo así la comparación entre el modelo proporcionado por la descomposición genética y los resultados derivados del análisis de las actuaciones del estudiantado. Este ciclo de investigación es recurrente y puede repetirse según sea necesario para realizar ajustes y mejoras en la investigación.

En particular, los resultados de este trabajo contribuyen al análisis teórico de un ciclo de investigación de un estudio más amplio. Como ya se mencionó, se estudian las concepciones previas que tienen los estudiantes que ingresan a la universidad sobre los números complejos asociadas a la resolución de ecuaciones cuadráticas.

Grupo de estudio.

En el estudio participó un grupo integrado por 17 estudiantes de nuevo ingreso a la carrera de ingeniería mecánica eléctrica de una universidad pública en México, con edades entre 18 y más años. Los estudiantes son egresados de diversas instituciones del nivel medio superior en México, pero en su mayoría del estado de Jalisco de dicho país.

Material y técnicas para la recopilación y el análisis de datos.

El análisis que se realiza en esta investigación es de tipo descriptivo, se centra en las actuaciones de los estudiantes. Para la recolección de datos, se aplicó de forma presencial al inicio del semestre -por el contexto universitario mexicano- una pregunta en el que los estudiantes debían formular una ecuación cuadrática y resolver, en este caso, las soluciones son números complejos (Figura 2a). Dicho cuestionamiento está basado en el estudio de Pardo y Gómez (2007).

a) Cuestionario al inicio del semestre

Instrucciones:

Resuelve el siguiente problema sin buscar información o comentar con tus compañeros. El problema debe ser resuelto durante el tiempo establecido y subir la respuesta al Classroom, ya sea a través de una foto o un archivo Pdf, pero que todo sea legible. El archivo debe contener todo el procedimiento y tu nombre.

Problema:

Encontrar dos números cuya suma sea 10 y cuyo producto sea 40.

b) Parte del cuestionario de mitad de semestre

Cuestionario Parte 1

Nombre: ____________________

Instrucciones

Sin hacer búsqueda de información o consultar cualquier fuente, en una hoja en blanco y usando pluma de color azul, responde lo siguiente. Todos los procedimientos que hagas deben estar en la hoja.

1. Resolver la siguiente ecuación $x^2 - 2x + 5 = 0$

Figura 2: Cuestionario aplicado a los estudiantes: a) al inicio del semestre, b) a la mitad del semestre.

A la mitad del semestre, después de haber estudiado ecuaciones cuadráticas y antes de iniciar formalmente la instrucción sobre los números complejos se aplicó otro cuestionario, de este se tomó en cuenta un cuestionamiento que involucra la resolución de una ecuación cuadrática con soluciones complejas (Figura 2b). Las respuestas de ambos cuestionamientos fueron subidas y almacenadas en Google Classroom.

RESULTADOS

Los resultados se organizan en dos partes. Primero se expone sobre las actuaciones del estudiantado al responder el primer cuestionamiento que implica modelar y resolver una ecuación cuadrática. Posteriormente se expone el análisis de las respuestas al segundo cuestionamiento, en el cual se debe resolver una ecuación cuadrática.

Respuestas al primer cuestionamiento.

Este cuestionamiento fue respondido por todo el grupo de estudio. Como puede verse en la Figura 3, ningún estudiante obtuvo como respuesta las soluciones complejas de la ecuación cuadrática que se debía modelar. Intencionalmente, en las instrucciones del problema no se especifica el dominio, a fin de identificar si los alumnos reconocen que las soluciones de las ecuaciones cuadráticas pueden ser números complejos.

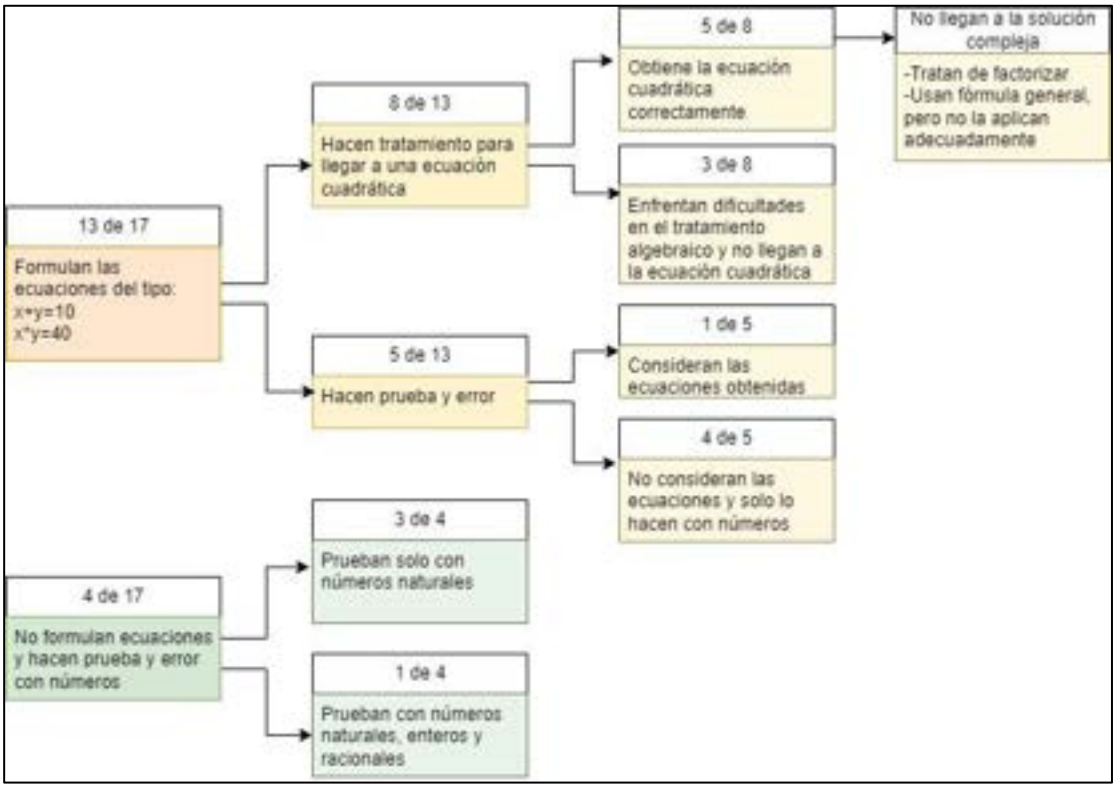

Figura 3: Clasificación de respuestas al cuestionamiento de inicio de semestre.

Además de no haber identificado concepciones asociadas a los números complejos, pudo observarse que 2 de los aprendices que lograron modelar la ecuación cuadrática tienen dificultades para su resolución (Figura 4a). Sin embargo, en dos de los casos (Figura 4 c y d), parece que sus errores fueron provocados por no saber qué hacer con el signo negativo dentro de una raíz, por ello cambiaban los signos a su conveniencia para tratar de dar una solución. En cambio, el estudiante que intentó resolver la ecuación por el método de factorización asegura que no existe solución sin especificar el dominio (Figura 4b); para él, de manera general, las ecuaciones cuadráticas pueden no tener solución.

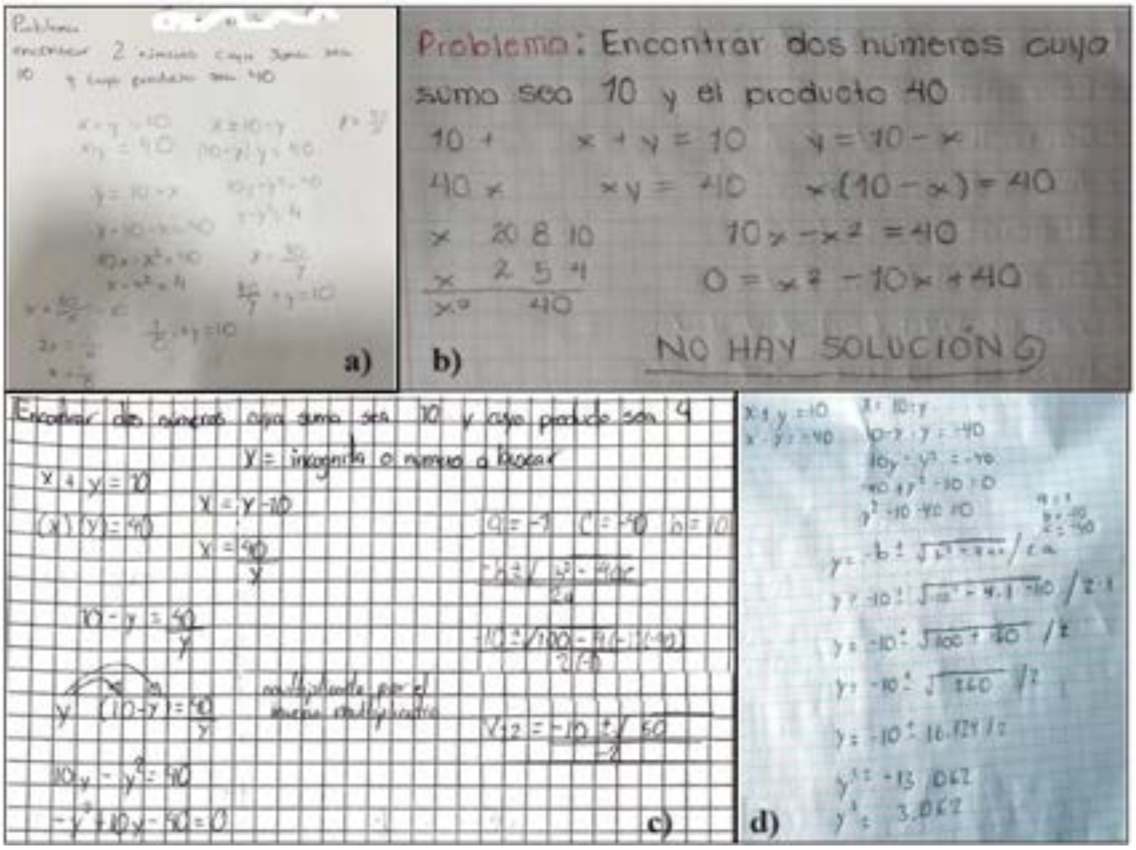

Figura 4: Respuestas de los estudiantes que modelaron la ecuación cuadrática.

Entre los 5 estudiantes que formulan las ecuaciones del tipo $(x + y = 10;\ x * y = 40)$ pero que hacen prueba y error, se tienen dos casos. El primero se refiere a los alumnos que dejan expresadas esas ecuaciones, pero después no hacen un tratamiento algebraico con ellas, sino que prueban solo con números naturales o enteros (Figura 5a).

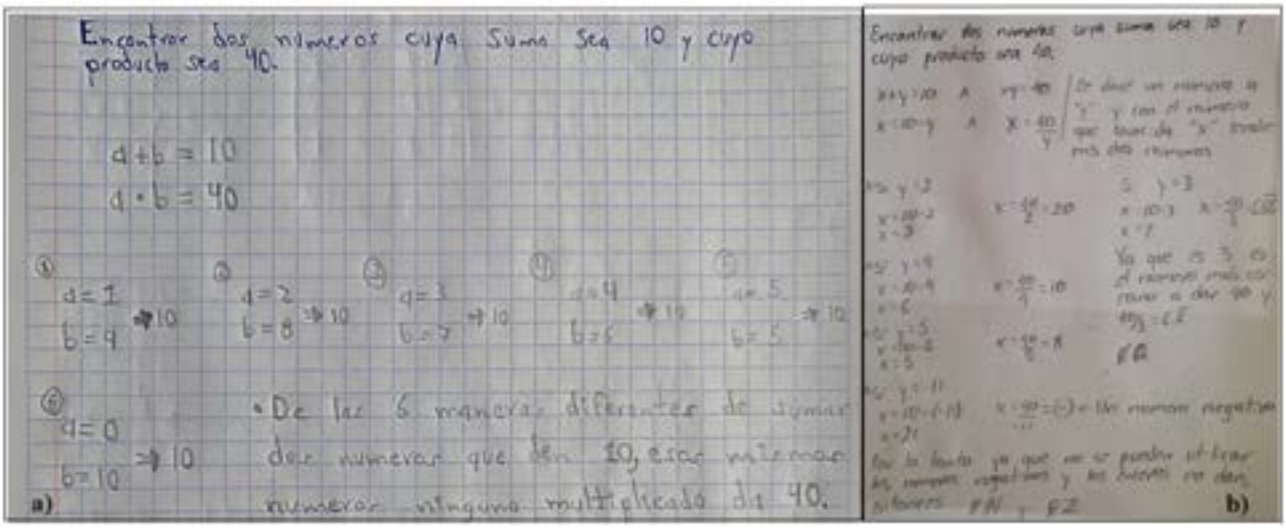

Figura 5: Respuestas de estudiantes que plantearon ecuaciones e hicieron prueba y error.

El otro caso se refiere a un alumno que utiliza esas ecuaciones, hace un tratamiento algebraico despejando la misma variable en ambas ecuaciones para que una de las variables sea dependiente de la otra. Posteriormente prueba con valores arbitrarios de la variable independiente; su análisis lo hace por casos en distintos conjuntos numéricos, pero no en los complejos, sino que como se ve en la Figura 5b, solo hace referencia a los números naturales, enteros y racionales.

Por último, cuatro estudiantes no formularon ecuaciones y directamente se dispusieron a probar con números. Tres de ellos solo con números naturales y un estudiante también probó con números enteros y racionales, similar a los procedimientos de la Figura 5, pero sin formular las ecuaciones.

Los resultados de este primer cuestionamiento permiten concluir que el grupo de estudio, al ingresar a la universidad, no ha desarrollado concepciones sobre los números complejos asociados a la solución de ecuaciones cuadráticas. Hay evidencia de dificultades, pero también hay concepciones que a través de una instrucción pueden permitir al estudiantado construir estructuras mentales sobre los números complejos.

Respuestas al segundo cuestionamiento.

Como ya se mencionó, el segundo cuestionamiento que se hizo al estudiantado fue unas semanas después de haber sido instruidos en el tema de ecuaciones cuadráticas, y antes de comenzar el estudio formal y amplio sobre los números complejos. Sin embargo, cuando se estudió el tema de ecuaciones cuadráticas se discutió un caso en el que la solución de la ecuación no era real, sino compleja. Así, el segundo cuestionamiento requiere la solución de la ecuación $x^2 - 2x + 5 = 0$. En las respuestas (Figura 6), ya se observaron concepciones sobre los números complejos.

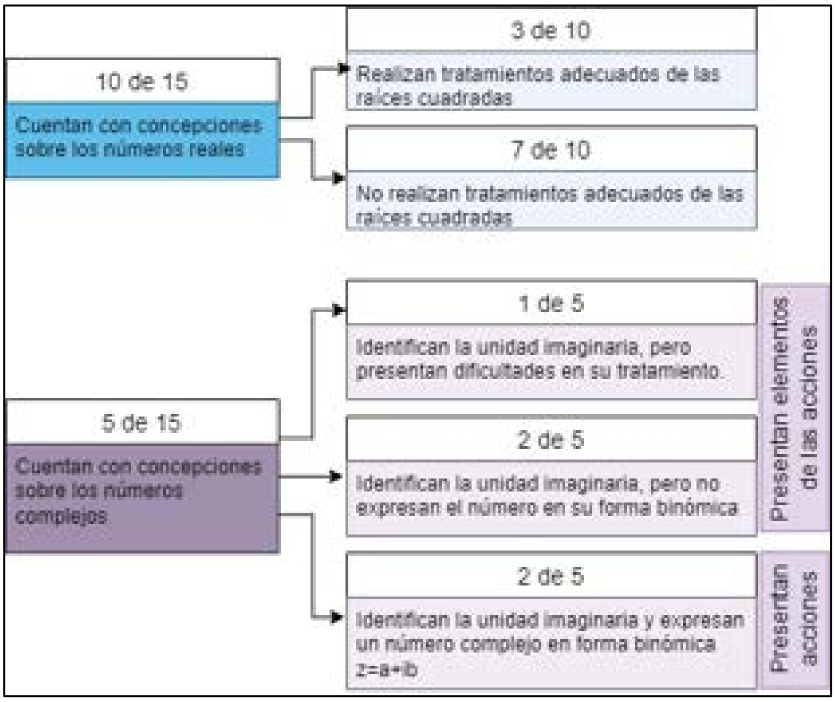

Figura 6: Clasificación de respuestas al cuestionamiento de mitad de semestre.

La pregunta fue respondida solo por 15 de los 17 estudiantes, ya que dos de ellos desertaron. Las respuestas se dividen en dos grupos principales (Figura 6). El primero está conformado por 10 de los 15 estudiantes y se caracteriza por no presentar estructuras mentales asociadas a los números complejos. El segundo grupo está compuesto por 5 de los 15 estudiantes, ellos presentan estructuras mentales asociadas a dicho concepto, en particular hay evidencia de elementos de las acciones y acciones.

En el primer grupo, 3 de los 10 estudiantes presentan concepciones sobre las ecuaciones cuadráticas en el dominio de los números reales, que les permite realizar tratamientos para intentar presentar una solución de la ecuación. Sin embargo, ellos no reconocen la unidad imaginaria en dichas soluciones, y las dejan expresadas con raíces negativas (Figura 7a). En cambio, en este mismo grupo, 7 de los 10 estudiantes tienen problemas con los tratamientos de las ecuaciones cuadráticas, en particular al aplicar la fórmula general para su resolución o al realizar operaciones (Figura 7b).

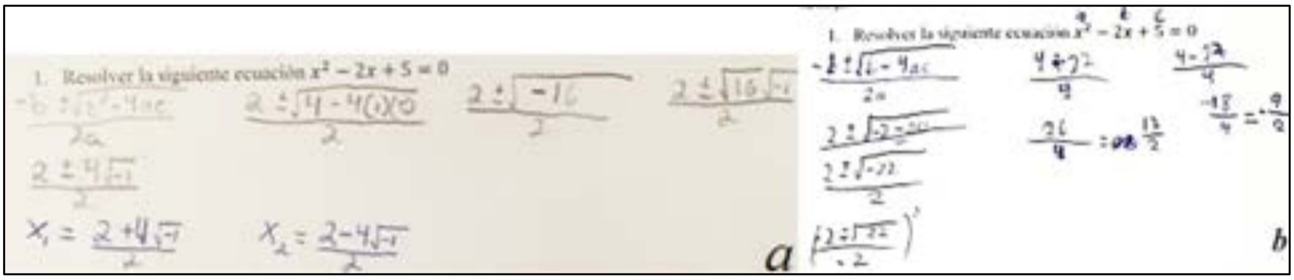

Figura 7: Respuestas de estudiantes que no muestran concepciones sobre los números complejos.

Pese a que en el grupo dos los 5 estudiantes muestran concepciones asociadas a los números complejos, éstas se distinguen como elementos de las acciones o acciones. Para la clasificación de respuestas en el grupo 2 se identificaron tres casos. En el primero, se observa que 1 de los 5 estudiantes reconoce la unidad imaginaria, pero aún no distingue la parte real de la parte imaginaria de un número complejo, es decir, no los interpreta como dos elementos que constituyen al número complejo. Por lo anterior, este alumno realiza los tratamientos de las operaciones erróneamente, ya que suma la parte real con la parte imaginaria (Figura 8b). Este tipo de errores también ha sido descrito por Aznar et al. (2010).

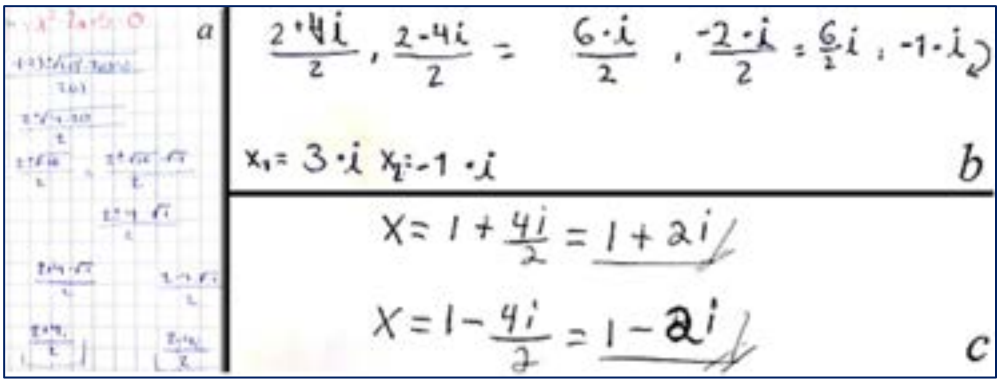

Figura 8: Respuestas de estudiantes que muestran concepciones sobre los números complejos.

El segundo caso lo presentan 2 de 5 estudiantes, ellos muestran reconocimiento por la unidad imaginaria, reconocen que la solución no está en los números reales, y realizan un tratamiento algebraico adecuado para dar solución a la ecuación (Figura 8a). Ellos no separan la parte real de la parte imaginaria, es decir, aún no expresan al número complejo en su forma binómica. Sin embargo, no cometen el error de sumar o restar los elementos que constituyen a esos números, como lo hizo el alumno del primer caso.

Las concepciones de los estudiantes de los casos uno y dos (Figura 8 a y b) se identifican como elementos de las acciones, porque aún no se interiorizan para representar a un número complejo en la forma binómica. Sin embargo, esas concepciones son fundamentales para la construcción de acciones.

El tercer caso se presenta con 2 de los 5 estudiantes, quienes muestran la solución de la ecuación cuadrática como un número complejo expresado en su forma binómica (Figura 8c). Además, ellos identifican la raíz cuadrada de -1 como i, con lo que realizan las operaciones respectivas. Por lo anterior, las concepciones de estos alumnos se pueden describir como acciones.

CONCLUSIONES

De acuerdo con los resultados del primer cuestionamiento, se puede reforzar la hipótesis que se plantea en esta investigación, es decir, que a pesar de que los números complejos se proponen para su enseñanza desde el nivel medio superior en México, los estudiantes ingresan a la universidad sin concepciones sobre este tipo de números, específicamente como posibles soluciones de ecuaciones cuadráticas. Esto debido a que el grupo de estudio no presentó evidencia sobre posibles estructuras mentales asociadas a los números complejos. Incluso, hay evidencia sobre la concepción de que, si no hay solución en los reales, entonces la ecuación no tiene solución. Además, los alumnos que lograron modelar la ecuación cuadrática presentan errores para aplicar la fórmula general, así como dificultades para su interpretación.

Para la enseñanza de las ecuaciones cuadráticas desde el nivel medio superior, es fundamental plantear ecuaciones cuya solución son números complejos. Esto es una pauta importante para activar mecanismos mentales que permite al estudiante construir nuevas estructuras, tales como acciones o elementos de las acciones asociadas a la parte real y parte imagina de un número complejo, tal como se expone en la Figura 1.

Con la explicación de un solo ejemplo de una ecuación cuadrática con soluciones complejas, unas semanas después, los alumnos evidencian concepciones sobre los números complejos. Hay elementos de las acciones cuando los alumnos 1) hicieron explícito que no existen raíces cuadradas de números negativos y concluyeron que la solución no pertenece al conjunto de los números reales, 2) identifican la unidad imaginaria, aunque no hayan realizado tratamientos adecuados. Hay muestra de acciones cuando los alumnos pudieron expresar un número complejo específico indicando la parte real e imaginaria escrita en su forma binómica. A su vez, basado en la instrucción, esas concepciones son fundamentales para la construcción de otras estructuras mentales asociadas a otras formas de representar a los números complejos.

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Students’ conceptions on the notion of a polynomial

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Polynomials are essential objects in mathematics, and hold a fundamental place in education, particularly at the secondary-tertiary transition. Our research focuses on students' conceptions about polynomials, and we present the results of a questionnaire covering three main themes: definition and recognition of a polynomial, equations solving and roots of polynomials, polynomials functions and changes of semiotic register. We identify schemes of high-school and higher education students, and highlight potential underlying difficulties.

Keywords: Teaching and learning of linear and abstract algebra, Transition to, across and from university mathematics, Curricular and institutional issues concerning the teaching of mathematics at university level, Teachers’ and students’ practices at university level, Polynomials and polynomial functions.

POLYNOMIALS AT THE SECONDARY-TERTIARY TRANSITION

The transition between secondary and tertiary education is an issue highlighted by mathematics education research. This transition has been identified as a source of numerous challenges, and potential ruptures. Research pointed out some recurring difficulties faced by students, along with epistemological and didactical obstacles (Gueudet, 2008). Thus, general patterns have been identified, complemented by research on specific subjects such as linear algebra (Dorier, 1997), probabilities (Doukhan, 2020), or functions (Vandebrouck, 2011). However, we found limited research on the teaching and learning of the notion of polynomial, although it is a fundamental concept in mathematics, being at the crossroad of multiple fields.

Initially underlying polynomial equations, polynomials then become elements of polynomial algebras. In analysis, polynomial functions are used as reference functions. They are important tools in numerical analysis, due to their computational properties and regularity that make them easier to study and implement. The universal property of polynomial algebras provides a specialization morphism that allows to consider matrix or endomorphism polynomials for instance. The diversity of points of view makes polynomials a central concept in the teaching of analysis and algebra in France. Students first encounter with polynomials happens in junior high school, through the manipulation of first-degree algebraic expressions. The concept of polynomial is still developed in tertiary education, with the definition and use of formal polynomials and will lead to formal series as well as to algebraic geometry.

However, even though they may seem basic, students often struggle to understand and use polynomials, which complicates various tasks for them. Building upon this observation, we aim to identify potential continuities and ruptures in the teaching of polynomials, by examining students' conceptions about them. We address this question by presenting an experiment carried out in 2023 (Veuillez--Mainard, 2023). We first provide a brief overview of prior research related to polynomials, and sum up the stakes identified in a curricular analysis we conducted. Then, we describe the methodology of our experiment and present an *a priori* analysis of the questionnaire we designed. Finally, we share the results of the experiment.

PREVIOUS WORKS ON POLYNOMIALS

Previous studies highlighted that the teaching of polynomials is likely to involve challenges stemming from various domains. Most of studies on polynomials either focus on the teaching of specific families of polynomials (tangent lines (Montoya Delgadillo et al., 2016), second-degree polynomials (Chaachoua et al., 2022)), or on the relations between these families (Buck, 1995). Polynomials are also encountered in literature on other subjects: expansion and factorisation of algebraic expressions (in the beginning of high-school), real functions, calculus, Taylor expansions … This implies a wide array of potential difficulties and obstacles in the teaching of this notion. Specific studies on these objects and their teaching are therefore necessary.

Bolondi et.al (2020) demonstrated that the very definition of a polynomial can be ambiguous in textbooks, by analysing the *definition schemes* used for terms: variable, algebraic expression, literal equation, algebraic sum and monomials. Dede and Soybas (2011) examined the *concept images* of polynomials for preservice mathematics teachers. They identified several concept images in conflict with the formal definition of a polynomial, with regard to the definition domain of coefficients, or to the degree of a polynomial for instance. They also showed that some students define polynomials as equations. In her PhD, Plestina (2023) conducted a study of the teaching and learning of polynomials in Croatia. She describes the genesis and development of the notion of polynomials in mathematics and carries out an analysis of the knowledge to be taught in several institutions.

Following on from these studies, we aim to identify students' conceptions of this subject at the transition from secondary to tertiary education.

CONTEXTUAL ELEMENTS ON THE FRENCH CURRICULUM

The French secondary high school is divided in three classes: seconde (10^{th} grade), première (11^{th} grade), terminale (12^{th} grade). In tertiary education, mainly two institutions offer advanced mathematics courses: university and “classes préparatoire” (preparatory classes), the latter leading students to engineering schools. In high school and “classes préparatoires”, the curriculum is national whereas each university chooses its own syllabus. We summarize in Table 1 the contexts where students meet polynomials, from 11^{th} grade to the beginning of tertiary education.

It should be noted that in high school, only the notion of a polynomial function is introduced, and called a “polynomial”:

> “The notion of a polynomial function can be freely used, more simply called polynomial” (Ministère de l’Éducation nationale et de la Jeunesse, 2019)

11^{th} grade - mathematics major	12^{th} grade - mathematics major	12^{th} grade - option "experts" mathematics	Scientific "classes préparatoires" (first year of higher education)
- Solving in $\mathbb{R}$ of second-degree equations - Reference functions for derivation and table of variation	- Reference functions for limits, continuity, convexity and primitives	- Solving in $\mathbb{C}$ of second-degree equations - Definition of any real polynomial function	- Formal polynomial, arithmetic of polynomial rings - Example of vector space - Reference function in calculus - Reference approximation function (Lagrange polynomials, series expansions…).

Table 1: curriculum regarding polynomials

We sum up here the stakes of the teaching of polynomials that we identified (Veuillez-Mainard, 2022). In high school, all the textbooks' tasks deal with polynomials of degree 2 or 3, and exceptionally of degree up to 6. The properties at stake are mostly specific to second-degree polynomials, e.g. in quadratic-equation solving. In tertiary education, students encounter high-degree polynomials, explicit polynomials of arbitrary degrees, and arbitrary polynomials. Moreover, polynomials appear though new mathematical theories, such as polynomial arithmetic, vector spaces or series expansions. Formal polynomials are introduced in higher education, sometimes with different definitions in different classes. Finally, we noticed that in 12^{th} grade "spécialité mathématiques", no new specific work on polynomials is initiated. Polynomial functions are supplanted by other reference functions, and it is not explicit in the textbooks that these new functions are not polynomial functions.

METHODOLOGY AND RESEARCH QUESTIONS

Theoretical framework and research questions

Our research aims to investigate student's conceptions on polynomials. For this purpose, we use the *theory of Conceptual Fields* (Vergnaud, 2009). It offers a cognitivist perspective on didactic questions, which appears to be relevant for studying individuals' conceptions. We rely on the concept of *scheme*, that emphasizes the operational aspect of knowledge, as "the invariant organization of activity for a certain class of situations" (Vergnaud, 2009, p. 88). In our study, we investigate schemes through the search of operational invariants, such as theorems-in-action (propositions held as true by the students in their activity) or action rules (implicit rules that guide the action of the student). In this framework, to address the various representations of polynomials, we will draw on the *semiotic registers* (Duval, 2017).

This leads us to investigate three research questions: What are the students' schemes on polynomials? In what way do these schemes evolve during the secondary-tertiary transition? What specific challenges do students face in the learning of polynomials?

Methodology

In order to identify students' conceptions, we designed two questionnaires, one for high school and one for higher education, which allowed us to collect a great number of data to analyse. The questionnaires address similar tasks, considering the academic contexts, so as to be able to compare the answers. We hypothesize that contrasting the answers to the two questionnaires is relevant to understand some of the issues of the learning of polynomials at the secondary-tertiary transition. The themes addressed were guided by the issues raised by literature, the epistemological analysis of the subject, and the analysis of the French curriculum. We conducted an *a priori* analysis of the questionnaires, in order to identify which operational invariants are likely to appear, and then compared this analysis with the collected answers.

For high school, the questionnaires were submitted to 11^{th}- and 12^{th}-grade students following advanced mathematics options. For tertiary education, it was given to students in first year of bachelor of mathematics and to students in economics "classe préparatoire", who also learn advanced mathematics. Overall, the sample is composed of 31 high-school students and 87 higher education students.

Design of the questionnaire

The questionnaire was designed around 3 themes: definition of a polynomial, solving of polynomial equations and roots, and polynomial functions. The final questionnaire for university students can be found in the appendix of this document.

In the first theme, we investigate the student's proposal for a definition of a polynomial, and the recognition of polynomials. In order to identify students' operational invariants in the second task, we selected polynomial and non-polynomial expressions with various characteristics: linear combinations involving square roots; reciprocal and rational functions; polynomials with integer, rational, and irrational coefficients; polynomials of low, high and arbitrary degree; and polynomial equations. We also varied the forms of the proposed polynomials: factored, expanded, or hybrid forms; and a polynomial written with the summation symbol Σ.

For the second theme (equations and roots), we first asked students to solve quadratic equations. We chose 4 equations: a zero product of factors, a zero difference of two squares, an equality of a square with a negative number, and the expanded form of the equation $(3x-1)^2=0$. The subsequent questions allow to examine the connections between roots, specialization, and factorization: one question focuses on the existence of roots of various polynomials, and another asks to describe the set of polynomials (or second-degree polynomials in high school) with a given root.

The last theme (polynomial functions) explores the properties of polynomial functions graphs. This allows us to observe changes of semiotic registers that students may use. We first asked to draw the graphs of monomial functions, based on the parity of the exponent. Then, we focused on the recognition of graphs of polynomial functions. We selected graphs of polynomial functions of degree 1,2,3 and 4, as well as the graphs of the exponential, sine, reciprocal and rational function. This type of task is uncommon for students, as we have not observed any textbook or exercise sheet offering a similar task, neither in high school nor in higher education. Finally, students were asked to study the optimum of a polynomial function of degree 2.

We detail in the following section some of the analyses of our experiment.

RESULTS

We will focus on the results regarding three questions, selected to represent the three themes addressed. In this way, we will highlight some students' difficulties regarding the understanding of polynomials. All the percentages given in this section are expressed in relation to the number of students who addressed the question.

Definition of a polynomial

Various definitions of a polynomial are given in high school and higher education. As a result, we did not expect high-school students and economics "classes préparatoires" students to give a definition of a formal polynomial, whereas university students are supposed to know this notion. We distinguished three groups of definitions in students' responses, characterized by the nature of the defined polynomial: polynomial as a null sequence from a certain rank, polynomial as a function, polynomial as an "algebraic expression" satisfying certain properties.

In the first group of definitions, students characterize a polynomial as a function, and most of them provide an algebraic expression of the form "$P = \sum_{k=0}^{n} a_k x^k$", or written in expanded form "$P = a_0 + a_1 x + \cdots + a_n x^n$". This definition is consistent with the one given in 11th grade for quadratic polynomials and with the one given in 12th grade (option "mathématiques expertes") for polynomials of any degree. The term "function" appears for 56% of 11th-grade students, 29% of 12th-grade students and a third of higher education students. Several higher education students specify in this definition that polynomials are continuous, differentiable functions.

In a second group, students use a null sequence from a certain rank to define a polynomial, and provide an algebraic expression of the form "$P = \sum_{k=0}^{+\infty} a_k X^k$". This definition is given by 28% of university students. Within this group, two types of use of a null sequence can be distinguished. Some answers define a polynomial as an expression of the form "$P = \sum_{k=0}^{+\infty} a_k X^k$" where (a_k) is a real sequence that vanishes from a certain rank. In other responses a polynomial is defined as a sequence, and the expression "$P = \sum_{k=0}^{+\infty} a_k X^k$" is a notation. This definition is the closest to the expert definition of a polynomial as an element of a polynomial ring. However, these definitions aren't operational for some of the other tasks of the questionnaire.

In the third group (comprising 43% of high-school students and 40% of higher education students), the form of a polynomial is described in natural language, in particular the operations of the variable needed to obtain a polynomial. The nature of the polynomial is neither a function nor a sequence; it can be a "mathematical object", "an expression", or may not be defined. Thus, few students resort to the formal definition of a polynomial: most only mention the general expression of a polynomial, and/or consider it as a function.

We have also identified definitions that exhibit characteristics from several of the groups described above. For instance, some students provide an algebraic expression of a polynomial without explicitly stating that it is a function, and some add that the expression is continuous and differentiable. Some university students define a polynomial as a formal object but without employing the notion of a null sequence: they use the uppercase X, sometimes specifying that it is called the indeterminate.

To conclude, we'd like to point out that the work on definitions is a at the heart of the secondary-tertiary transition. Indeed, very few high-school students specified the nature of all variables involved in their definition: the unknown x, its exponents, and the coefficients. In higher education, 37% of students still did not specify that the coefficients are real, and 20% did not explicitly consider the case of integer powers.

Equation solving

In the textbooks, the majority of the tasks focus on the use of the discriminant for solving quadratic equations. However, all the equations that we selected for the questionnaire can be solved without employing this technique. We noted that when it is possible to compute the discriminant after only one expansion, it is often used by students. Indeed, 30% of students used this technique for the equation $(x-4)(x-5)=0$ and over 70% used it for the other equations. This seems to indicate a lack of connection among students between roots and factorization, and the study of high-school textbooks confirms that the proposed tasks do not emphasize this link. These results are consistent with previous studies on the teaching of second-degree polynomials (Chaachoua et al., 2022). Some students also used the discriminant outside of its field of validity, for example on equations of degree 3, 4, and even of arbitrary degree. This suggests that some students, even in higher education, engage in purely syntactic work when solving equations.

Another technique of equation solving (that is proposed in textbooks from 10$^{\text{th}}$ grade) is the factorization of the polynomial expression to reduce it to a product of first- or second-degree factors. It can be applied to two of the equations of the questionnaire, using binomial squares formulas. In high-school textbooks, the use of those formulas for equation solving is systematically guided by the statement, leaving little room for initiative. Consequently, on the non-factored equation we proposed, no high-school student used this technique. In comparison, 30% of university students factored the polynomial $(x^2+2)^2-9$ to find its roots. Note that some students both factored the polynomial $9x^2-6x+1$ and calculated the discriminant to find the solution of the equation $9x^2-6x+1=0$. This may be because the discriminant technique is highlighted in high school, which could make it the only valid technique for students. One can also assume that calculating the discriminant allows them to prove the uniqueness of the solution found by factoring, or to verify their solution.

Finally, for the equation $(x^2+2)^2-9=0$, 80% of 12$^{\text{th}}$-grade students (i.e. 4 out of the 5 who tackled the question) and 24% of university students transform it into an equality of two squares. This kind of equation, worked since 10$^{\text{th}}$ grade, poses difficulties for students. Most of them use erroneous theorems-in-action, such as:

1. The solutions of the equation $x^2 = \alpha$ where $\alpha \in \mathbb{R}^-$ are $x = \pm\sqrt{|\alpha|}$
2. The solution of the equation $x^2 = \alpha$ where $\alpha \in \mathbb{R}^+$ is $x = \sqrt{\alpha}$
3. If $(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}$, the equations $(x^2+\beta)^2 = \alpha$ and $x^2 + \beta = \sqrt{\alpha}$ are equivalent.

These theorems-in-action sometimes coexist within the resolution of the same task with the valid theorems. For instance, in higher education, out of the 9 students who transformed the equation into an equality of two squares, 6 used the third theorem-in-action to deduce that $x^2 + 2 = 3$, then all of them concluded with the implication $x^2 = 1 \Rightarrow x = \pm 1$. It can be assumed that some of these mistakes arise from classical misconceptions about the square root.

Properties of polynomial function and semiotic registers

In the third theme, we presented tasks involving polynomial functions. One of these tasks is the recognition of curves of polynomial functions. This provides a context for observing the semiotic registers that students implement in their answers. Indeed, they can mobilize algebraic properties of polynomial functions (number of zeros) and functional properties (limits, continuity) to justify some attributes of the graphs. They can also use the register of algebraic expressions if they provide an expression of the given function. This is an unusual task for students, and consequently 52% of high-school students and 35% of higher education students answer at least one of these questions without providing justifications.

We identified that some students identify graphs as polynomial graphs only if they have the characteristics of monomial graphs. This leads them to use criteria like symmetry to identify polynomial graphs. Besides, one student states that "a polynomial is either a parabola (if the degree of the polynomial is even) or of this form […]" and draws the graph of an even-degree polynomial: this student does not differentiate between monomial and polynomial graphs.

Students were more successful in demonstrating that a graph does not represent a polynomial function. For the graph of the exponential function, 31% of 11^{th}-grade students, 86% of 12^{th}-grade students, and 53% of university students explicitly identify the exponential. However, only 3 of those students give a justification that the exponential is not a polynomial function. Among the other students, 6 higher education students (16%) justify that the graph is not polynomial by examining its limits at infinity. The continuity of polynomial functions is mentioned by a greater number of students, who succeed in justifying that the graph of a function with a discontinuity cannot represent a polynomial function. Finally, 7 students (18%) manage to justify that the graph of the sine function is not that of a polynomial function (noticing that a nonzero polynomial function has a finite number of zeros).

CONCLUSION AND OUTLOOKS

The experiment that we conducted gave some answers to the research questions we introduced. The first operational invariants that we identified concern the recognition of a polynomial expression and the recognition of the graph of a polynomial function. For the first point, students use invariants that apply either to the overall form of the expression ("a polynomial is not factored" or "an expression is a polynomial if a degree can be identified"), to the operations of the variable ("the variable cannot be in the denominator of a fraction", "an expression containing a power is a polynomial"), or to the coefficients in front of the variable ("the coefficients of a polynomial cannot be fractions", or "should not contain a square roots"). While some of these operational invariants are valid, they are not sufficient to identify polynomial expressions in general. For the recognition of polynomial graphs, some students seem to identify graphs of polynomial functions with those of monomial functions. When solving a polynomial equation, most students aim to identify the coefficients of a second-degree polynomial, and then calculate the discriminant. This rule is sometimes extended to polynomials of degrees greater than 2, provided that the expression comprises 2 or 3 terms, which they identify as the coefficients of a polynomial. Specific theorems-in-action have also been highlighted in solving equations of the form $x^2 = \alpha$ or $(x + a)^2 = \alpha$.

Throughout the different tasks, we have also noticed that the action rule "to answer a question about a polynomial, I start by expanding it" is used by students beyond the context of equation solving, for instance for setting up a table of variation. We also observed it in the question "is -2 a root of $(x + 2)^2 - 1$?" that 45% of 12th-grade students expanded the expression. For most of the questions, this method is not effective, and leads to more computation errors.

We then question the evolution of students' schemes during the secondary-tertiary transition. Regarding the definition of a polynomial, the emergence of a new definition (formal polynomial) at university leads students to produce new types of definitions, including hybrid ones that combine a functional and an algebraic vision of polynomials. We also observed that higher education students are better able to define all the parameters involved in the definitions they provide. Higher education students performed significantly better on the recognition tasks, being more equipped with properties and tools on polynomial functions. For example, more than half of high-school students answer that $x + 1$ is not a polynomial, and 38% of them answer that the graph of an affine function is not a polynomial graph. Students in higher education mostly correctly handle these questions. Both secondary and higher education students are familiar with quadratic equation solving, but only higher education students managed to handle the 4th-degree equation.

These analyses provide clues about potential difficulties students face during the transition from secondary to higher education. The definition of a polynomial seems complex for students, especially since only the definition of a real polynomial function is provided in high school. In contrast, in higher education, the definitions of a formal polynomial, a real polynomial function, and sometimes a polynomial function in any ring coexist. Furthermore, few tasks involve non-real polynomials, which may latter confuse students when regarding the utility of the concept of a formal polynomial. For equation solving, the systematic use of the discriminant makes computation errors more likely and might prevent polynomial arithmetic comprehension. Moreover, the technique od second-degree discriminant cannot exactly be used in higher-degree equations, while students sometimes try to. This is exacerbated by the near disappearance, in high-school curriculum, of theorems linking polynomial roots to their factorization. Finally, we have noted difficulties regarding the graphical representation of monomial and polynomial functions, both at the high school and university levels. Students seem to be familiar with the general shapes of monomial graphs, but face challenges in determining their relative positions. This raises questions about their ability to mobilize different semiotic registers, and about the links they make between the relative positions of monomial curves and classic inequations such as $x \leq x^2$ when $x \geq 1$. Moreover, if some students manage to apply properties on limits and zeros of polynomial functions, few succeed in recognizing graphs of polynomial functions of degree greater than 3.

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APPENDIX 1: QUESTIONNAIRE (UNIVERSITY VERSION)

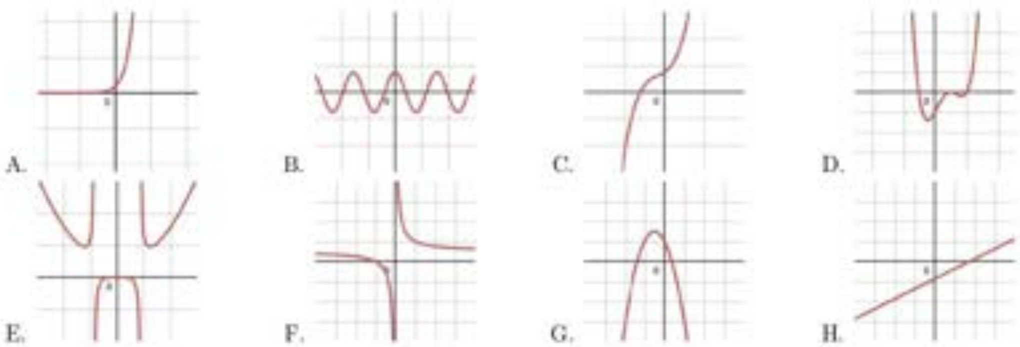
1. A student in your class was absent during the lesson about polynomials. Write them a definition of a polynomial. Provide examples.
2. How would you explain the use of polynomials in mathematics to a student in your class ?
3. Which of the following expressions are polynomials ?

   ○ $\frac{x^2}{2}$ ○ $x^2+\pi$ ○ $3x^{12}+\sqrt{x}-\sqrt{2}$ ○ $\frac{x}{2}+ax+x^5$ $(a \in \mathbb{R})$ ○ $\sqrt{x}+5$ ○ $\frac{1}{x}-\frac{1}{x^2}$ ○ $\frac{1}{x}+\frac{2}{x^4}-x^5$

   ○ $\frac{x+5x^2}{x^3-x^{23}}$ ○ x^n-x^{n-1} $(n \in \mathbb{N}^*)$ ○ $x^5-3x^2=4$ ○ $x+1$ ○ $x^2-2x+1=0$ ○ $(x-5)(x+12)(x^2-x)$

   ○ $(x-1)(x+\sqrt{2})+3x^4$ ○ $(x-1)^2+5$ ○ $\sum_{i=0}^{5}(-1)^i x^i$ ○ $x^{10^6}+1$

4. Solve the following equations with unknown $x \in \mathbb{R}$ providing the details of the solution.
   A. $(x-4)(x+5)=0$ B. $(x+1)^2=-12$ C. $(x^2+2)^2-9=0$ D. $9x^2-6x+1=0$
5. Do the following polynomials have (at least) one real root ? Justify your answer.
   A. $P(x)=(x-\sqrt{2})(x^n+x-1)$ $(n \in \mathbb{N})$ B. $Q(x)=x^n-x^{n-1}$ $(n \in \mathbb{N}^*)$ C. $R(y)=y^5-3y^2+2$ D. $S(x)=x^{1000}+2$
6. Which polynomials vanish at 5 ?
7. Let P be a polynomial such that for all $x \in \mathbb{R}$, $P(e^x)=0$. What can be said about P ?
8. Plot on the graph below the shapes of the functions $x \mapsto x^n$. Differentiate the cases based on the parity of $n \in \mathbb{N}^*$.
9. Can the following graphs represent polynomial function ? Provide a justification for each curve.



10. Does the function $f : x \mapsto (x+3)^2-1$ have an optimum (maximum, minimum) ? Justify. If so, determine it.

Form-Function Relations for Eigentheory in Quantum Mechanics

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Eigentheory concepts are central in mathematics and physics; they serve multiple functions, such as symbolizing physical phenomena and facilitating mathematical computations. Words associated with eigentheory develop and vary over time (e.g., eigenvector, eigenstate), as do associated symbols (e.g., $Ax = \lambda x$, $H|E_n\rangle = E_n|E_n\rangle$*). In this study, we investigate how "eigen" develops over time for a quantum mechanics classroom community by analyzing form-function relations (Saxe, 1999) for eigen concepts over 22 class sessions. We share results concerning our microgenetic and ontogenetic analyses of the creation of form-function relations and their shifts over time. We illustrate how the professor's use of prior functions with new forms helped shape the class community's common ground for eigentheory in quantum mechanics.*

Keywords: Teaching and learning of mathematics in other disciplines, teaching and learning of linear and abstract algebra, quantum mechanics, eigentheory.

INTRODUCTION

Eigentheory concepts are central in mathematics and physics. They serve multiple functions, such as symbolizing physical phenomena and facilitating mathematical computations. Students can reason about physics and linear algebra concepts in inextricable ways, almost simultaneously mathematizing physical phenomena in terms of their corresponding mathematical objects and interpreting the mathematical symbols in terms of the physical phenomena they symbolize (Serbin & Wawro, 2022). Eigenequations often first take form in mathematics classes as $Ax = \lambda x$ for $n \times n$ matrix A, x in $\mathbb{R}^n$ or $\mathbb{C}^n$, and scalar λ, and then later as $T(v) = \lambda v$ for linear operator T on vector space V. Physics students also encounter eigentheory in quantum mechanics, where eigenvalues of various Hermitian operators represent possible measurement values of corresponding observables. Furthermore, quantum mechanics uses Dirac notation, in which eigenequations take on forms, such as $S_z|+\rangle = \frac{\hbar}{2}|+\rangle$ and $H|E_n\rangle = E_n|E_n\rangle$, and convey information related to spin and energy, respectively. In quantum mechanics, it is "eigen, eigen, eigen all the way" (Shankar, 2012, p. 30).

In this paper, we pursue the research question: *How does the concept of "eigen" develop over time for a quantum mechanics classroom community?* [1] We leverage a form-function analysis (Saxe, 1999), analyzing the public displays of form-function relations used in the classroom community. Several verbal, symbolic, and written forms can be associated with the same eigentheory concept, and many of these forms can serve different functions in both math and physics. For a classroom community, form-function relations develop over time through the negotiation of the community's common ground (Saxe et al., 2015). The development of these form-function relations for eigentheory concepts in a Quantum Mechanics course is the focus of this study.

LITERATURE REVIEW

There is a growing body of literature on student understanding of eigentheory (e.g., Altieri & Schirmer, 2019; Salgado & Trigueros, 2015; Serbin et al., 2020). There are conceptually complex aspects to a deep understanding of eigentheory. For example, interpreting $Ax = \lambda x$ could involve relating the matrix-vector product Ax as equivalent to the scalar-vector product $\lambda \boldsymbol{x}$ (Thomas & Stewart, 2011), conceptualizing eigenvectors as the x that are stretched by A (e.g., Sinclair & Gol Tabaghi, 2010), or imagining A as a function on some x to produce λx (Larson & Zandieh, 2013). These useful interpretations for $Ax = \lambda x$ have been documented with quantum mechanics students (Wawro et al., 2019; Wawro et al., in press). Eigentheory can take on additional meanings in quantum mechanics contexts (e.g., Gire & Manogue, 2012), and making sense of multiple valid interpretations can be nontrivial. For example, Wawro et al. (in press) found that when asked to interpret the meaning of the eigenequations $Ax = \lambda x$ and $S_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$, some physics students were unsure how to resolve the disconnect between their geometric interpretation of the first equation and their quantum mechanical interpretation of the latter equation.

In science, "symbols mediate the connection between the physical world and how we think about phenomena, a process that is not trivial, due to the complexity of symbolic notations and the abstract relationship between mathematical expressions and the phenomena they reflect" (Rodriguez et al., 2018, p. 2115). Students merge math and physics by mathematizing physical phenomena and interpreting mathematical symbols in terms of physical referents (Serbin & Wawro, 2022). Research suggests interpreting mathematical symbolic expressions in terms of physical phenomena may be nontrivial for students (Caballero et al., 2015; Her & Loverude, 2020). In quantum mechanics, students need to reconcile their understanding of eigentheory symbols in terms of both the mathematics and the physical phenomena the symbols represent.

The literature illustrates the complexity associated with interpreting eigentheory symbols and the varied meanings these symbols convey, particularly in physics. Most of these studies focused on individual students' reasoning about eigentheory; our study contributes by focusing on a class's collective ways of reasoning. We build on all of these studies' findings through our analysis of the meanings a quantum mechanics class community develops in written, verbal, and symbol form for eigentheory over time.

THEORETICAL BACKGROUND

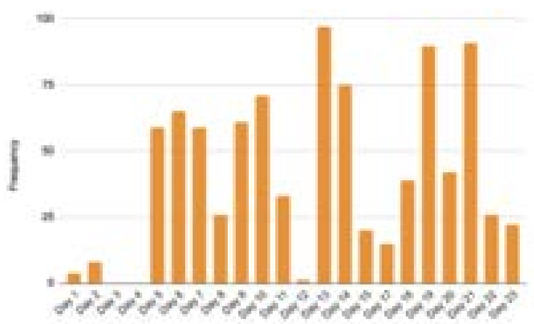
Through anthropological work on cultural development of mathematical ideas, Saxe (1999) developed a framework for investigating the form-function relations created by individuals and communities over time. A *form* is a verbal, symbolic, graphical, or physical representation that takes on mathematical meaning. As individuals engage in activity or communication, they tailor forms to serve certain functions in activity, thereby establishing form-function relations. *Functions* are defined as the "purposes for which forms are used as individuals structure and accomplish practice-linked goals" (p. 20). Forms can be adapted to serve several different functions, and functions can be served by different forms. For example, the form $y = 5x$ can function to convey a constant rate of change or exhibit covariation of two variables, and the forms e and 1 can both serve the function of symbolizing a group identity (Plaxco, 2015). Form-function relations can be used to analyze the "reproduction and alteration of a common ground of talk and action over lessons in classroom communities" (Saxe et al., 2015, p. 71). *Common ground* refers to "shared knowledge of word meanings and norms for communication" that "enables successful communication and coordinated action" (Saxe & Farid, 2021, p. 8). Common ground is created as community members interact to "produce and interpret displays of mathematical thinking, making use of representational forms (linguistic, graphical, gestural) to serve communicative and problem-solving functions" (Saxe et al., 2015, p. 4). To identify form-function relations for eigen and how they shift over time, we use microgenetic and ontogenetic analysis.

Microgenesis is the process by which individuals construct representations by tailoring forms to serve functions that accomplish goal-directed activity (Saxe et al., 2015). This often occurs in public displays and contributes to the alteration of a common ground. Individuals are enabled and constrained by the common ground as they create new form-function relations. They can use familiar forms to serve new functions or recruit new forms to serve existing ones. Saxe et al. (2015) referred to the use of familiar forms or functions as continuity and the use of new forms or functions as discontinuity. *Ontogenesis* is the developmental shifts in relations between the forms used and the functions that they serve (Saxe, 1999) and is characterized by shifts in microgenetic displays. Ontogenetic analysis involves an "analysis of continuities and discontinuities as individuals reproduce and alter form-function relations" (Saxe et al., 2015, p. 13).

METHODS

The data come from an in-person, senior-level Quantum Mechanics course taught in a public research-active university in the northeast US. Class sessions occurred three times weekly for 50 minutes each. Class sessions from the first nine weeks (23 days) of the semester-long course were video recorded, with a focus on capturing the professor and whole-class discussions. The professor was an experienced quantum mechanics instructor and physics education researcher, and the course had 17 students. Data sources were video recordings and associated transcripts. Only exchanges that occurred with the entire class (as compared to small groups) were analyzed.

We imported transcripts into MaxQDA, which is a qualitative and mixed methods data analysis software that allows for the creation of a multi-tiered codebook, as well as code tracking and counting across multiple transcript documents (Kuckartz & Rädiker, 2019). We inductively coded (Miles et al., 2013) both transcript and screen captures of slides and boardwork. We coded together while watching the videos, discussing our codes, and resolving inconsistencies as needed. We coded instances in which "eigen" concepts were explicitly leveraged. Our coding system had a nested organization according to form. First, at the categorical level, forms were separated according to what was characterized by "eigen" (e.g., eigenstate, eigenbasis). The second organization level separated categorical forms into the specific forms found in the data. For example, the form "eigenvector - verbal" was assigned if "eigenvector" was said, the form "eigenvector - written" if "eigenvector" was written on a board or slide, and the form "eigenvector - $|+\rangle$ symbolic" if the symbol $|+\rangle$ was used to convey eigenvector meaning. The final level of coding corresponds to the function that we interpreted the form to have when the form-function relation was detected. For example, any of the aforementioned eigenvector forms could have associated functions such as being an element of a basis that diagonalizes a spin operator or being associated with an expectation value for a measurement. If an instance involved more than one form, such as both saying and writing "eigenvector," the same function was assigned twice and yielded two form-function pairs. Our coding of instances when a form was tailored to serve a function comprised our microgenetic analysis (Saxe et al., 2015). Our ontogenetic analyses involved examining how forms, functions, or form-function pairs were used throughout the data. We created tables to visualize the frequencies of forms, functions, and form-function pairs in all class sessions, which we used to identify continuities or discontinuities in the class's use of said forms and functions over time.



(a)

Categorical Form	# of unique form-function pairs in form category			# of unique form-function pairs in categorical form
	Symbolic	**Written**	**Verbal**	
Eigenvalue [223]	30	11	60	101
Eigenvector [118]	20	9	52	81
Eigenstate [381]	74	21	77	172
Eigenbasis [27]	3	1	12	16
Eigenequation [113]	41	7	23	71
Eigenfunction [31]	7	3	11	21
Eigen(misc) [11]	0	0	11	11
Total unique form-function pairs				473

(b)

Figure 1: Daily count of form-function pairs (a) and summary (b) of the various types.

RESULTS

In total, our analysis of the whole-class discussions of 22 class sessions [2] resulted in a total of 904 instances in which a form-function relationship for the concept of "eigen" was communicated; Figure 1a gives their distribution over the 22 days. Our analysis resulted in seven categorical forms: eigenvalue, eigenvector, eigenstate, eigenbasis, eigenequation, eigenfunction, and miscellaneous. Column 1 of Figure 1b gives the number of times each categorical form appeared in the data set; for instance, "eigenstate" accounts for over one-third of the total coded forms (381 out of 904). Within each categorical form, we identified specific symbolic, written, or verbal forms for an eigen concept (e.g., the symbol a_n as a form for eigenvalue). Finally, every specific form in the data was coded with what function it accomplished. For example, the form "eigenvalue - a_n - symbolic" could function as a value on a diagonal of a matrix operator expressed in an eigenbasis or as the expectation value of a measurement corresponding to an eigenstate; these two form-function pairings account for 2 of the 30 different symbolic form-function pairs in the eigenvalue categorical form. Other forms could also serve the same functions; for instance, "eigenvalue - $\frac{\hbar}{2}$ -

symbolic" pairs with those same two functions and accounts for another 2 of the 30 eigenvalue symbolic form-function pairs. In total, there were 473 different form-function pairs across all seven categorical forms (see Figure 1b). Many form-function pairs appeared multiple times across the data set; for example, the aforementioned form "eigenvalue - $\frac{\hbar}{2}$ - symbolic" and function "expectation value of a measurement corresponding to an eigenstate" pairing appeared three times; in total, counting repetition, there were 904 form-function pairs for "eigen" in our analysed data set.

We organize the remainder of the Results section by: an exemplar of microgenetic analysis of form-function relations, ontogenetic analysis of continuity of forms, and ontogenetic analysis of discontinuity of forms.

Exemplar of microgenetic analysis of form-function relations

On Day 5, the professor led a discussion of a prepared slide entitled "Postulates of Quantum Mechanics," sometimes pointing at the words and symbols on the slide while talking, asking questions, and responding. First, we examine the slide. From the typed postulate, "The only possibly result of a measurement of an observable A is *one* of the **eigenvalues** $\{a_n\}$ of that observable", we coded forms "eigenvalue – written" and "eigenvalue – a_n – symbolic" with the function "possible result of a measurement." The slide then had "eigenvalue equation ($Av = \lambda v$)"; the written word form served the function of "labelling subsequent physics equations as eigenequations," and the symbolic eigenequation form functioned as a "referent conveying similar structure" to those subsequent physics equations. Finally, $S_z|+\rangle \doteq \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = +\frac{\hbar}{2}\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar}{2}|+\rangle$ and its spin-down equivalent were symbolic forms with the function of "exemplifying two eigenequations for spin-1/2." Next, we examine the dialogue:

Professor: So uh when you have a matrix [points to A], you have eigenvectors [points to v in λv], right. Do people remember what, what's special about an eigenvector, for a matrix, for a given matrix? What, what makes it eigen…esque…ish?

Student: When it's operated on by A it only scales.

Professor: That's right … So, eigenvectors don't rotate at all, they only scale, right. If you operate A on this particular eigenvector [points to Av], all you get is some scaling factor times the original vector [points to λv], alright. That's the, that's the magic. Those are magic vectors, right.

The "eigenvector – verbal" form in lines 1 and 3 and the "eigenvector - v - symbolic" form in line 2 serve the function, "structure inherently tied to an operator," meaning that these uses of eigenvector were to call attention to it as a property or aspect of an operator or matrix. In line 4, we coded the professor's invented word "eigen-esque-ish" as a miscellaneous verbal form functioning to "describe a quality inherent in an eigenvector." The students responded to his inquiry (line 5), which we coded as the functions, "can be operated on" and "does not rotate only scales," for the eigenvector symbol v; we repeated the latter function for the professor's restatement with the verbal utterance of eigenvectors (line 6). The professor concluded the conversation by connecting their ideas back to the eigenequation $Av = \lambda v$. We coded the eigenequation symbol form with the function, "operating A on an eigenvector scaled the vector." We coded the particular aspects he brought out by stating "eigenvector" verbally while pointing to the symbol v with the function, "can be operated on," and the symbol λ for eigenvalue with the function of "being a scaling factor." He closes his explanation by again calling out eigenvectors as special, pointing to the symbol v and calling them "magic vectors." Altogether, there were 17 form-function pairs in this 75-second clip.

Ontogenetic analysis: Continuity of forms

Following Saxe et al. (2015), our ontogenetic analysis involved examining continuities and discontinuities of forms, functions, and form-function pairs. One such aspect of our ontogenetic analysis was our investigation of the classroom community's continuity of forms for *eigenstate* over time. The various eigenstate forms that existed more than once in the data are given in the rows of Table 1, with the frequency of that form each day shown in the columns. The "eigenstate - verbal" form had the highest frequency; it and the "eigenstate – written" form had a high degree of continuity, which is sensible given their generality. Furthermore, the forms $|\pm\rangle, |a_n\rangle$, and $|E_n\rangle$ were relatively continuous, and sometimes served the same function over time. For example, $|E_n\rangle$ functioned in linear combinations of the state $|\psi\rangle$ nine times over days 13, 14, 15, and 19. Continuity of form-function relations serves to reinforce the class's common ground of the relationship between symbol forms and the functions they serve.

Class Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	17	18	19	20	21	22	23	Total
Eigenstate – verbal						5	2	7	11	24	17		20	19	4	6	2	16	9	19	14	6	181
Eigenstate – written		1				1	3	2	3	5	2		6	6		2		6	2	3			42
$\vert+\rangle$ and $\vert-\rangle$ – symbolic		1				4	2		1		1					1		1		3			14
$\vert+\rangle_n$ and $\vert-\rangle_n$ – symbolic							1										1						2
$\vert a_n\rangle$ – symbolic						4		1	1	6				3	1					4			20
$\vert a_n\rangle$ state – verbal									2	2													4
$\vert E_n\rangle$ – symbolic													29	26	3			6	2	12	3		81
$\vert E_n\rangle$ – verbal													1	1	1			1					4
$\vert x_i\rangle$ – symbolic																		3	7	10			20
$\Phi_{E_n}(x)$ – symbolic																				4			4

Table 1. Frequencies of various eigenstate forms over the 22 class days.

Ontogenetic analysis: Discontinuity of forms

Table 1 demonstrates discontinuity of forms by indicating new forms that developed over time; in some cases, these served the same functions that were previously served by other forms. For example, the symbols $|\pm\rangle$ and $|a_n\rangle$ first appeared in the beginning of the course to represent eigenstates in a spin-1/2 system. The class's common ground then expanded to include $|E_n\rangle$ to refer to energy eigenstates on day 13, $|x_i\rangle$ for position eigenstates on day 19, and $\Phi_{E_n}(x)$ for eigenfunctions on day 21. The community's meanings of eigenstate developed over time as new symbol forms were introduced to the common ground to refer to the same eigenstate concept. For instance, the eigenstate symbol forms $|\pm\rangle$, $|a_n\rangle$, $|E_n\rangle$, and $|x_i\rangle$ all served the same function of conveying that when operated on, the resulting output is the product of an eigenvalue and an eigenstate (itself). Furthermore, $|a_n\rangle$ and $|E_n\rangle$ also functioned in linear combinations to compose $|\psi\rangle$; $|\pm\rangle$ and $|E_n\rangle$ were both used to calculate the probability of measuring an observable; and $|a_n\rangle$, $|E_n\rangle$, and $|x_i\rangle$ all functioned in aiding the computation of inner products. These examples illustrate how, over time, different forms were tailored to serve the same functions. This helped establish the common ground of recognizing $|\pm\rangle$, $|a_n\rangle$, $|E_n\rangle$, and $|x_i\rangle$ as instantiations of the same overarching eigenstate concept with the same structure and abilities to be used in certain computations.

Our ontogenetic analysis also revealed forms that resurfaced in the class community's common ground after not being used for several days. This was particularly the case for the eigenequation symbol forms, $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$. The professor used these two symbol forms again on day 19 to convey that eigenequations have a similar structure regardless of context (spin, energy, or position). This is evident in an episode in which the class discussed a task from small group work: "Write down an eigenvalue equation for an operator $\hat{X}$ that represents (1-D) position." The professor wrote $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ on the board (see Figure 2b) and asked:

Professor: If I want to write an eigenvalue equation where that's my operator, what is it going to tell me? What do these eigenvalue equations tell me, in general? [points to $\hat{H}|E_n\rangle = E_n|E_n\rangle$ eigenequation]

Students: If you can operate with that, what your vector gets scaled by.

Professor: Right, right. So, if I operate on an eigenvector with that operator, what is this then? [Points to the eigenvalue E_n in $\hat{H}|E_n\rangle = E_n|E_n\rangle$].

Student: A scalar.

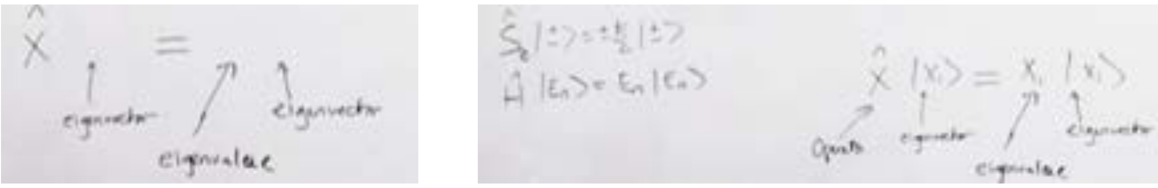
Professor: So, I need, what do I need here? I need an eigenvector, right, and I need that same eigenvector here, right, and I need, what do I need there? [points to space next to eigenvector on the right side of the equal sign] (see Figure 2a).

Students: A scalar. The position.

Professor: The position, right? But yeah, that's the eigenvalue, right? So, the, yeah, the general answer is the eigenvalue, but in this case, if this is my operator, the eigenvalue is a position…I'm going to call it x_i because it's a spot, right? It's a point. What should I use as how to represent an eigenvector of position? Like given some of the conventions we have for writing stuff, what would be a? [Students: x_i.] Yeah, like, well first, it better look like that, right [writes empty kets in equation $\hat{X}|\square\rangle = x_i|\square\rangle$], and what do I want to put in here?

Students: x_i

Professor: x_i right? [completes $\hat{X}|x_i\rangle = x_i|x_i\rangle$] ... Because what the convention was, this is the value of, the eigenvalue for the eigenvector, right? So, this expression is not like, in itself, uh, out of the realm of your ability to write, right? Like in the sense that it's just like this [points to $\hat{H}|E_n\rangle = E_n|E_n\rangle$].



(a) (b)

Figure 2. The professor's symbol forms written on the board during a Day 19 episode.

Here, two prior eigenequation forms, $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n\,|E_n\rangle$, resurfaced in the community's common ground as they created a new eigenequation symbol form, $\hat{X}|x_i\rangle = x_i|x_i\rangle$. Both prior forms served the function of being a structure inherently tied to operators ($\hat{S}_z$ and $\hat{H}$) to exhibit that the new eigenequation they were creating was also inherently tied to an operator, $\hat{X}$. In juxtaposing $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n\,|E_n\rangle$ with the new symbol form $\hat{X}|x_i\rangle = x_i|x_i\rangle$, the class community tailored the prior forms to serve the function of being a referent to convey a similar structure that eigenequations have the same vector on both sides of the equal sign. Overall, the community leveraged their previously established symbol forms to create a new symbol form with the same function. This contributed to the class community's development of common ground for various eigenequations by establishing the recognition of different symbol forms from different quantum mechanical systems as instantiations of the same overarching concept with the same functions.

CONCLUSION

In this study, we performed microgenetic and ontogenetic analyses of a class community's forms, functions, and form-function relations about eigentheory concepts over 22 class sessions in a quantum mechanics course to answer our research question regarding how the class developed meanings of eigentheory concepts over time. We exemplified how shifts in the forms used in class aligned with the progression of the course content from spin to energy to position, informing our research question of how "eigen" developed over time. We found that some form-function relations resurfaced in the collective common ground (Saxe & Farid, 2021) after not being referenced in several class days to help convey that different instantiations (e.g., $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n\,|E_n\rangle$) of the same overarching concept (e.g., eigenequations) can have the same function or convey the same meaning as a newly developed form for that concept (e.g., $\hat{X}|x_i\rangle = x_i|x_i\rangle$). Our analysis allowed us to understand how the professor supported his students in developing meanings for eigentheory concepts in ways inextricably related to quantum mechanical concepts (Serbin & Wawro, 2022). The juxtaposition of the class's established forms and associated functions with newly introduced forms was productive pedagogically and seemed to help the class develop meanings of the shared structure or functions that different eigentheory forms may have. Furthermore, most of the form-function relations that contributed to the constitution of the class's common ground came from microgenetic displays by the professor. Future research could focus on identifying members of the class community that were key contributors to the constitution of the common ground and how this provision of contributions is shaped by other aspects of the class's common ground, such as the class's social and sociomathematical norms (Saxe et al., 2015).

NOTES

1. A preliminary presentation about our study was given prior to the completion of our coding (Wawro & Serbin, 2023).
2. There was a technological issue on Day 16, so no data were collected. We omit that day from figures and analyses.

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La ruta cognitiva sobre la noción de Isomorfismo de Espacios Vectoriales: el caso del profesor de álgebra lineal

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El presente reporte de investigación aborda la implementación de un cuestionario para caracterizar la ruta cognitiva sobre la noción de Isomorfismos de Espacios Vectoriales (IEV). La investigación fue realizada con profesores chilenos y mexicanos que imparten la asignatura de álgebra lineal en un programa de Ciencias. El diseño de la investigación se sustenta en el ciclo metodológico propio de la teoría APOE. Los resultados evidencian la construcción conceptual de IEV de un profesor al resolver un problema, mediante la coordinación de conceptos como matriz asociada, región en el plano $\mathbb{R}^2$ *y base generadora, que subyacen al modelo multinterpretativo de IEV.*

Keywords: Learning of linear algebra, Learning of specific topics in university mathematics, Isomorphism vector space, APOS theory

INTRODUCIÓN

El estudio del álgebra lineal está presente en la mayoría de los programas del área de Ciencias e Ingenierías. La importancia de esta disciplina para la formación de un científico ha motivado que investigadores, asociados en comunidades (RUMEC[1], LACGS[2]), se propusieran develar los problemas propios que se asocian al proceso de enseñanza y aprendizaje, reportando dificultades y obstáculos que son propios del AL (Sierpinska, Dreyfus & Hillel, 1999; Dorier y Sierpinska, 2001).

El desafío que tiene la enseñanza del álgebra lineal, para aquellos que utilizan sus objetos de saber, es poder comprenderlos en forma unificada. En este sentido, la noción de IEV, es un objeto que puede ser interpretado desde tres perspectivas -funcional, matricial y geométrica-figural- con la cual, se puede evidenciar la interacción de las distintas nociones que marcan una ruta cognitiva a seguir para resolver situaciones problema.

De acuerdo con el escenario anterior, el objetivo de esta investigación es caracterizar una ruta cognitiva de un profesor de la asignatura, bajo la perspectiva de la teoría APOE (Acción, Proceso, Objeto y Esquema), a través de la interacción de tres esquemas, representados por las interpretaciones –funcional, matricial y geométrica-figural– de la noción de IEV. De la coherencia de esos tres esquemas emerge el modelo cognitivo multinterpretativo, que ayuda a caracterizar las interacciones entre los niveles de Esquemas Intra, Inter y Trans de la noción de IEV para describir la comprensión integral del concepto.

LA INVESTIGACIÓN Y SU OBJETIVO

Este reporte de investigación, es parte de un estudio doctoral cuyo el objetivo es indagar las construcciones mentales que muestran profesores de la asignatura de álgebra lineal asociadas a un modelo multinterpretativo sobre la noción de IEV. Para cumplir dicho propósito utilizamos el ciclo metodológico propio de la teoría APOE. En él, consideramos tres momentos importantes, el primero referido al estudio teórico para el levantamiento de las DG hipotética y determinar los niveles Intra, Inter y Trans. Luego, se diseña e implementa un

[1] RUMEC. Research in Undergraduate Mathematics Education Community
[2] LACSG. Linear Algebra Currículo Study Group

cuestionario que nos permitiera investigar sobre las construcciones mentales y la interacción de los Esquemas y, finalmente, analizar las respuestas a la luz del constructo teórico de APOE. En este sentido, la aplicación de las acciones investigativas es para responder al cuestionamiento:

¿De qué manera interaccionan los diferentes esquemas relacionados con el modelo multinterpretativo de IEV en el proceso de solución de situaciones problema?

Los investigadores decidieron que, para responder a esta pregunta de investigación se deben responder dos sub-preguntas, (1) ¿Cuáles son los mecanismos y construcciones mentales que determinan el nivel de Esquema Intra, Inter y Trans del informante al responder situaciones problema? Y ¿Cuáles serán las construcciones mentales que determinarán la interacción de los Esquemas?

Responder la pregunta 1, nos permitirá identificar los elementos matemáticos que constituyen a las interpretaciones relacionadas al modelo de multinterpretación de IEV. Mientras que, la pregunta 2, permitirá identificar los elementos matemáticos que permiten caracterizar a la ruta cognitiva del informante.

Con ello, podremos cumplir el objetivo de investigación que caracterizar una ruta cognitiva de un profesor de la asignatura, bajo la perspectiva de la teoría APOE (Acción, Proceso, Objeto y Esquema), a través de la interacción de tres esquemas, representados por las interpretaciones –funcional, matricial y geométrica-figural– de la noción de IEV.

ANTECEDENTES DE LA INVESTIGACIÓN

El Isomorfismo es una noción importante para la construcción de la matemática. Los conceptos y propiedades que subyacen a él, tales como: la función, el dominio y recorrido, y propiedades como la inyectividad y sobreyectividad, entre otros, determinan su naturaleza abstracta. En particular, para el álgebra lineal, la comprensión del IEV requiere de la construcción conceptual de las nociones de vector, dependencia e independencia lineal, base, espacio vectorial y transformación lineal, así como de la interacción de diferentes esquemas (funcional, matricial y geométrica-figural).

Por otro lado, las dificultades y obstáculos que subyacen al IEV, solo por estar al interior del álgebra lineal han sido documentadas por diversos investigadores, reportando información importante que dará sustento al presente estudio.

A continuación, se presentan aquellas investigaciones que han sido reportadas por la Didáctica de la Matemática y que contribuyen a profundizar en la comprensión del objeto matemático IEV.

Desde el punto de vista de los conceptos que subyacen al IEV, se ha podido identificar investigaciones que, desde una perspectiva cognitiva, contribuyen a enfrentar las problemáticas ligadas al proceso de enseñanza y aprendizaje. Por ejemplo, el espacio vectorial es una noción importante para el álgebra lineal, las investigaciones realizadas desde una perspectiva funcional, reportan la importancia que tiene la estructura algebraica de un espacio vectorial para la comprensión axiomática, que definen sus operaciones. Además, el rol que tiene la combinación lineal para la determinación de las bases que la constituyen, en pro de la comprensión de las características estructurales de los espacios y subespacios vectoriales (Kú, Trigueros y Oktaç, 2008; Parraguez, 2009; Parraguez y Oktaç, 2012; Parraguez, 2013).

Desde el punto de vista geométrico, Rodríguez y Parraguez (2013), construyen cognitivamente el espacio vectorial $\mathbb{R}^2$ a través de una descomposición genética, en la que se destaca la importancia que tienen la noción de vector y los conceptos que subyacen a él, como lo es el parámetro, el segmento dirigido y la función que la definen para su visualización en dos dimensiones.

Otra noción importante es la de transformación lineal (TL). En este sentido, se han desarrollado modelos cognitivos llamados descomposiciones genéticas que, desde una perspectiva funcional, se destaca como un concepto unificador para el álgebra lineal. Estos modelos han servido de base para distintas refinaciones que ha considerado la importancia de las bases y las operaciones internas y externas que la definen (Roa, 2008; Roa y Oktaç, 2012).

Una investigación importante y que es un antecedente para la presente investigación, es la realizada por Parraguez, Lezama y Jiménez (2016) en torno a la construcción cognitiva del teorema de cambio de base de vectores. En ella, realizan una descomposición genética con base en la interpretación funcional y matricial del objeto de estudio. En este mismo sentido, otras investigaciones que profundizan en la articulación conceptual de las formas de representar a una TL. Por ejemplo, a partir del teorema del cambio de base, se construye una descomposición genética que busca determinar la comprensión cognitiva de la representación matricial y funcional de una TL (Trigueros, Maturana, Parraguez y Rodríguez, 2015; Roa y Oktaç, 2012). Por otro lado, a través de figuras concretas en el plano $\mathbb{R}^2$, se vincula la forma, funcional y matricial de la TL (González y Roa, 2017).

Tal como se ha podido observar, las investigaciones descritas destacan los esfuerzos realizados por investigadores en Didáctica del álgebra lineal para profundizar en las problemáticas detectadas desde una perspectiva funcional, dejado de lado la construcción de un modelo que integre otras formas de interpretar los conceptos relacionados con el IEV, como lo es, la interpretación matricial y geométrico-figural.

Por lo tanto, esta investigación se propone caracterizar la ruta cognitiva de un profesor de asignatura sobre la noción de IEV desde tres interpretaciones, para la determinación de un modelo de construcción cognitiva multinterpretativa, que considere una interacción entre los conceptos que subyacen a la interpretación funcional (IEV_f), matricial (IEV_m) y geométrico-figural (IEV_{gf}).

MARCO TEORICO

A continuación, se presentan las estructuras mentales que define la teoría APOE para poder describir la construcción mental de un individuo que se enfrenta a un concepto matemático. En este sentido, para la presente investigación se construye un modelo multinterpretativo que describe la construcción hipotética del concepto de IEV y, que se utilizan como base para el análisis y descripción de la ruta cognitiva.

Una Acción, según Arnon et al. (2014), es una transformación sobre un objeto u objetos que está dirigida al individuo de forma externa, a través, de un estímulo externo, con la cual, la transformación estará guiada de forma explícita por las instrucciones externas. Por ejemplo, Oktaç (2019) menciona que una construcción acción de la TL, significa que el individuo calcule la imagen de un vector, conocida la TL.

La estructura de acción, según Arnon et al. (2014), es la más básica y primitiva de todas, puede evidenciarse en las actividades de la enseñanza inicial del álgebra, cuando por ejemplo a un individuo se les solicita calcular, sustituir o aplicar algoritmos. Ella es importante y necesaria para el desarrollo de otras estructuras mentales, como la de proceso.

Una estructura proceso, según Arnon et al. (2014), se evidencia cuando un individuo a medida que repite y reflexiona sobre las acciones, puede dejar de depender de las instrucciones externas tomado el control interno sobre lo que realiza. Con ello, adquiere la capacidad de imaginar la realización del procedimiento y pasos sin depender de forma explícita de ellos, entonces, se dice que el individuo a interiorizado una acción en un proceso.

Por ejemplo, una construcción proceso del concepto de transformación lineal T permite imaginar el efecto de la determinación del $Ker(T)$ en el dominio de la TL. Según Oktaç (2019), la generación de nuevos procesos, también puede ser efecto de la coordinación de dos o más procesos. Además, de la reversión de un proceso en otro nuevo. Ello es posible debido al dinamismo propio de esta estructura.

En Asiala et al. (1996), se menciona que en la construcción objeto un individuo reflexiona sobre el dinamismo de un proceso, logrando entenderlo como un todo, realiza transformaciones sobre él, ya sea a través de acciones u otros procesos, pudiendo construir esas transformaciones. Dado lo anterior, se puede decir que el individuo ha encapsulado el proceso en un objeto cognitivo. Por ejemplo, un estudiante tiene una construcción objeto de la TL cuando es capaz de determinar si ella es un IEV (Maturana, 2015; Oktaç, 2019)

Según Oktaç (2019), un esquema es una colección coherente de estructuras asociadas con un concepto en la mente de un individuo. Y, por ejemplo, para esta investigación, un estudiante muestra una estructura de esquema de IEV si puede establecer relaciones entre los conceptos asociados al modelo multinterpretativo de IEV, es decir, puede interaccionar los esquemas funcional, matricial y geométrico-figural del IEV, coordinando las estructuras que subyacen a cada esquema.

Cabe destacar que Arnon et al. (2014), menciona que el esquema es una colección coherente de acciones, procesos, objetos y otros esquemas, la cual se puede transformar en una estructura estática y/o una estructura dinámica que permite la incorporación de otros objetos o esquemas similares.

Según Piaget y García (1989), los esquemas evolucionan conceptualmente, en concordancia con la interacción de otras estructuras. Con ello, se reconocen tres niveles que muestran la evolución conceptual de un esquema. El primero, denominado Intra, se relaciona con aquella construcción de acciones, procesos, objetos y esquemas relacionados con un mismo concepto de manera aislada. En este sentido, por ejemplo, si un estudiante logra la construcción nivel Intra del IEV en lo matricial, no necesariamente eso significa que se relaciona con el concepto desde lo funcional.

El nivel Inter, se caracteriza por la existencia de relaciones entre acciones, procesos, objetos y esquemas entre diferentes conceptos. Ello conlleva a proponer que cuando un estudiante muestra este nivel Inter del IEV, será capaz de relacionar conceptos asociados a los esquemas con diferentes del modelo multinterpretativo de IEV en esas componentes.

Y finalmente, el nivel Trans, propone identificar alguna conservación que le de coherencia al esquema, en el sentido de que el individuo sea consciente de cuando es pertinente su uso y cuando no en la solución de situaciones problema. Para el estudio del IEV, el estudiante que tenga un nivel de esquema Trans de IEV, será capaz de establecer relaciones entre los conceptos asociados a los esquemas funcional, matricial y geométrica-figural del modelo multinterpretativo de IEV.

Una parte conceptual importante que nos entrega la Teoría APOE para el análisis de los datos en esta investigación, es el significado de interacción de Esquemas. En este sentido, Arnon et al. (2014), menciona que un individuo puede construir esquemas que cambian constantemente y que pueden estar en distintas fases del desarrollo. Con ello, un individuo puede mostrar que para resolver problemas necesita coordinar diferentes Esquemas.

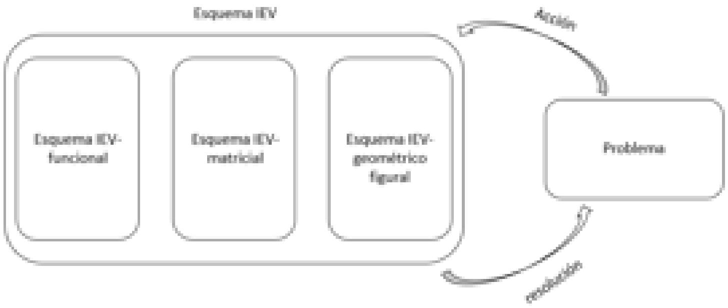
Modelo Multinterpretativo de la noción de IEV.

En el contexto de la teoría APOE, según Arnon et al. (2014), una DG es un modelo hipotético que describe las estructuras y mecanismos mentales que un individuo evidencia para construir un concepto matemático. El modelo de DG hipotético se construye con base en la experiencia del investigador en aspectos propios de la matemática, las investigaciones previas del concepto y la profundización histórica sobre él, para poder comprender la evolución del concepto en estudio.

Para está investigación, se ha determinado generar un modelo multinterpretativo sobre el IEV, que involucra la creación de tres Esquemas que subyacen a la noción: interpretación funcional de IEV_f, matricial de IEV_m y geométrico-figural de IEV_{gf}.

El Esquema IEV_f se describe por los niveles Intra, Inter y Trans. A raíz de ellas se desprende la construcción conceptual basada en el análisis sobre las bases de espacios vectoriales, la transformación lineal y sus propiedades de inyectividad y sobreyectividad, desde la interpretación de las expresiones algebraicas asociadas. Así mismo, el Esquema IEV_m esta interpretación se sustenta en la construcción conceptual de la matriz asociada a la TL y las propiedades que de ella se desprenden (matriz invertible, determinantes, entre otros). Por último, en el Esquema IEV_{gf}, las formas graficas en plano $\mathbb{R}^2$ y en el espacio $\mathbb{R}^3$ se construyen en coherencia con las propiedades de la noción de IEV. (Figura 1)

Figura 1. Interacción mental de subesquemas



En la Figura 1, un problema ejerce una acción externa que activa al Esquema general de IEV. Con ello, un subesquemas se activa para comenzar a resolver y permite que coordinaciones entre subesquemas interactúen para la caracterización de la ruta cognitiva utilizada para resolver el problema.

A la luz de la base teórica del modelo APOE, en el siguiente apartado se presenta el trayecto metodológico utilizado para realizar este estudio, en concordancia con el ciclo de investigación propuesto por este referente.

METODOLOGÍA

La presente investigación se ha propuesto profundizar en la comprensión conceptual de un profesor de asignatura escogido por conveniencia (Monje, 2011), interpretada a través de las estructuras y mecanismos mentales que muestra en la construcción cognitiva del objeto matemático de IEV. Para ello, nos apoyaremos en la teoría APOE, lo que nos entregará un soporte conceptual robusto en la comprensión de la construcción mental del objeto y de los mecanismos mentales que permiten la incorporación y dinamismo entre ellos.

Se utiliza el ciclo metodológico de APOE que nos permite secuenciar y determinar los procedimientos de investigación. En este sentido, (1) el ***análisis teórico*** considera el estudio sobre las estructuras y mecanismos que darán origen a la DG hipotética del modelo multinterpretativo de IEV. Una vez desarrollada la primera etapa, sigue (2) el ***diseño y aplicación*** de un cuestionario y una entrevista semi estructurada que considere profundizar en la interpretación funcional, matricial y geométrico-figural de la noción de IEV. Por último (3), el ***análisis de los datos*** nos permitirá validar el modelo de construcción conceptual o refinarlo, de acuerdo con los elementos emergentes que se presenten en las respuestas de los informantes.

En este sentido, podemos destacar que el modelo funcional-IEV se caracteriza porque los elementos a interpretar subyacen al concepto de Función, es decir, conceptos como transformación lineal (TL), espacio vectorial, conjunto Ker y con junto imagen son parte de la DG hipotética. Así mismo, el modelo matricial-IEV se caracteriza porque los elementos a interpretar subyacen a los temas de matrices. En este sentido, conceptos como matriz asociada a la TL, matriz invertible, entre otros son parte de la DG hipotética.

Finalmente, el modelo geométrico figural-IEV, se caracteriza porque el estudiante puede interpretar al IEV a través de los conceptos que subyacen al conocimientos de las formas en el plano $\mathbb{R}^2$.

RESULTADOS Y DISCUSIÓN

La investigación tiene por objetivo caracterizar la ruta cognitiva de un profesor que imparte la asignatura de álgebra lineal en programas de formación científica. En este sentido, analizamos la interacción de los Esquemas Intra, Inter y Trans cuando responde a una situación problema. La intención es evidenciar la ruta cognitiva para caracterizar las estructuras y mecanismos mentales que la propician.

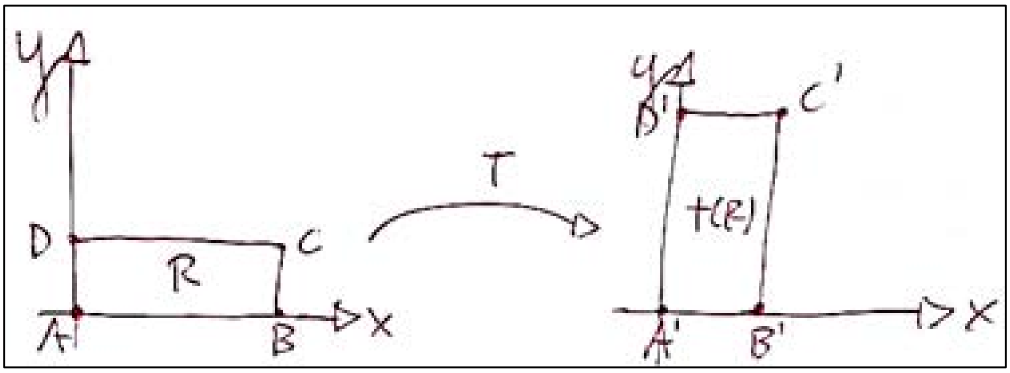
LA SITUACIÓN PROBLEMA

Sobre la región delimitada por los puntos $A(0,0), B(4,0), C(4,2)$ y $D(0,2)$, actúa una transformación T, tal que, $T(0,0) = (0,0)$, $T(4,0) = (2,0)$, $T(4,2) = (2,6)$ y $T(0,2) = (0,6)$. Con esta información, determina la transformación lineal T explícitamente y argumenta sobre si T es un isomorfismo de espacios vectoriales. Además, determina la matriz asociada a la transformación lineal T y describe los movimientos que produce T en la figura inicial.

El objetivo de la pregunta es producir en el informante la activación de los Esquemas IEV_f, IEV_m o IEV_{gf}. Con ello, podremos evidenciar la ruta cognitiva que sigue y podremos caracterizar las interacciones realizadas. Así, las respuestas obtenidas son:

A partir del análisis del enunciado, el profesor Informante (PI) activa el Esquema IEV_{gf}. Evidencia de ello es cuando realiza un diagrama que es coherente con los datos del enunciado. En este sentido, el PI parte ubicando puntos en el plano cartesiano que al coordinar con la estructura mental Proceso de vector, le permite construir el Proceso de región (R). Con ello, la Acción de evaluar los puntos a través de la TL, permite Encapsular en la noción de Imagen de la región. Esto habla del conocimiento que tiene sobre las formas en el plano $\mathbb{R}^2$ y, con ello, la acción que produce la TL en un conjunto de puntos de $\mathbb{R}^2$ en sí mismo (Figura 2).

Figura 2. Representación de la acción que genera TL



A partir del análisis de la Figura 2, durante la entrevista al informante los Investigadores (I) realizaron las siguientes preguntas:

I: ¿Qué significa R, T(R)?

PI: R corresponde a la región generada por dos vectores linealmente independientes (L.I), y T(R) es la región Imagen al aplicar la TL.

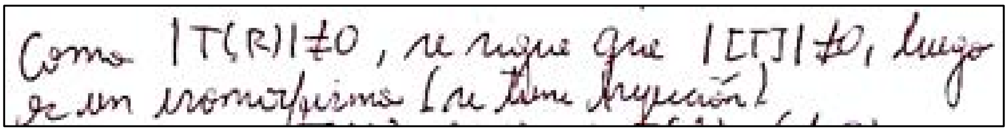
I: ¿Podrías explicar que te hace pensar que la TL está bien definida?

PI: hay dos características del dibujo que me hacen pensar que la TL está bien definida. La primera, es que el vector cero va únicamente al cero $T(0{,}0) = (0{,}0)$ y que a cada vector le asigno un único vector.

En los argumentos mostrados en este fragmento de entrevista, podemos deducir que su explicación está basada en la coordinación que existe entre los Procesos de base generadora y de TL como una función.

A continuación, el PI reflexiona sobre el diagrama representado y atiende a la tarea de mostrar que la TL es isomorfa. Para ello, la respuesta del PI es a través de la construcción de la matriz asociada a la TL. En este sentido, la coordinación se produce desde el Proceso de región R, representada por la figura que forma los vectores L.I y el Proceso asociado al determinante de la matriz $[T]_{\alpha}^{\beta}$ para poder argumentar sobre que esta es invertible y referirse a la biyección (Figura 3).

Figura 3. Coordinación de la Noción de Área y Determinante



A raíz de esta respuesta, los investigadores profundizaron a través de las siguientes preguntas.

I: Explica el por qué desde la figura puedes concluir que el determinante es distinto de cero y, por tanto, te hace concluir que es un isomorfismo.

PI: …la figura se forma por dos vectores que parten del origen, entonces, lo que yo pensé es que el área de esta figura antes de transformar es el determinante de los vectores puestos en columna, entonces el área de la nueva figura va a ser igual al determinante de la transformación de estos dos determinantes.

I: ¿Qué propiedades fundamentas para esta afirmación?

PI: Utilizo las propiedades de determinantes. Me refiero a lo siguiente:

$$\begin{aligned} Area\ (R) = |\vec{v}_1, \vec{v}_2| \implies (T\big(Area(R)\big) &= |T(\vec{v}_1), T(\vec{v}_2)|) \\ &= |A \cdot (\vec{v}_1, \vec{v}_2)| \\ &= |A| \cdot |(\vec{v}_1, \vec{v}_2)| \\ &= |A| \cdot Area\ R \end{aligned}$$

PI: Como el área de la región R es distinta de cero, entonces, la única posibilidad es que el determinante de A es distinto de cero.

A raíz de las respuestas entregadas en el fragmento de entrevista, podemos establecer que para argumentar sobre si la TL es isomorfa, la matriz asociada a la TL representada por la letra A, se construye a través de la coordinación de los Procesos representación de la región R y el determinante como un área, lo que permite argumentar sobre la biyección de la TL, sin necesidad de construir a la TL.

A continuación, la tarea solicitaba que pudiese construir la TL explícitamente. En este sentido, el PI a raíz del dibujo inicial, puede reflexionar sobre las bases a isomorfizar, con ello, los Procesos de base generadora y TL como una función puede construir al proceso de TL representada por la expresión algebraica. Esta construcción se encapsula cuando a cada vector de la base de partida, le asocia un único vector en la base de llegada (Figura 4).

Figura 4. Construcción mental Objeto de la TL

$$T(4,0) = 4T(\hat{i}) = (2,0) \Rightarrow T(\hat{i}) = \left(\tfrac{1}{2}, 0\right)$$
$$T(0,2) = 2T(\hat{j}) = (0,6) \Rightarrow T(\hat{j}) = (0,3)$$
$$T(x,y) = x\,T(\hat{i}) + y\,T(\hat{j}) = \left(\tfrac{1}{2}x, 3y\right)$$

En este sentido, los investigadores realizaron las siguientes preguntas:

> **I:** ¿La expresión encontrada es un isomorfismo?
> **PI:** Claramente es una TL isomorfa.
> **I:** ¿Qué es lo claro? Puedes argumentar
> **PI:** Bueno, ya lo probe arriba… pero es una TL que cumple con que $T(0,0) = (0,0)$ y además, porque es una TL de $\mathbb{R}^2$ en si mismo. Un endomorfismo.

Los argumentos del PI muestran que a partir de la construcción mental Objeto de la TL, se desencapsula las propiedades de inyectividad y sobreyectividad que le ayudan a argumentar sobre si la TL es isomorfa.

Finalmente, se le solicita encontrar la matriz asociada a la TL y, con ello, explicar los movimientos que produce la TL en la gráfica. A raíz de los argumentos utilizados en la Figura 4, y en coherencia con la estructura mental Objeto sobre la noción TL, el PI demuestra que al realizar la Acción de asignar a un vector $\vec{v}$ de la base α en otra base β, puede coordinar los Procesos de base ordenada y vector, a través de la combinación lineal, para determinar las coordenadas de los vectores $[\vec{v}]_\beta$. Con ello, interactúa con el esquema Inter-*IEV$_m$* por medio al construir la matriz asociada a la TL, definida por $[T]_\alpha^\beta$ en la Figura 5.

Figura 5. Construcción de la matriz asociada a la TL.

$$[T] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}$$

Producto de la respuesta entregada, se realizaron las siguientes preguntas:

> **I:** ¿Cuál es el sentido geométrico que puedes establecer?
> **PI:** … de la matriz y del dibujo, puedo decir que existe una contracción de un factor de $\frac{1}{2}$ con el eje $\overline{OX}$ y una expansión de un factor de 3 con el eje $\overline{OY}$.
> **I:** ¿Cómo puedes fundamentar esas afirmaciones?
> **PI:** lo fundamento en los realizado en la Figura 3. Puedo establecer las coordenadas del vector de la base de partida, escrita como una combinación lineal de los vectores de la base de llegada.

El fragmento de entrevista, nos muestra que el Proceso matriz cambio de base, definida por $[T]_\alpha^\beta$, existe una reversión en el Proceso vector coordenada para poder argumentar sobre la acción que produce la TL isomorfa en la gráfica de ella.

Tal cómo ha sido expuesto en las respuestas y fragmentos de entrevista, la pregunta genera una ruta cognitiva que le permite argumentar sobre cómo resolver la situación problema, que se caracterizada en el apartado siguiente.

CONCLUSIÓN

En coherencia con la pregunta de investigación, podemos mencionar que la ruta cognitiva del PI está determinada por la triada de subesquemas que interactúan según el modelo multinterpretativo de IEV **(Trans-*IEV_{gf}*, Trans-*IEV_m*, Inter-*IEV_f*)**. En este sentido, podemos mencionar que, a partir de la pregunta se activa el Esquema Trans-*IEV_{gf}*. Evidencia de ello, es que el informante puede argumentar gráficamente los efectos que produce la TL en relación con las estructuras, mostrados la gráfica de los vectores y su transformación.

La interacción del **Esquema Trans-*IEV_{gf}*** con el **Esquema Trans-*IEV_m***, se da en coherencia con la estructura mental Objeto del plano $\mathbb{R}^2$. En este sentido, demuestra que al realizar la Acción de calcular el área de la región que generan dos vectores L.I le permite coordinar los Procesos de determinante de una matriz y del proceso de matriz asociada a la TL, definida por $[T]_\alpha^\beta$. Ello le permite dar coherencia al discurso para argumentar que la matriz asociada a la TL es invertible, a través, de que el determinante es distinto de cero, debido a que el área de la región identificada no es nula.

La interacción del **Esquema Trans-*IEV_{gf}*** con el **Esquema Inter-*IEV_f***, ocurre cuando a partir de la gráfica existe una reflexión sobre las bases a Isomorfizar. En este sentido, desde la coordinación de los Procesos de base generadora y de TL como una función, permite construir al Proceso de TL a través de su expresión algebraica.

La interacción del **Esquema Inter-*IEV_f*** con el **Esquema Trans-*IEV_m***, se produce desde la estructura mental Proceso de la TL. En ella podemos evidenciar que la Acción de asignar un vector $\vec{v}$ de la base de partida α en otra base β, permite que los escalares puedan ser ordenados para definir la matriz asociada a la TL $[T]_\alpha^\beta$.

Por último, la estructura mental Proceso de la matriz asociada a la TL, se Revierte para dar coherencia sobre los movimientos que genera la TL en función de los vectores de la base de partida. Así, el informante puede reconocer que existe una contracción respecto al eje $\overline{OX}$ y una dilatación con respecto al eje $\overline{OY}$ (interacción del **Esquema Trans-*IEV_m*** con el **Esquema Trans-*IEV_{gf}***).

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Potentials of the use of geometric visualizations in the learning of linear algebra

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Keywords: Digital resources in university mathematics education, teaching and learning of linear algebra

INTRODUCTION

Linear algebra introduces students to basic linear concepts and operations in different dimensions. The concepts and operations covered in the course can to a large degree be visualized in two and three dimensions using a dynamic geometry system (DGS). My PhD research project is aimed at investigating what potentials lie in the use of the DGS GeoGebra to construct and visualize linear structures from the course lectures in student groups.

The main research question in the project will be:

"How can the connection between the geometric and arithmetic aspects of certain linear concepts and operations be developed for STEM-students in linear algebra when working in groups with the construction of visual representations?"

PREVIOUS RESEARCH

In Turgut (2018) it was found that dynamic geometry systems might be used as effective tools of semiotic mediation for teaching learning 3D linear transformations.

Dogan (2018) found that the instruction supported by dynamic visual representations had shaped the knowledge of the participating students, resulting in them making sense of more abstract algebraic ideas using their geometry-based knowledge.

My research approach is different to already conducted studies in the sense that the student groups work with task sheets completely without teacher involvement.

THEORETICAL FRAMEWORK

My research project is based on the belief that learning happens in social and cultural contexts. For this reason, *sociocultural theory* (Vygotskij et al., 1978) is chosen as a grand theory. Hence, it is understood that a consistent didactical approach requires the students to be involved in social group activity during the project.

The theory of semiotic mediation (Bussi & Mariotti, 2008) will be used as a mid-range theory in the project. The theory of semiotic mediation is based on Vygotsky's notion of semiotic mediation in sociocultural theory. The theory relates to the students' use of artifacts to construct mathematical meanings in a social context.

The relation between the artifact and the subject is of special interest in the project since the use of GeoGebra by the students is the defining feature of the intervention. The relation can be explored in greater depth by use of the *instrumental genesis* (Rabardel, 2003).

METHODOLOGY

The study is categorized as a quasi-experimental evaluation study. The project aims to study and evaluate the effects of the intervention where a sample of the students taking the linear algebra course participate in group workshops throughout the semester.

The transcribed video recordings will be analysed by recognizing and categorizing the shared signs produced by students into three categories:

(i) *Artifact* signs (aS) refer to the context of the use of the artifact, often referring to one of its parts and/or to the action accomplished with it.

(ii) *Mathematical* signs (mS) refer to the mathematics context, they are related to the mathematical meanings.

(iii) *Pivot* signs (pS) may refer both to the activity with the artifact and to the mathematical domain.

The analysis is done to measure indexes related to the move from personal sense to mathematical meaning.

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Understanding linear combination with digital tools

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Palabras clave: linear algebra, linear combination, digital tools.

Linear algebra has been recognized as a difficult subject because of the formality with which the concepts are taught (Dorier et al., 2000). Part of the answer to this problem is to develop new approaches to teaching it. Researchers have found that contextual problems allow formal definitions to be signified (Wawro et al., 2012). Others have observed that digital tools can help students develop a geometric understanding of concepts (Tabaghi & Sinclair, 2013). This study presents a sequence of tasks to introduce the concept of linear combination in $\mathbb{R}^2$ and $\mathbb{R}^3$. We chose this concept because it is fundamental to learning linear algebra. The question guiding this study is how does students' understanding of linear combination progress when interacting with tasks based on a contextual problem and the use of digital tools?

THEORETICAL FRAMEWORK

This study used the C&P principles (Cuevas & Pluvinage, 2003) and the emergent model heuristic (Gravemeijer, 1999) for task design. The C&P principles propose to start from a problem in context, promote student participation, validate results contextually, implement inverse operations, and promote the articulation of diverse representations. Emergent models define the transition from intuitive to formal reasoning through four levels of activity (Gravemeijer, 1999): situational, referential, general and formal. We use these levels to characterize progress in students' understanding of linear combinations as they work through the proposed tasks.

METHOD

We present a sequence of four tasks to introduce the linear combination of vectors (see Fig. 1). For each task, we developed a virtual scenario (see Fig. 2) and Exploration and Guided Learning Sheets (EGLS). The EGLS contain instructions for manipulating the virtual scenarios and activities that guide the student in constructing the concept. The tasks were implemented with 20 linear algebra students. The experiment was conducted in three two-hour sessions where students worked in pairs by computer. The data collected consisted of the responses to the EGLS, computer screen video recordings, and audio of each pair of students.

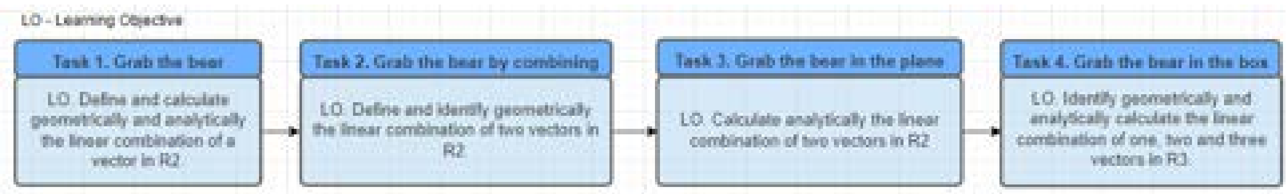

Fig. 1: Task sequence

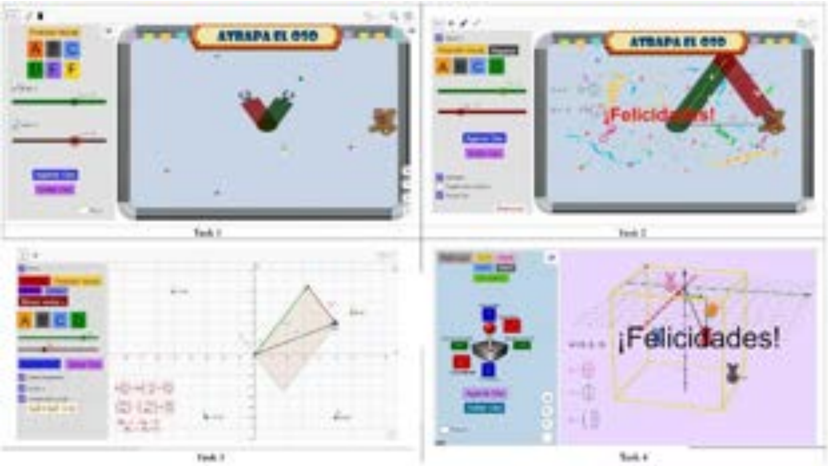

Fig. 2: Virtual scenarios

From the analysis of the data, we can draw some preliminary conclusions. The results show that the levels of understanding developed sequentially. Students developed the situational level when interacting with the contextual simulation of the movement of a robotic arm in a claw machine. They transitioned to the referential level when they worked with the geometric representations of the concept and interpreted the definition of linear combination in context. At the general level, they detached from the context and focused on the geometric representations of the linear combination. Finally, they transitioned to the formal level when they applied linear combination to define whether a vector is a linear combination of others without using the virtual scenarios.

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Continuing the dialogue between semiotic and praxeological analysis: the case of eigentheory

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Keywords: Teaching and learning of linear and abstract algebra, novel approaches to teaching, eigentheory, ostensives.

INTRODUCTION

A dialogue between APC-space theory (Action Production and Communication) and the Anthropological Theory of the Didactics was initiated in 2008 with a work (Arzarello et al., 2008) focusing on how these two approaches complementary frame the semiotic (or ostensive) dimension in mathematical activity. We aim to continue this dialogue, using analytical lenses pertaining to the two approaches, namely the semiotic bundle and praxeological analysis, in a research project regarding the study of eigentheory. Following a pilot study and a design phase (Piroi, to appear), we implemented a teaching sequence on eigentheory in a linear algebra course for a degree program in mechanical engineering in an Italian university. Italian universities are characterized by a standardization of first years mathematics courses in scientific undergraduate degree programs. The linear algebra courses offered to engineering students, thus, do not different from those taught in other degree courses. The geometrical interpretation of linear algebra concepts is seldom presented, and certainly never emphasized. The way eigentheory is presented mirrors this tendency. Eigenvalues and eigenvectors of a linear transformation T are introduced through their formal definition as those values λ and vectors v (if existing) such that $Tv=\lambda v$. Subsequently, the algebraic algorithm to compute them is given, without further exploration of the meaning of these concepts.

RESEARCH QUESTION AND METHODOLOGY

The sequence was designed with the aim of fostering the use of different semiotic resources, to enrich the study of eigentheory. The design had to adapt to the teaching setting typical in the institution, characterized by a very high number of students attending classes. It spanned four two-hour sessions. After completing different tasks assigned during the classes, students could post in the padlet their work to be shared with the whole class and the teacher. For the analysis, we collected recording of the entire classes, pictures posted on the padlet, and video recordings of three small groups of students solving the proposed tasks. We address the following research question: How to analyze the instructional proposal and its effects on students' activities on eigentheory linking the semiotic and the praxeological analyses? Specifically, how do the diverse semiotic registers employed by students and the encountered praxeologies influence one another?

We conducted a fine-grained semiotic analysis of the recordings of each small group, examining how each sign used in their activity emerged, developed, and interacted with other signs used. Complementarily, we performed a praxeological analysis, detecting different praxeologies emerging and evolving during the sequence. In this type of analysis we highlighted the use of different ostensives in these praxeologies, and their role in the techniques used to solve the different tasks. A helpful tool for the analysis, was the construction of a *reference epistemological model* in the form of a question and answer map (Florensa et al., 2020). In such map, we have depicted the connections between possible questions and answers that are likely to arise from an initial question Q0: "Can I find a diagonal matrix similar to the one given? How? Is it always possible?". We have then reproduced the same map to highlight what types of questions, answers, praxeologies emerged, and ostensives used to carry out them actually appeared in our proposed sequence. In the poster we will present this map, where the dominant model for teaching eigentheory, and the one accomplished via our proposal, are visually confronted. Showing pictures of students written productions, drawings and gestures, we will illustrate how the interaction of the different semiotic resources has been beneficial in the development of each praxeology.

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TWG3: Mathematics and other disciplines

Multivariable integrals for physicists – a concept or a tool?

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Abstract. We investigate the use of multivariable integrals in the course Classical electrodynamics as part of research on mathematics education for physics students. The baseline is given by an epistemological model for multivariable integrals in mathematics programs. To understand pedagogical choices for the development of mathematical knowledge within the physics course we interview two teachers. We find that the typical tasks are formulated in highly symmetrical situations which simplifies the calculations, although some tasks require more elaborated techniques. So, the course demands quicker development of mathematics results as tools. The concept of multivariable Riemann integral is reduced to an iterated integral. When this is not sufficient, lack of formalism is compensated by use of physical interpretations.

Keywords: Teaching and learning of analysis and calculus, teaching and learning of mathematics in physics, multivariable integrals, classical electrodynamics.

INTRODUCTION

Multivariable calculus represents a central and significant domain within mathematics, incorporating diverse theories and results derived both from single-variable calculus and linear algebra. At the same time, it is intricately connected with fundamental aspects of topology, differential equations and differential geometry. Its importance extends to physics, playing a foundational role in classical mechanics, electrodynamics, quantum theory and relativity. Even more, the development of multivariable calculus has historically been determined by close intertwinement of mathematics and physics. However, despite these relationships, it is widely acknowledged that "non-specialist students encounter difficulties with mathematics" (González-Martín, Gueudet, Barquero, & Romo-Vázquez, 2021). Regarding physics, there is some evidence on student difficulties described by means of the theoretical framework of the Anthropological Theory of the Didactic (ATD), thus e.g., Hitier & González-Martín (2022) compare students' knowledge concerning the concept of derivative in physics (mechanics) and mathematics, and difficulties that students meet when transferring from one field to another.

This paper is a contribution to ongoing research into learning and teaching curves and surfaces, which are fundamental geometric objects in multivariable calculus. Additionally, it serves as a starting point for a new research direction focused on enhancing the learning and teaching of mathematics for physics students. We focus on the role of integrals of multivariable functions in the (undergraduate) study program of physics. Each such program has its own institutional specificities - some focus more on the experimental side of physics, while others are more theoretically inclined or combined with the study of mathematics. Having a separate course on multivariable calculus taught by a mathematician, who develops the theory systematically, might lead to compartmentalization and weak connections of the taught knowledge to its applications in other courses. In other programs, where there is no such course, the different pieces of multivariable calculus are taught within other (physics) courses. We study the situation in the latter case.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

We formulate the study in the language of ATD (Chevallard, 1991), which describes (mathematical) knowledge in terms of *mathematical organizations* or *praxeologies* as the set consisting of *types of tasks*, *techniques* needed to carry them out, *technologies* that justify the techniques, and *theories* that sustain technologies. Praxeologies describing a piece of knowledge, form a *reference epistemological model* (REM), a relative and hypothetical model set by a researcher, established depending “on the didactic problem approached and the phenomena one wishes to study” (Bosch, Gascon, & Trigueros, 2017, p. 43). ATD further emphasizes institutional construction of knowledge, i.e. it describes the position of an individual towards a piece of knowledge taught or learned in a certain institution. Our case concerns two institutions – institution of mathematics courses and institution of physics courses. When examining two institutions of knowledge construction, the focus is specifically also on the exchange of praxeologies between them (Castela & Romo Vazquez, 2011; Chevallard, 1999; González-Martín et al., 2021).

For undergraduate courses on multivariable calculus carried out for students of mathematics, a specific REM that describes the organization of knowledge related to multivariate integration was developed by Bašić & Milin Šipuš (2022) to study geometrical aspects that are intrinsic to the calculation of multivariable integrals. The proposed REM comprises two regional praxeologies named I1 *Integral Calculus* and I2 *Vector Calculus*, each divided into local praxeologies organized around a definition of a concept or a theorem. In I1 Integral Calculus there are four local praxeologies concerned with Riemann integration: Definition of the Riemann integral, Lebesgue’s theorem, Fubini’s theorem and the Change of variables theorem. In I2 Vector Calculus there are five local praxeologies: Curve integrals, Green’s theorem, Surface integrals, Gauss’ theorem and Stokes’ theorem. Furthermore, five local praxeologies are identified delimiting the use of geometric techniques in the integration tasks:

- G1. Identification of equations representing geometrical shapes,
- G2. Identification of symmetry,
- G3. Identification of relative positions and intersections,
- G4. Conversion between the parametric and the implicit representation, and
- G5. Use of coordinates in a polar and spherical system.

In this study, our intention is to delineate the role of multivariable integrals within the undergraduate physics study program, with a particular emphasis on their relevance in the context of classical electrodynamics. This branch of physics has historically been the origin of the key classical results and applications of curve and surface integrals. In this paper we focus on the mathematics praxeologies interacting with the physics praxeologies as part of the latter's theory, technology or technique. Our main hypothesis posits a notable distinction in the position of multivariable calculus in the education of physicists compared to the education of mathematicians. For instance, we anticipate that the traditional multivariable calculus course offered in a mathematics program may not be essential or adequately comprehensive to meet the mathematical requirements of a course on classical electrodynamics. This is stated even though the mathematical demands of the latter course inherently fall within the domain of multivariable calculus.

Therefore, we question: In which way the mathematical praxeologies of multivariable integrals integrate with the physics program? More precisely, which local praxeologies of the REM for multivariable integrals are present in the course of classical electrodynamics? In cases when this reflects a pedagogical choice of a teacher what are the possible reasons for these decisions, and if some part of the mathematical theory is missing how is that addressed or compensated?

CONTEXT OF THE STUDY AND METHODOLOGY

We investigate the situation at the Department of Physics at the Faculty of Science at the University of Zagreb, Croatia. As a first step we have analysed the contents of the courses given in the first three years of the physics program to identify use of multivariable integrals. There used to be only one elective course in which the mathematical formalism of multidimensional calculus was introduced and presented, but even this course was recently taken out of the program. Hence, we consider a study program in which the multivariable calculus is taught by physicists throughout a few different courses. For students, the first encounter with multivariable integrals is already in the first year in which Riemann integrals in two and three dimensions are used to calculate inertial moments in some typical cases. In the second year, multivariable calculus is used through the context of differential equations.

In this study we focus on the use of multivariable integrals in the course Classical electrodynamic (CED), taught in the third year. As a preparation for our study, we have considered the course materials consisting of the lecture notes written by the professor and exam questions with the solutions written by the teaching assistant. The lecture notes are written in a very pedagogical style, introducing mathematical and physical concepts gradually, and supplemented with appendices on mathematical theory. They are based on the well-known and widely used international textbooks. The notes comprise of more than 380 pages. Integration techniques are not treated separately. We have focused on the three chapters covering the Maxwell equations, in both the differential and integral form.

From the exams posed in the academic year 2018/2019, we have first selected tasks for which the solutions contain any type of (multivariable) integrals. Based on these criteria, we have identified and selected five tasks for further analysis. In the analysis of the solutions, we have used the description of the local geometrical praxeologies to determine which are present and have identified techniques and technologies that are present in the solutions, but not described by our mathematical REM.

After these preparatory steps, we have conducted the interviews with the professor and the teaching assistant of CED to investigate the reasons for the differences between the use of multivariable integrals among the physicists and mathematicians. The purpose of the interviews was to gather insights into practices of the institution and the experience of the students in dealing with mathematical problems related to multivariable integrals. Interviews were conducted by the first author and lasted for about 30 minutes each. Prior to the interview, the teachers were asked to gather examples of students' difficulties in the course concerning the use of mathematics, and more specifically multivariable integrals. The interviews were structured by the following questions:

- In which courses physics students encounter multivariable integration?
- Is the concept of Riemann integration in multiple dimensions discussed?
- In which way do the students learn how to approach integration problems and choose appropriate coordinates? Which coordinate systems are used?
- How do students use and interpret Gauss and Stokes theorem? Are these theorems presented as instances of the same formalism?
- What are, in your opinion, differences among mathematicians' and physicists' understanding of multivariable integrals?

These questions were constructed with the goal to organize the analysis in the following topics: 1) The teaching and learning of the required mathematical techniques, 2) The position of the mathematical theory of multivariable calculus in the course CED, 3) Specific requirements for the teaching of mathematics in physics courses and teacher's reasons for certain pedagogical decisions. The answers were noted during the interviews and then rephrased using the language of ATD and the developed REM. The text was checked by the interviewees to confirm reliability. To refine our findings and confirm our interpretations, we have organized one more interview with the professor of CED and asked for clarifications based on the analysis of the selected five tasks. The interview was again organized by the first author and lasted for 90 minutes. The synthesis of the findings is presented in the next section.

RESULTS AND DISCUSSION

The task analysis and the use of geometrical praxeologies

The analysis of exam tasks and their solutions provide some illustrative situations. We present selected mathematical details from the solutions to five exam tasks (denoted T1-T5) that were singled out during our analysis as those with considerate mathematical aspects. These mathematical details are our shortened descriptions of the solutions, formulated due to the space limitations of the paper. Our analysis resulted in the identification of the local geometric praxeologies as presented in Table 1. The tasks also served as a prop for the final interview with the professor of CED.

Task T1 is formulated in a typical geometric situation in which the appropriate coordinate system becomes evident. The students are expected to use the formulas given in the lectures. The computation is lengthy, but if the spherical coordinates are plugged in carefully, it turns into a simple integral of one variable.

In task T2, the main work is to describe a configuration of four spheres using its symmetries to obtain a system of algebraic equations. The solution is characteristic because of the use of pre-existing formulas for the electric potential corresponding to a specific geometrical shape.

Tasks T3 and T4 differ only in the geometric curve over which the integration takes place. In T3, the choice of the polar coordinate system makes the geometrical considerations easier and enables the use of symmetry. The calculation of the integral along the circle becomes trivial as the integrating function is constant. In T4 the integration is over a hyperbola (instead of a circle) and the solution requires integration of trigonometric functions.

In T5, the main ideas of vector calculus about the electric and magnetic potential are to be used to provide a strictly mathematical consideration of introducing a new vector potential. In this task only the mathematical techniques are required, and the new vector potential has no physical meaning (known to students in advance). Furthermore, the solution requires many aspects that are not part of the mathematical REM (i.e. of a typical mathematics course in multivariable calculus): the familiarity of the Poisson differential equation (which the students have encountered in the course but in a different context) and the technique of writing the integrated function as a series of Legendre’s polynomials.

Task	Context	Theory	Local geometric praxeologies
T1	Electrical force of the sphere on a sphere cap	Riemann integral	G1, G2, G5
T2	Total charge of a system of four spheres	Gauss' theorem	G1, G2, G5
T3 and T4	Magnetic field produced by the electric current in a curved wire	Biot-Savart’s law (curve integrals)	G2, G5
T5	Abstract formalism of the electric and magnetic potential	Divergence and curl operators of vector calculus	N/A

Table 1: Geometric praxeologies in selected tasks.

Teachers' perspective on the use of mathematical praxeologies

In the interviews, the teachers have confirmed our analysis of the courses in the first three years of the physics program. They added that, in informal discussions, the students report that CED is the most difficult course for them so far. One of the reasons why it stands out as demanding in terms of required mathematical techniques is that some of these techniques are not developed in any other part in the study program. For the part on Riemann integration, the theoretical part of the (regional) praxeologies is not presented. Instead, when there is a need to integrate over geometric objects the task is directly reduced to parametrizations and iterated integrals. As the teaching assistant claims, opposed to mathematics programs, in physics programs students only need to master one type of conversion between representations (e.g., parametrizing a geometrical object), whereas algebraic equations of geometric objects are not used.

The theory of Fubini's theorem (about calculating Riemann integrals as iterated integrals) or the Change of variables theorem is not mentioned explicitly, although it is sometimes used implicitly, and their use is never justified. For the first theorem the professor explains that for the physics students, multivariable integrals are introduced as iterated integrals and the students are not introduced to the classical definition of multivariable Riemann integral. For the change of variables, there is no need for any non-standard choices of coordinate systems, and for the standard ones (e.g., polar or spherical) students learn how to use them during mechanics courses in their first year (simply by memorizing the differentials in each system and understanding how the parameters are considered in these systems). The professor concludes that for his course "it would be valuable that the students are aware of these results explicitly and understand when they use them, but that there is no need for the proofs of these results".

The teaching assistant also explains that the use of specific coordinate systems is taught from the first year 'on the go' in the scope of various courses, e.g., General physics, Classical mechanics etc. When comparing to the tasks in the mathematical REM, we may notice that the first encounter with such tasks (e.g. find the center of the mass, the inertial moments etc.) is similar to one given in the mathematics courses. The assistant continues to explain why not all praxeologies of the mathematical REM are used in the course and why the tasks are suited for a particular kind of techniques:

> There are three coordinate systems used by physicists: Cartesian, cylindrical and spherical. Other coordinate transformations (e.g., plane rotation by 45 degrees, which can be described $(x, y) \mapsto (x + y, x - y)$) are familiar only to a few students and shown as peculiarities. The students learn how to use them situationally, e.g., the spherical system is used for a sphere or a point source. These examples take advantage of the presence of symmetry. After making a choice of an appropriate coordinate system, the integration

> problems are usually such that the iterated integral has separated variables. For the problems that are not posed in such a way, the students are given the instruction in the form of a rule they are advised to follow: in spherical coordinates integrate in the order phi-theta-r. To graduate from the physics program (5 years) the students need to use only iterated single variable integrals.

Similar phenomena occur for the theory and main theorems concerning surface integrals. For a physics student at the level of CED, Gauss' and Kelvin-Stokes' theorem are two separate and different results, representing "the constraint equations and the evolution equations". The teachers point out that it is not of such great importance to observe the same math formalism. Moreover, these results are considered by physicists as "technical tools that enable practical considerations and simplifications of integrals and are used mechanically". Instead of proofs, a simplified argumentation for the theorems is provided based on two main ideas: curves and surfaces are represented as unions of linear objects (segments or rectangles) and then the results are deduced by passing to the limit (considering 'infinitesimal' partition of the geometric object). The professor says:

> For a physicist, this argument is acceptable because of its plausibility and accordance with the intuition about space and geometry. The issue for the students appears as the concepts of surface orientation and differential forms are not developed, so the rules for changing the sign in the surface integral (e.g., $dy \wedge dx = -dx \wedge dy$) remain vague and unexplained. In general, the solutions of equations should in addition consider 'natural' boundary conditions, e.g., they vanish at infinity (as fast as needed).

Our analysis of tasks T1-T4 shows that in the course students rarely encounter objects other than simple and highly symmetrical rods, cylinders, points and spheres. The second order surfaces (e.g., paraboloids and hyperboloids) and the intersections of multiple surfaces of that kind is not considered. Hence, the geometric praxeologies related to the relative position of the surfaces (G3) are neither necessary nor present. Furthermore, in the solutions to tasks we did not encounter instances of conversion between the implicit and parametric equations (G4). In the interviews, this is confirmed by the teachers and justified. After discussing the examples in the interview, the professor pointed out that he was not aware, but agrees that the tasks considered in the course are rich with (geometrical) symmetry that simplifies the calculations and that most of the tasks could be solved with single integrals. He continues in arguing that the less symmetric examples will not occur as they seem "physically ugly", could be avoided by a change of a coordinate systems, or are usually physically not relevant.

We learn from the interviews that the special techniques needed for T5 are shown during the course, as the professor explains: "In general, integrating a power series term by term is not possible always and requires a mathematical justification, which we do not require in this course." From this we see that the calculations performed in physics are not always justified at the level of mathematical rigour and that sometimes more advanced mathematics is required by the discipline. The professor points out that there are many more instances in which, in general, there is a need to consider the justification for a mathematical step in the calculation. Examples of such steps are taking the derivative of the integral, commuting the integral with the limit, or commuting the integral with summation (as in the example). During the course lectures, these steps are explained more rigorously, but this is not required from the students during the exams.

We interpret these considerations by asserting that physicists may not require all the mathematical praxeologies of our REM. However, the program might be improved by reinforcing the theoretical components that are relevant to physics. The professor emphasizes the primary distinction between the mathematics and physics courses:

> Mathematicians often adopt a more systematic approach, gradually developing concepts, while physicists tend to emphasize interpretations, using integrals as tools and generating results in a limited number of situations of interest. Mathematics courses progress too slowly, and the required results should be familiar to physics students early in their university education.

Unsurprisingly, we also see that the needs for mathematical knowledge related to multivariable integrals in the discipline of physics extend the usual organization of multivariable calculus courses in standard mathematics programs. As to the views on the specific program, we learn from the professor that the institution withholds the tension between theoretical and experimental physicists, which is jokingly described as: the former are inclined to discuss ‘tensors’ and the latter focusing on ‘sensors’. This dynamic results in ongoing discussions about the position of mathematics courses.

This seems aligned with an intriguing, and probably well-known, phenomenon, in our view. Physics courses demand a high level of proficiency in using mathematics as a tool, yet the study program falls short of the formalism and proofs commonly found in mathematics programs. Paradoxically, this apparent gap does not hinder the learning and teaching of physics if the main concepts and technologies of the theoretical block (logos) are effectively presented. The absence of mathematical formalism is compensated by the incorporation of physical concepts and interpretations, which provide additional meaning to the calculations.

CONCLUSION

In this paper we have investigated the use of multivariable integrals in physics courses. Our analysis provides some insight into the use of mathematical knowledge on integrals in multivariable calculus in the course Classical electrodynamics (CED), provided by textbooks, exam questions and solutions and the interviews with the teachers in the course. Knowledge on integrals required in CED is analysed with respect and compared to the developed epistemological model (Bašić & Milin Šipuš, 2022) for teaching mathematicians.

Interviews with a CED professor who commented that mathematics courses are "too slow" for the requirements of physics education, have prompted us to validate a very general assertion that mathematics education of physicists as non-specialists demands a distinct *didactic contract* compared to that of mathematicians. This is particularly pertinent to the deductive "theorem-proof" approach commonly employed in mathematics courses by means of which mathematical results are not developed "quickly enough". However, in the considered physics study program, the multivariable calculus is taught by physicists throughout many different courses. Students are not exposed to a typical mathematics course, moreover the multivariable calculus results are not systematically developed. This results in the absence of some mathematical praxeologies. Concerning the use of integrals, the observed absence of mathematical praxeologies (e.g., tasks dealing with general curves and surfaces, taking a theorem as a definition) is justified by being "physically not relevant". Since explicit formulation of mathematical results is also sometimes missing or remains limited only to a narrow type of cases (presented as tasks), students' calculus knowledge remains unrelated. They reach for various tools offered by a physical context. This could also be the focus of future research – to gain insights into how physics students manage incomplete mathematical content.

The compensation of mathematical requirements in physics courses encompass a broader set of tools (*techniques* and *technologies*) than typically presented in mathematics courses, drawing from the discipline of physics. The techniques often rely on the careful choice of coordinate systems to exploit the symmetries of the involved objects and reduce the calculations to single variable integrals. Furthermore, there is no concept of a Riemann integral over multidimensional domains other than the iterated integral based on the parametrization of the domain in specific standard coordinates. As a consequence, integration is performed as a technique without theoretical justification, since Fubini's theorem and Change of variables theorem are used only implicitly. If this leads to a potential difficulty in solving a task (for example, if the integral cannot be calculated in one way and one of the two theorems should be used), the lack of theory is substituted by argumentation based on physical interpretation. Moreover, argumentation in physics can be followed by students without knowing the (mathematical) proofs of the results used as tools. Hence, we speculate that further enhancements of mathematics education for physics students may be achieved by providing solid foundations of the basic mathematical concepts along with the explicit presentation of the theorems, as well as a clear indication when they are used and why that is justified.

The mathematical knowledge to be taught to physicists requires a deeply thought-out *didactic transposition* that will provide at least some parts of the theory for the students to feel confident about understanding the procedures they follow and to carefully select the techniques and types of tasks that are relevant to most typical physical situations. The students might be encouraged to pursue deepening their knowledge on their own or through additional (elective) courses, depending on their future academic choices. In the end, there is never a perfect balance, but we might certainly hope that the multivariable calculus for physicists will remain being taught either by mathematicians that are aware of the requirements of the physics program, or a slightly theoretically inclined physicist that is enthusiastic about the systematic development of mathematical knowledge.

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Students' Understanding of Time Series and Trajectory of Solutions to Differential Equations

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Abstract. The study of modeling and dynamical systems is becoming increasingly important in the life sciences. The concepts of time series and trajectory are central to understand the behavior of a dynamical system described by differential equations. A critical skill is to be able to navigate between these two types of representations as they provide complementary information. In this exploratory study, I use the covariational reasoning framework to analyze to what extent students who have taken an undergraduate course on modeling have mastered the skill of sketching the trajectory associated with a time series, and vice versa. Analysis of interviews show that students do not systematically exhibit the same level of covariational reasoning when completing these related tasks.

Keywords: mathematical modeling, mathematics in other disciplines, differential equations, covariational reasoning

INTRODUCTION

Over the last decades, the use of mathematics in biology and the life sciences has become more and more pervasive (May, 2004). This evolution has been accompanied by high profile reports from professional associations in the United States calling for the reform of the mathematics instruction of biology and life science students (e.g., National Research Council, 2003; American Association for the Advancement of Science [AAAS], 2009). These documents call for mathematics courses that focus on modeling and dynamical systems. For example, the AAAS describes the "ability to use modeling and simulation" (AAAS, 2009) as a core skill for undergraduate biology students. Similarly, the Association of American Medical Colleges (AAMC) lists the ability to "make inferences about natural phenomena using mathematical models" as a core competency that students applying to medical school should master (AAMC, 2009). These recommendations have also been used outside of the United States, for example in Australia (Matthews et al., 2010).

In this context, one question that naturally emerges is how students understand the concepts necessary to interpret and analyze models described by differential equations, in particular the concepts of time series and trajectory. These two types of representations of solutions to differential equations are central for "quantify[ing] and interpret[ing] changes in dynamical systems" (AAMC, 2009) as well as for determining the long-term behavior of a system. A time series shows the graph of each variable as a function of time. Thus, when given a time series it is easy to determine the value of each variable for any given time. However, with a time series it is hard to predict how a system would react to a perturbation or how the time series starting from a different initial condition would look like. A trajectory shows the how the variables change in the state space, which is the space formed by the variables describing the system[1]. With a trajectory it is easier to interpret changes in a system and to predict how it would evolve after a perturbation. It is a critical skill to be able to go back and forth between these two types of representation, more specifically to be able to construct the trajectory associated with a time series, and vice versa.

Over the last decades there has been a growing interest in the teaching and learning of differential equations. Lozada et al. (2021) identified in their literature review less than twenty articles published on this topic in the 1970s but more 160 articles in the 2010s. While many articles examine the use of specific teaching methods, activities or software to teach differential equations, some authors have focused on student thinking of differential equations. For example, research has investigated student reasoning about the notions of equilibrium solution, asymptotical behavior, and stability (e.g., Rasmussen, 2001; Zandieh & McDonald, 1999). Other authors have investigated how students interpret solutions to differential equations when using a direction field (Ortiz et al., 2010) or other graphical representations (KarimiFardinpour & Gooya, 2018). However, to my knowledge no study has examined how students navigate between the time series and trajectory representations. The goal of this exploratory study is therefore to analyze to what extent students who have taken an undergraduate mathematics course focusing on modeling have developed the skill of constructing the trajectory associated with a time series, and vice versa.

THEORETICAL FRAMEWORK

In order to analyze how students build and make sense of these two types of graphical representations I use the covariational reasoning framework as described by Thompson and Carlson (2017). Covariational reasoning is defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002). Several authors have argued that covariational reasoning is central to understand the concept of function and to engage in mathematical reasoning (e.g., Thompson, 1994; Carlson et al., 2002). The six levels of Thompson and Carlson's covariational reasoning framework (2017) are given in Table 1.

Level	Description
Smooth continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous	The person envisions changes in one variable's value as

1 In two dimensions, the state space is also often called the phase plane.

covariation	happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together.
Pre-coordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

Table 1: Levels of covariational reasoning.

Chunky continuous variation, that is used to define chunky continuous *co*variation, means that a person thinks of variation of a variable's value as changing by intervals of fixed size. For example, the person thinks of the variable's value changing from 0 to 1, then 1 to 2, and so on. The values between 0 and 1 come along because they are part of the chunk but the person does not envision the variable having these values in the same way as for 0, 1, 2, and so on (Thompson & Carlson, 2017).

Using the covariational reasoning framework, my research question can be formulated as: What level of covariational reasoning do students exhibit when sketching the graph of the trajectory associated with a time series, and a time series associated with a trajectory?

METHOD

I conducted think-aloud interviews with 8 participants. All participants were undergraduate students who had previously taken a mathematics course for biology and life science majors. This course focused on modeling and dynamical systems, and one topic covered is how to build time series and trajectories. In this course students also learn to code in SageMath in order to simulate solutions to differential equations. All students had taken the course during the previous quarter, except for one who had taken it two quarters before the interview.

For the interviews, students were given several exercises to solve including one task asking them to sketch the trajectory associated with a time series, and another one asking them to sketch a time series associated with a trajectory. Both the times series and trajectory came from two-variable models. Such models are widely used in the life sciences and beyond; for example, to describe the evolution of predator-prey populations (e.g., Lotka-Volterra model), competition between species or the movement of a mass attached to a spring. The time series and trajectory given to the students are shown in Figure 1 and Figure 4, respectively. Students were asked to sketch the corresponding trajectory or time series on paper while saying aloud what their reasoning was. The interviews were recorded and transcribed and copies of students' written work were retained. I conducted a thematic analysis (Braun & Clarke, 2006) of the transcriptions focusing on the level of covariational reasoning exhibited by the students while completing these tasks.

RESULTS

In this section I focus on the interviews of 2 students with pseudonyms Stan and Carly. These two students were selected because they both reasoned at different covariational levels when completing these two tasks. Moreover, their answers are representative in the sense that all students who knew how to approach these tasks reasoned at either the smooth continuous or chunky continuous covariational levels.

Exercise 1: Sketching the trajectory associated with a time series

In this exercise students were asked to sketch the trajectory associated with the time series shown in Figure 1. The two variables represent the evolution of Romeo's love (or hate) for Juliet and of Juliet's love for Romeo.

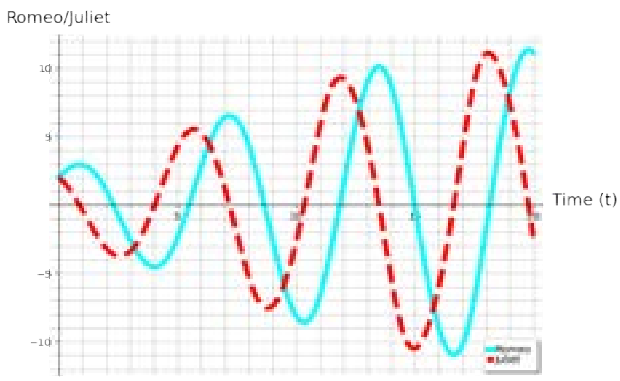

Figure 1: Time series given to the participants.

Stan started solving this exercise by observing that "this is a time series and there is oscillations". He then mentioned that "one way to take the trajectory is to look at where the points are at some time unit. I'll be going in units of 5 to make it easy." He then went on picking points on the time series and wrote down the values of the two variables in a table. Next he sketched the points and connected them with line segments. The fact that the points be connected by line segments (see Figure 2, the original trajectory is sketched with solid lines with arrows indicating the direction) and not smooth curves strongly suggests chunky covariational reasoning. When asked whether he could create a more precise graph, Stan answered:

Stan: If I wanted to go for the most accurate, I would most likely realistically (sic) go by units of three or two, most likely units of two because it would give us 10 points and that might be enough to create more representative trajectory.

In other words, Stan would keep using the same process but with a smaller time interval. Stan then sketched how a smaller time step could give a different trajectory. Notably, he still connected the new hypothetical points he drew with line segments. These line segments are in dashed line in Figure 2 (whereas his original trajectory is composed of the solid line segments with arrows on them). This answer further shows chunky continuous covariation.

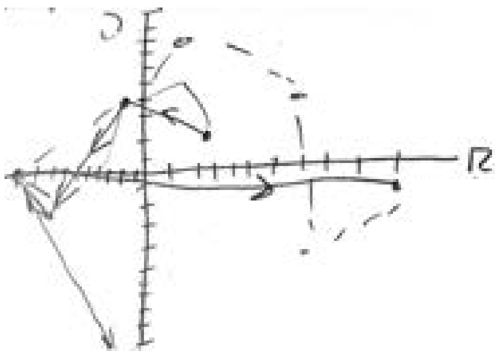

Figure 2: Stan's trajectory.

We can contrast Stan’s method and reasoning with Carly’s. Carly also started by picking points on the time series and writing down the values of the two variables in a table. However, when asked how she had decided which points to pick, she explained: “I'm just going by each of the points when they kind of just change direction in general or when they cross the line [Note: meaning when they are zero] or hit a max or min.” In other words, instead of picking points at fixed time intervals, as Stan did, Carly picked points that are important to draw the trajectory. She then connected them with smooth curves as we can see in Figure 3. This trajectory made out of smooth curves suggests that Carly reasoned at the smooth covariational level.

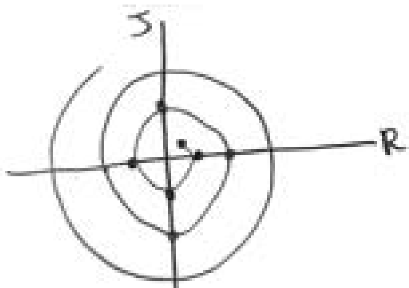

Figure 3: Carly's trajectory.

Exercise 2: Sketching a time series associated with a trajectory

The second exercise asked the students to sketch a time series associated with the trajectory shown in Figure 4. This exercise is in a way the inverse of the previous task. It would thus seem reasonable to expect students to exhibit the same type of covariational reasoning as they did in the first exercise.

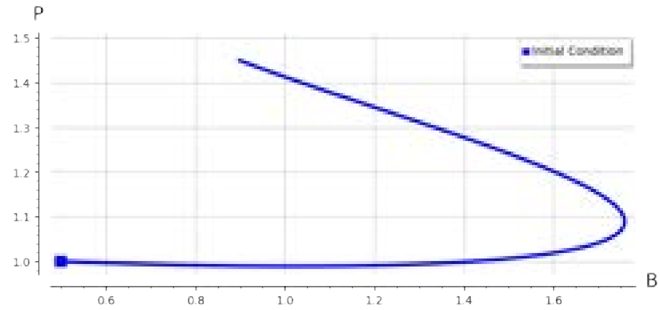

Figure 4: Trajectory given to the participants.

For this task, Stan noticed from the start:

Stan: The first thing I notice is that from the initial condition until B=1.4, it remains relatively consistent at P=1. […] So, I'll just take some of the points [between the initial condition and B=1.4] because we could assume that everything between these points are (sic) going to be relatively the same.

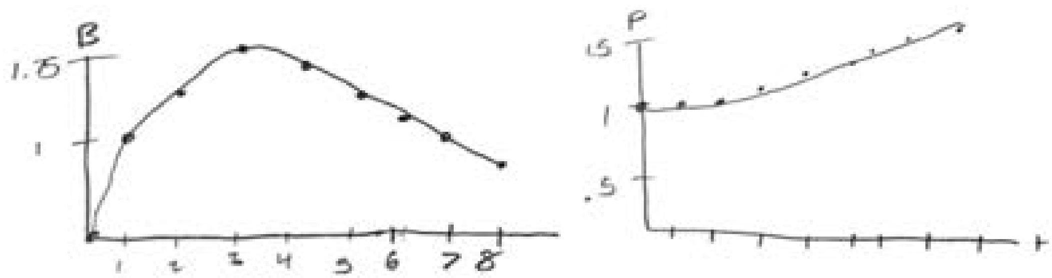

Figure 5: Stan's time series.

While Stan picked "only" three points on the first part of the trajectory that goes from the initial condition to B=1.4 (he picked the initial condition, the point at B=1, and the point at B=1.4), he still highlighted every point where the trajectory crosses a gridline. Stan then highlighted the point where the trajectory turns back "because it starts to go backwards, where B is reducing, and P is continuing to increase". This is in clear contrast to what he did in the first exercise where he picked points at regular time intervals regardless of the features of the two functions making up the time series. After that, though, he picked all the points where the trajectory crosses gridlines without attending to the features of the trajectory (similarly to what he did for the first exercise). After writing the coordinates of the points he had picked on the trajectory in a table, he sketched the points on a graph. He then connected the points with *smooth* curves and not line segments in contrast to what he did in the first exercise (see Figure 2). For this second task we thus see that Stan exhibited behaviors that suggest smooth covariational reasoning while also showing actions suggestive of chunky continuous covariation (such as picking points at regular intervals for the second part of the trajectory).

Carly started this task by writing down in a table the coordinates of points on the trajectory. When asked how she had picked the points she answered "more so just, go by [the gridlines] cause they're easy to… [Note: probably means the numbers are easy to read] and then I figure out the P." She then graphed the points and connected them with line segments (see Figure 6). So, unlike what she did for the first exercise where she picked points based on important features of the graph and then connected them with smooth curves, this time she picked points based on how easy it was to read their coordinates. One exception is the point (1.75, 1.05) where the trajectory turns back that she also noted in her table. Then, instead of connecting the points with smooth curves, she used line segments. Taken together these actions suggest that Carly works at the chunky continuous covariational reasoning level for this task unlike what she did in the first exercise.

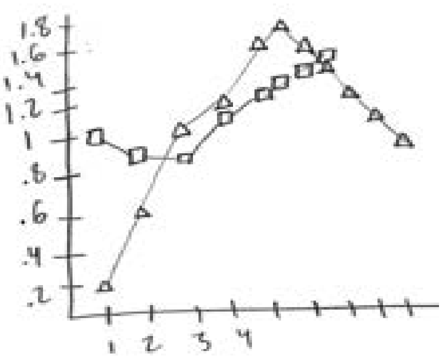

Figure 6: Carly's time series.

These two tasks show that students do not automatically reason at a consistent covariational level even when completing tasks that are directly related to each other. An obvious question that needs to be further investigated is why the students seem to reason at different levels for two tasks that can be seen as inverse of each other. Related is the question of whether students see these two tasks as deeply related to each other or whether they see them as separate (albeit similar) procedures. Another point that needs to be studied is to what extent examples that students have seen in class influence their answers. For example, did Carly remember an example of an unstable spiral (drawn with smooth curves) or did she make a conscious choice to use smooth curves? Had she learned that oscillations with increasing amplitude correspond to unstable spirals?

Influence of teaching methods

Both Stan and Carly were in the same class. They had the same instructor, attended the same lectures and had the same homework. Based on the slides used in class, the topic of navigating between these two types of representation was introduced with one example done by the instructor of starting with a time series and building the associated trajectory (but no example of starting with a trajectory and sketching a time series was shown in class). Students were given a protocol that starts with "Choose the important points (start, end, extreme & middle points)" on a time series or trajectory. Then "Make a table with time and state variables as columns". Next, "Draw your axes (both axes state variables)" and finally "Plot the points for each time […] & connect them." Students then completed a small-group activity where they were given a time series and had to sketch the associated trajectory. The solution for this activity showed a trajectory with a smooth curve. While the example shown in class exhibits a smooth curve, one can see that the method focuses on sampling points and then connecting them. The idea of smooth continuous covariation of the two variables is not explicitly mentioned or discussed. It could easily be overlooked by the students and thus support a chunky continuous covariation point of view. This idea is reinforced by the homework. The first two exercises on this topic show hypothetical time series and trajectories that are chunky continuous. The solution to the first exercise is discrete (a collection of points) while the second one is chunky continuous. The two other exercises on the topic show a smooth trajectory and a smooth time series. However, the solution to the last exercise shows a chunky continuous trajectory, which could signal to the students that a smooth time series can be associated with a chunky continuous trajectory.

There is another important element that can influence students into adopting a chunky continuous covariational reasoning perspective: coding. All students in the course learn to code with the computer algebra system SageMath. In particular, students learn to draw time series and trajectories. To do so in SageMath, one first needs to create a list of time points for which the values of the variables will be numerically approximated. Then the points are connected with *line segments*. Thus, when there are few time points, a trajectory or time series appears chunky continuous, in other words, a series of points linked by line segments. To make a trajectory or time series more precise, one increases the number of time points which makes the graph look smoother. In this context, it would be easy for a student to think that a time series or trajectory is a collection of points connected by line segments and that the graph "becomes" smooth when the number of time points is large enough.

This idea that adding points to a trajectory will make a collection of line segments "into a curve" can be observed in Stan's interview. When asked how confident he is about the sketch of his trajectory, he answered that while he is very confident about the process he feels that there are "so many missing points" in his trajectory. Then pointing to a line segment between the first two points on his trajectory he said that "because this is just a straight line, there could realistically be a line that is somewhere or a dot that makes this a curved line (sic) rather than a straight line or something like that." While saying this he drew extra points and connected them with what line segments (see dashed lines in Figure 2). It is interesting to note that he talks about “curved line” or that by adding points and connecting them with line segments, on makes a straight line become “curved”. I note that this still exhibits a chunky continuous covariation perspective, only with a smaller time step.

CONCLUDING REMARKS

This exploratory study is a first step into understanding how students reason when constructing the time series or trajectory graphs of a solution of a system of differential equations. We have seen that such situations are opportunities for students to reason covariationally and that they operate between the chunky continuous and smooth continuous levels. Notably students do not automatically exhibit the same level of covariational reasoning for the two exercises presented here (I note, however, that other participants did show the same level of covariational reasoning for both exercises). Future research should analyze this apparent discrepancy. Do students really use different levels of covariational reasoning or have they learned that a graph should look smooth but without knowing why? Can they explain why they used one level rather than the other? Another area that needs more research is how teaching methods and the use of computer software influence students’ understanding of how these graphs are drawn. Would an approach that focus more on qualitative sketches of time series and trajectories better support students in acquiring smooth continuous covariational reasoning rather than an approach focusing on a procedure using time sampling?

ACKNOWLEDGEMENT

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Analysing statistical teaching practices in a specific institutional context

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We study the teaching practices of Statistical Sampling Theory in a specific context of professional Education in France. We focus on a specific moment of teaching practices, called SAÉ (Learning and Evaluation Situations). After discussing the SAÉ as the main object of the study, our aim is to characterise the links made in the teaching between the SAÉ and 1) the Ressources (courses), 2) the professional competencies to be developed by students and 3) the professional environment. From an anthropological approach, we investigate these links in terms of personal didactic praxeologies developed by a teacher. This empirical study sheds light on several aspects of the teaching practices, like the didactic choices made by the teacher to take into account the students' needs in this specific institutional context.

Keywords: Teachers' and students' practices at university level, Curricular and institutional issues concerning the teaching of mathematics at university level, Anthropological approach to didactic, Resources, Statistical Sampling Theory.

INTRODUCTION

The present study focuses on teachers' practices in tertiary education. Our research work is part of *Dida-Stat$_{Expl}$* project, which aims to carry out an exploratory study to analyse and characterise the teaching practices in Statistical Sampling Theory for future practising statisticians in France. We aim to bring elements of understanding on teaching Statistics in a context of post-secondary professional Education. The scope of our research is understudied, the context is related to a recent reform; we therefore followed an empirical approach, based on a case study (single case) to explore the field. We seek to build a theoretical framework and a method for studying practices to identify the didactic choices made by a teacher, as well as the reasons, which explain these choices in this institutional context. We focus on a specific moment of teaching practices, called *Situations d'Apprentissage et d'Évaluation* (Learning and Evaluation Situations) – designated by *SAÉ* in the following to keep the French acronym. Our aim is not to study the evaluation practices in teaching the Statistical Sampling Theory, but rather the way in which the *SAÉ* offers the possibility of understanding teaching practices. This type of situation is reminiscent of the recommendations for curricular reform proposed by Pepin et al. (2021) for the training of future engineers.

After presenting the specific institutional context of the study and in particular our object of study *SAÉ*, we will describe the theoretical framework based on the Anthropological Theory of Didactic (Bosch & Gascón, 2014). We will then define our methodology of data collection and analysis. The final sections are devoted to analysis, results and conclusion.

INSTITUTIONAL CONTEXT

Curricular analysis of the Technological University Bachelor in France

From the start of the 2022 academic year in France, the Technological University Institutes (*Instituts Universitaires de Technologie* in French), a component of the university, offers three-year undergraduate programs called the Technological University Bachelor (*Bachelor Universitaire de Technologie* (*BUT*) in French). These programs are designed to provide students with knowledge and practical skills. This professional bachelor, which is a post-secondary education, is based on a national curriculum, which is then specified according to various specific fields (e.g. legal careers, chemistry, information-communication, etc.). One of the major innovations of this curriculum is the 'Competency-Based Approach'. As part of the curriculum reform, each pathway is certified by 4 to 6 final competencies, and competencies are "understood as 'complex knowledge of how to act' implemented in a professional context and which mobilise and combine resources acquired during the course"[2] (MESR, 2022, Appendix 1). In the rest of this paper, when we talk about competencies, we are referring to the competencies targeted in the curriculum. As Bolou-Chiaravalli et al. (2022) point out, this refers to the meaning of competency as "complex knowing how to act based on the effective mobilisation and combination of a variety of internal and external resources within a family of situations"[2] (Tardif, 2006, p. 22).

With the aim of developing the competencies targeted in the program, the teaching modules are structured around *Ressources*[1] and *Situations d'Apprentissage et d'Évaluation* (*SAÉ*). The *Ressources*, which take the form of courses, enable "the acquisition of fundamental knowledge and methods" and the *SAÉ* encompass "the professional situations in which the student develops the competence" (MESR, 2022, Appendix 1). *SAÉ* are "situations which have an integrative aim, confronting the student with activities similar to those encountered by professionals in the field"[2] (Bolou-Chiaravalli et al., 2022). *SAÉ* are also intended to be assessed for certification purposes. *SAÉ* seek to be "authentic", even if this type of situation in training (i.e. outside professional environments) cannot really exist as such, so it is recommended to design and to organise them pedagogically to encourage learning and the development of the targeted competencies (Bolou-Chiaravalli et al., 2022). These situations are added to those encountered during work placements in order to increase the potential for professional development of the diploma (Bolou-Chiaravalli et al., 2022). The place of *SAÉ* is quantitatively important, they represent between 40% and 60% of the content of the national curriculum. *SAÉ* are described in the national curriculum by means of recommendations. They are therefore an important part of Bachelor (*BUT*) students' education. Moreover, although some projects may have existed in the past in Technological University Institutes, this new format, closely linked to the Competency-Based Approach and the professionalisation aspect of the diploma, is a new and original feature that needs to be taken into account in the study of teaching practices. The *SAÉ* are at the interface between the competencies to be developed in the training, the *Ressources* provided in the training (the courses) and the professional environments for which the students are destined.

In this paper, we are going to focus on the *SAÉ* as a research object to understanding teaching practices in this Bachelor.

The specific case of an *SAÉ* in Data Science

Our study focuses specifically on the Bachelor (*BUT*) "Data Science", specialising in "Statistical Exploration and Modelling", which aims, according to the national curriculum, "to train professionals skilled in the collection, processing and statistical analysis of data"[2] (MESR, 2022, Appendix 24). This pathway is based on four final competencies, which are: Processing data for decision-making purposes; Analysing data statistically; Adding value to a product in a professional context; Modelling data within a statistical framework.

The *SAÉ* that interests us in our research is studied in the second year of the Bachelor in the third semester. It aims to develop these four competencies and is entitled "Data collection and analysis by sampling or experimental design" (SAÉ 3.EMS.01). The objectives of this *SAÉ*, as stated in the national curriculum, are to:

- "deepen the notion of survey and polling in a more general framework" than students have already been able to develop in the first year;
- "make students understand the difference involved in a draw without replacement, the most common situation in a survey";
- "encourage students to think about setting up a design of survey experiment"[2] (MESR, 2022, Appendix 24).

The description of the *SAÉ* in the national programme only states:

> "The student is put in the situation of setting up a survey based on a data collection plan to respond to a defined problem. This SAÉ provides an opportunity to study survey methodology in greater depth. The student must be able to define the target survey population, be able to choose a sample judiciously before drawing up the data collection plan, determine the data collection plan [...], determine the sample size, design and draw up the questionnaire [...], collect the data, judge whether the sample needs to be adjusted [...] and judge the quality and reliability of the sample survey."[2] (MESR, 2022, Appendix 24, p. 136).

This *SAÉ* is mainly linked (but not exclusively) to the *Ressource* entitled "Survey sampling techniques and methods" (R3.EMS.10), the contents of which include the different survey sampling techniques (simple random sampling without replacement, stratified, multi-stage, clustered random sampling), adjustment methods, sources of biases, etc. The principal *Ressource* associated with the *SAÉ* is officially 20 hours long. However, teachers could suggest additional sessions depending on the need and the content they decide. These additional sessions are considered – following the curriculum – as complementary *Ressources*. *Ressources* and *SAÉ* from the first year of the Bachelor may also be useful for the *SAÉ*, according to the national curriculum.

Figure 1 illustrates the links recommended institutionally, according to the curricular analysis, between this specific *SAÉ*, the competencies, the *Ressources* and professional environments targeted by the training.

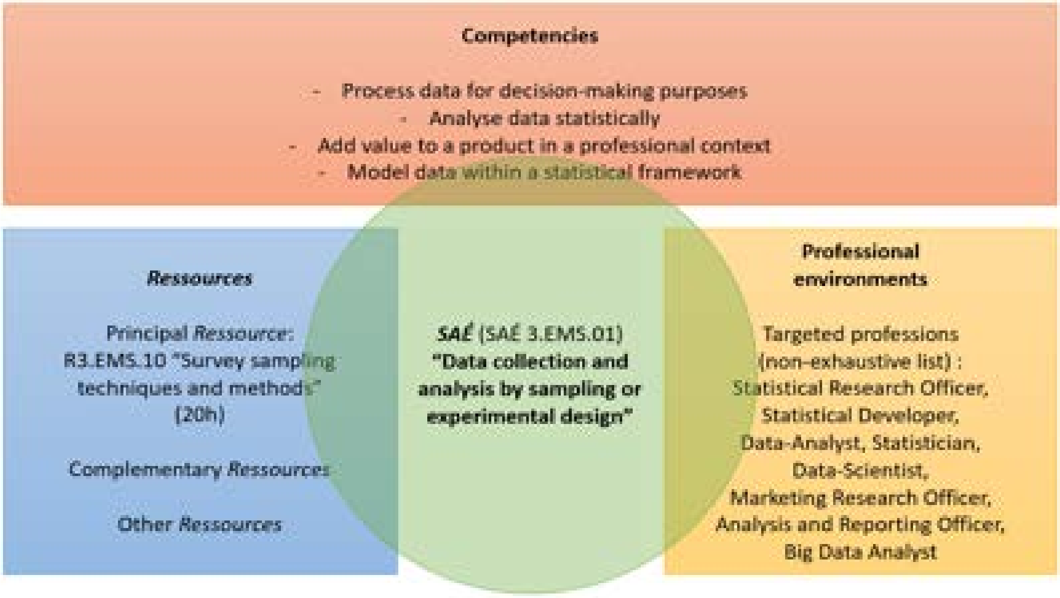

Figure 1: *SAÉ* at the heart of training

In the light of our preliminary analysis, we make the working hypothesis that *SAÉ* is a hub of teaching practices. It makes it possible to highlight the links that the teachers establish between the theory and its applications, the content across the *Ressources*, and the future workplace of the students. We also consider that the *SAÉ* functions in the considered context, intricately guides and drives teaching activities and choices. Within this context, the preparation and the implementation of the *SAÉ* serves as a "valuable" moment that allows to explore several aspects of teaching practices, and teachers' choices on different levels, including institutional, didactic and epistemological considerations.

We are therefore seeking to understand the teacher's practices of Statistical Sampling Theory through the implementation of the *SAÉ* "Data collection and analysis by sampling or experimental design" in relation to these three inputs:

- What links are made between the *SAÉ* and the different *Ressources*? how? why?
- What links are made between the *SAÉ* and the competencies to be developed by students? how? why?
- What links are made between the *SAÉ* and students' future professional practices? how? why?

THEORETICAL FRAMEWORK

In order to investigate teaching practices in the institutional context of the Bachelor "Data Science" described above, we use the Anthropological Theory of the Didactic (ATD, Bosch & Gascón, 2014). Indeed, the particularity of this study seems to be the very strong link that exists between the teachers' practices and the specificities of this institution. These links are due in particular to the existence of a national curriculum specific to this training, which is something quite rare in tertiary education. The ATD, which considers that the knowledge taught is shaped by the institutions, seems to us to be entirely appropriate for the purpose of this study. In the manner of González-Martín (2021) and Gueudet et al. (2022) who used the ATD to analyse teaching practices for non-specialists' students in tertiary education, we investigate the personal didactic praxeologies of a teacher for the specific case of the *SAÉ* "Data collection and analysis […]".

According to the ATD, any human activity can be modelled by a praxeology. Unlike mathematical praxeologies, which are specific to mathematics (e.g. "Determine the sample size"), didactic praxeologies model ways of teaching mathematical praxeologies (e.g. "Ensure that students complete personal work" (Gueudet et al., 2022)). For a given didactic type of tasks T, there is one or more ways of doing things, known as didactic techniques τ. These techniques can be associated with a technological rationale θ that justifies the technique's ability to perform the type of tasks considered. Finally, in the praxeology model (Bosch & Gascón, 2014), a theory Θ is used to justify the technology, but in the case of didactic praxeologies, this last component is implicit.

Following an empirical approach, and drawing on the results of the curriculum analysis (previous section), we focus our investigation on didactical types of tasks associated with the implementation of the *SAÉ* "Data collection and analysis […]". We then suggest focusing on: "Link the *SAÉ* with *Ressources*" (T_{ress}), "Link the *SAÉ* to institutional competencies to be developed" (T_{comp}) and "Link the *SAÉ* to students' future professional practices" (T_{profes}). In this exploratory study, we seek to describe the personal didactic praxeologies associated with these types of tasks.

Thus, our research question is the following: in the preparation and the implementation of the *SAÉ* "Data collection and analysis by sampling or experimental design", and in relation to the three types of tasks T_{ress}, T_{comp} and T_{profes}, what are the didactic praxeologies developed by the teacher (a case study)?

METHODS

Within the *Dida-Stat*$_{Expl}$ project, our general methodology consists of a combination of three types of analysis: an epistemological and a curricular analysis of knowledge at stake (Statistical Sampling Theory); an analysis of the content taught, the resources designed and used by the teacher; and an analysis of the teaching practices in terms of the epistemological aspects and the institutional context. Regarding our research question, we will focus in this paper on the latter type of analysis. We present an analysis based on declared practices from an interview of a teacher, called Carine in the following. We start in this section by presenting Carine, we continue by the presentation of the grid of the interview and of the data analysis method.

Profile of the teacher and local context

Our study of teaching practices focuses on a single case study, the case of Carine, who is a statistics teacher and biostatistics researcher. She has been teaching in the pathway "Data science" of the Technological University Institute at university for 13 years. She has been teaching the Statistical Sampling Theory course for four years (she began before the reform). In the context of this new curriculum, she is in charge of the 20-hours *Ressource "*Survey sampling techniques and methods", which she devotes to lectures and tutorials, to which she completes by ten hours of practical work (local choice). She is also in charge of the implementation of the associated *SAÉ*. In the following, when we use the term "principal *Ressource*", we are referring to lectures and tutorials (according to the meaning in the curriculum), otherwise we will refer to practical work.

Grid of the interview

The grid of the interview comprises five distinct parts. To explore the teaching context, and the practices of Carine in this context, we tried to cover by the interviews different didactic aspects. In the first part, we ask her to provide a comprehensive presentation of her profile and background. The second part delves into the context of teaching "Survey sampling techniques and methods", exploring institutional dimensions and organisational aspects that shape the content taught and influence the teaching process. The third section is dedicated to the teacher's perspectives on students' needs and difficulties, while also examining the connection between teaching decisions and students' future career opportunities. The fourth part focuses on the use of various resources, including those designed by the teacher for the principal *Ressource* and the practical work. In the fifth and final part we ask questions around the preparation and the implementation of the *SAÉ*, the main object of the study. We will develop this choice in the following.

As a methodological choice, the *SAÉ* plays a pivotal role in our exploration of the way the teacher tries to establish connections between the competencies, the *Ressources*, and the needs of the workplace. In the interview the types of tasks are prompted by the questions we asked. What interests us is more particularly to identify the "how" (techniques) and the "why" (technologies) of the tasks which fall within the types of tasks related to our centre of interest, namely T_{ress}, T_{comp} and T_{profes}. The questions in the grid of the interview related to each type of tasks make it possible to identify the techniques. The technologies that justify these techniques are sometimes determined within the answers where the techniques are identified. Otherwise, the technologies can be identified from the answers to the other related questions. We will develop this point for each of the retained types of tasks.

For T_{ress}, we asked the questions: "How do you consider the resource in relation to the *SAÉ*?", "Do you make connections between the *SAÉ* and the [principal] *Ressource*? Can you give an example?", "Do you make links with the lecturers and tutorials of other courses?". These last two questions lead us to distinguish two subtypes of tasks associated with T_{ress}: “link the principal *Ressource*, the practical sessions and the *SAÉ*” (T_{ress1}) and “link the *SAÉ* to other courses of the semester” (T_{ress2}). The first question provides us with elements of the technological rationale and the different techniques (noted τ_{ress_i}) are identified in the answers to the last two questions.

For T_{comp}, we asked the following questions: “What links do you make between the *SAÉ* and the institutional competencies?”, “What are your objectives in terms of students’ learning and skills development from the *SAÉ*?”. The first question provides answers more on the techniques (noted τ_{comp_i}), while the second question allows us to identify the technologies which justify the choices made.

For T_{profes}, we asked the question “What is the place of the *SAÉ* in your teaching?” and additional questions that allow us to identify the way or not the future professional practices of the students impact the teacher’s decision-making or choices (technique noted τ_{profes_i}) (the interplay between the teaching choices and the real-word needs of students’ future careers).

Carine's interview took place at the end of August 2023, before the start of the academic year. It took place over video and lasted 48 minutes. Two researchers (of the three authors) were present, but only one conducted the interview. The interview was video recorded and transcribed for analysis.

Data analysis method

We selected for our analysis the excerpts related to the questions presented above. We started by looking for the type of tasks mentioned by the interviewer or that specified the teacher in her answers. We then looked for the techniques used by the teacher to perform the type of tasks and the possible justifications for using these techniques. We consider these justifications to be an integral part of the praxeology related to the type of tasks, and we interpret them as elements of technology (logos).

RESULTS

Links between the *SAÉ* and the different *Ressources* (T_{ress})

To address the didactic subtype of tasks “link the principal *Ressource*, the practical sessions and the *SAÉ*” (T_{ress1}), Carine uses various complementary techniques. She explains that she takes care to design an *SAÉ* that makes it possible to apply the theoretical concepts encountered in the principal *Ressource* (τ_{ress1_1}). She justifies this by explaining that it enables her to ensure that the students have fully understood what is presented and seen in the *Ressource*. Conversely, she says that she adapts the methods and concepts presented in the *Ressource* according to what will be needed to study the *SAÉ* situation (τ_{ress1_2}). Carine explains that she has to proceed this way because of the *SAÉ* nature:

> Carine: The *Ressource* is the support for the *SAÉ*, without the *Ressource* [students] cannot do the *SAÉ*.

The element of technology that emerges from these two techniques can be summarised as follows: the principal *Ressource* provides theoretical elements while the *SAÉ* enables practical application; the *SAÉ* and the *Ressource* interact together. As concerns the practical sessions, she explains building these sessions in order to illustrate the situations and examples encountered in the *Ressource* (τ_{ress1_3}). She also adapts the practical exercises to complete what could not be illustrated in the *SAÉ* (τ_{ress1_4}). She justifies these techniques by pointing out that the *SAÉ*, being a situation that is intended to be realistic and close to a survey, does not allow all the concepts encountered in the principal *Ressource* to be tackled. Therefore, the practical exercises enable the students to put other situations into practice:

> Carine: It's not necessarily easy in a real survey [the *SAÉ*], to be able to apply all the concepts [encountered in the *Ressource*]. [...] So that's where it's complementary, practical sessions are an illustration of the course.

The second subtype of task we identified is the following: "link the *SAÉ* to other courses of the semester" (T_{ress2}). To do this, she proposes a situation that would network two *SAÉs*: the one associated with statistical sampling techniques and the one linked to data compliance and regulation (in France, we talk about the RGPD rules) (τ_{ress2}). There are two elements of technological rationale that justify this technique, firstly, for the teacher thinking about legal constraints when collecting data is part of the survey practice. Secondly, she explains this way of working is convenient for her and the other teachers as it allows the evaluation to be shared:

> Carine: We worked like this on the *SAÉ* because it's more convenient for everyone and it's a real case study, so we didn't have to do another *SAÉ* for the students.

These two subtypes of tasks (T_{ress1} and T_{ress2}) are associated with the didactic type of tasks "Link the *SAÉ* with *Ressources*" (T_{ress}). The description of these two related praxeologies allows us to emphasise elements of response to our research question in the case of the didactic praxeology associated with T_{ress}.

Links between the *SAÉ* and the competencies to be developed by students (T_{comp})

To address the didactic type of tasks "link the *SAÉ* to institutional competencies" (T_{comp}) Carine uses various techniques. She displays, at the beginning of her slideshow, competencies from the national curriculum that she is targeting in her course (the *Ressource*) (τ_{comp_1}). She explains that this enables students to find their bearings with regard to the expectations of the national curriculum. She also helps students to take an inventory of the competencies they have acquired or are in the process of acquiring (τ_{comp_2}). This inventory takes the form of a *portfolio*:

> Carine: Students are expected to report on the competencies they have acquired in relation to the professional situation encountered. [...] They list these competencies in something called a *portfolio*, I help them to build their own.

As part of the *SAÉ* assessment, the teacher asks the students to self-assess their institutional competencies (τ_{comp_3}). To do this, she asks them to position themselves on a competency grid and to produce, where appropriate, proof of mastery of this competency. She gives little justification for this practice, but she indicates that she is "obliged" to do so, due to the constraints of the institution and the competency-based assessment, which governed the course. The description of this didactic praxeology provides elements of answer to our research question in the case of the didactic praxeology associated to T_{comp}.

Links between the *SAÉ* and students' future professional practices (T_{profes})

To address the type of tasks "Link the *SAÉ* to students' future professional practices" (T_{profes}) Carine proposes a concrete situation in the *SAÉ*, linked to the professional world (τ_{profes_1}). She justifies this choice by explaining that she is trying to get the students to project themselves into a professional context:

> Carine: The idea is that in this *SAÉ* they should be able to project themselves as if they are in the professional world. (31'21).

A second technique we identified is to get the students to take on the role of surveyor during the *SAÉ* (gathering data, taking account of bias, collecting information) (τ_{profes_2}). She justifies this choice by making the link with the difficulties involved in carrying out a survey, it is an element of the technological rationale. According to her, these difficulties are encountered in the professional world:

> Carine: To produce statistics you need surveys. There are always biases, and these biases come through data collection, particularly in the world of surveys. Students need to realise that it's complicated to get information. (31'21).

The description of this didactic praxeology provides elements to answer our research question in the case of the didactic praxeology associated with T_{profes}.

CONCLUSION

The empirical study that we implemented enabled us to define the *SAÉ* as a research object. It also allows us to suggest a framework for analysing teaching practices in this specific context. It appears that the *SAÉ* is relevant teaching situations which that allow to highlight several aspects of teachers' practices (didactic choices, institutional constraints, perceptions of students' needs, students' future professional opportunities). Following an anthropological approach, we highlight the role of certain institutional or personal factors in the choices made by the teacher. However, this approach should be put into perspective with a cognitive one to better understand the "agency of teachers" and the dynamics of its evolution in this specific institutional context of post-secondary professional education. For this reason, we want to deepen our analysis, by considering the purpose of the student activity from the teacher's point of view. To do this, we will analyse practices through classroom observations. The institutional context of this study with the presence of *SAÉ* is specific, but it seems to us that the questions it raises go beyond this particular context and can be generalised to professional training courses (for example engineering training) in order to analyse the links that are made with real professional situations.

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NOTES

1. We will keep the French term *Ressource(s)* in italic used by the institution to designate the courses. We use the term "resources" (in English) to designate the term as it is understood in Mathematics Education, it could be curricular materials (a textbook, a teacher's guide) or anything that could be appropriated by teachers for preparing their teaching.

2. Our translations.

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Investigating teachers' practices for non-specialist students

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Our study concerns teaching practices in the case of non-specialist first-year university students. Referring to the Anthropological Theory of the Didactic and drawing on background literature and on our previous work, we identified five didactical types of tasks, which may especially concern teachers of non-specialist students. We designed a questionnaire in which we asked teachers if they perform these types of tasks, how, and what are the reasons for their choices. We analyse the answers (N=38) collected: When teachers declare they never address a type of tasks we investigate the reasons they present; otherwise, we investigate their declared didactical praxeologies. We observe that while the types of tasks are performed by a majority of teachers, a significant amount of them do not present reasons grounding their choices.

Keywords: Anthropological theory of the didactic, Teachers' and students' practices at university level, Teaching and learning of mathematics in other disciplines.

INTRODUCTION

The study presented here concerns university teachers' practices, when they teach mathematics to first-year science students, not specialised in mathematics. In Gueudet et al. (2022), we presented a first exploratory study: using the Anthropological Theory of the Didactics (ATD, Chevallard, 1999) we investigated the declared teaching practices of three teachers, and tried to identify aspects (more precisely didactical praxeologies, see below) specific for non-specialists students. Building further on this first step, we designed a questionnaire for university teachers and submitted it to teachers involved in the teaching of mathematics for non-specialists in France and in Argentina. These two countries have been chosen for a comparison that will be part of our further work, but is not considered here.

In the next section we expose our theoretical framework, namely ATD and in particular the concept of *didactical praxeology*. Then we present research about non-specialist students' difficulties, and our previous work, that led us to identify five didactical types of tasks potentially important in a teaching for non-specialist students. We designed a questionnaire in which we asked teachers if they perform these types of tasks, how, and what are the reasons for their choices. We present our methods for designing the questionnaire and coding the answers (N=38), and then expose our results. We investigate if the teachers actually perform these five types of tasks. If they declare they never perform a type of tasks we analyse their reasons; for the types of tasks they perform we analyse the declared didactical praxeologies (presented here for "Teach basic mathematics" and "Link mathematics and other disciplines"). We conclude by presenting the answers to our research questions and further research directions

THEORETICAL FRAMEWORK

The Anthropological Theory of the Didactics (ATD, Chevallard, 1999) proposes that all human activity regularly developed can be described using an essential and founding model: that of *praxeology*. This includes two key and inseparable elements: praxis and logos. The first refers to the *know-how part*, that is, the types of problems or tasks that are studied and the techniques that are used to solve them. The "logos" is identified with the *knowledge part*. It includes the technological discourse that gives meaning to the proposed problems, and allows the techniques to be interpreted. The theory justifies the technological descriptions and foundations. In this way, any praxeology consists of four elements: *tasks*, *techniques*, *technologies* and *theories*.

In the case of mathematical activity, Chevallard (1999) distinguishes two types of praxeologies: *mathematical organisations* (OM), which respond to "What mathematics to study", and *didactic organisations* (DO) which respond to "How this study is carried out", that is, ways to carry out the study of the OM or ways to achieve the teaching objectives of this OM.

In the case of didactic organisations, the components are called *didactic tasks*, *didactic techniques*, *didactic technologies*, and *didactic theories*. Didactic tasks represent a relatively precise object (Chevallard, 1999). For example, "teaching to model an economic system" and how to do, for example, "proposing the resolution of moderately open economic problems". In turn, this technique or way of doing, should appear as something both correct, understandable and justified.

The existence of a technique then presupposes the existence, around it, of an interpretative and justifying discourse of the technique and its context of applicability and validity. This discourse is called *technology* - for example, "because solving economic problems involves the construction of mathematical-economic models that describe and predict the behaviour of the system". Technologies can also generate techniques. In turn, a technology requires an interpretation and a justification. This is the level of *theory*, rarely appearing in the case of didactical praxeologies.

When a set of tasks shares a technique, they are grouped into *types of tasks*. Like tasks, types of tasks are also relatively precise objects. This common technique is relative. This means that, in a given institution and for a given type of task, there is in general at least one technique, or a small number of institutionally recognized techniques (Chevallard, 1999).

RELATED WORKS AND CHOICE OF FIVE TYPES OF TASKS

Research about the practices of university mathematics teachers for non-specialist students is scarce. González-Martín (2021) studies in terms of didactical praxeologies the practices of two teachers in two courses for future engineers (strength of materials; electricity and magnetism). He evidences that their use of integral is mostly implicit in these courses; this can create difficulties for students who need to make the link between their mathematics course and these other courses. In a previous study

(Gueudet et al., 2022) we interviewed three university teachers who teach mathematics for non-specialists. We identified in particular three didactical types of tasks that these teachers performed for their students: “Foster students’ interest and engagement in mathematics” (T_{iem}); “Restore students’ self-confidence in mathematics” (T_{scm}); and “Teach basic mathematics” (T_{bm}, basic mathematics means here mathematics that are taught at grade 10 or before). The literature about the difficulties met by non-specialist students confirms that these types of tasks are likely to be especially important for non-specialist students, who might not be interested in mathematics, and perhaps experienced difficulties in mathematics at secondary school (see e.g., Kürten, 2017). Investigating further the literature about non-specialist students, we identified two other didactical types of tasks that mathematics teachers could address. As evidenced for example by Hitier and González-Martín (2022) in their study about the derivative in calculus and in mechanics courses, significant differences exist between the mathematics present in a mathematics course and the mathematics present in the course for another discipline. These authors noted that textbooks, and the teachers (who closely follow the textbooks’ choices in their own course) propose a reduced number of tasks in the context of the other discipline; moreover, inconsistencies exist between the praxeologies linked with the derivative in mathematics and in mechanics. This suggests a need for “Link mathematics and other disciplines” (T_{lmo}) in mathematics courses for non-specialists. Moreover, non-specialist students need to work with models containing mathematics in the other discipline(s) they learn. Constructing, or even using a mathematical model proves especially difficult, if they never learned it in their mathematics courses (see e.g., for physics, White Brahmia, 2023). Thus “Teach mathematical modelling” (T_{mm}, we note that ‘modelling’ could be termed here ‘incomplete modelling’, since teachers can only address some aspects of mathematical modelling) is also a didactical type of tasks potentially important for these students.

Our study was thus designed to answer the following research questions, concerning the teaching of mathematics to non-specialist students:

RQ1. Do the teachers tackle these five types of didactical tasks?

RQ2. Why do some teachers never perform a type of tasks?

RQ3. For teachers who perform these types of tasks, which personal didactical praxeologies do they develop?

METHODOLOGY

In this section, we describe the methodology used for this research, from data collection to coding.

Data collection – Questionnaire

We designed a questionnaire organized around the five types of tasks presented above and consisting of three parts. The first (A) deals with general information about the respondent. The second part of the questionnaire (B) consists of five questions in the same format, corresponding to the five types of tasks mentioned above, presented in the form of an objective (e.g. “I set up procedures and activities to achieve this objective: ‘Teach mathematical modelling’”). For each, the respondents can answer on a Lickert’s scale how often: (1) always, (2) very often, (3) often, (4) sometimes, (5) never. If the answer is “never”, they are asked to specify a reason via a drop-down menu: (i) It's not my responsibility, (ii) My students aren't concerned by this problem, (iii) I don't have the time to do this in addition to the maths programme, (iv) I would like to do it, but I don't know how, (v) Other (open question). If the respondents did not answer "never", they then asked in two following open questions to describe the way they do it (“How do you proceed to reach this aim?”, technique) and then the reason for doing it this way (“Can you explain why you think this achieves this objective (why do you do it this way)?”, technology).

The last part (C) is optional and allows respondents to give comments and possibly volunteer for a future interview.

The authors have sent a link to the online questionnaire to colleagues in their universities, asking them to forward it to any teacher they know as mathematics teachers to non-specialists. The accompanying text of the questionnaire explained the aim of the research. We consider that the respondents were teachers interested by pedagogical and didactical issues linked with the teaching to non-specialist students.

Analysing the answers to section B

We collected 38 responses. In what follows, the respondents will be named R1, R2 … R38. We firstly focused for each type of tasks on the “never” answers and the associated reasons. In a second step, we tried to identify for the other answers (from “always” to “sometimes”) the personal techniques and technologies developed by the respondents. For each type of tasks, two researchers independently proposed a first coding, characterising the techniques (actions described by verbs with equivalent or similar meanings) and the technologies (justification of the choice of a technique). The two researchers confronted their coding, then these initial codes were discussed in the whole team and adjusted. We then followed a cycle of independent coding and confrontation until we reached agreement. In some cases, the answer to the question about the personal technology “why do you do it this way” was irrelevant: typically, the respondent said that the students need this, but without any explanation about the particular technique chosen. We classified such justifications as "no technology".

RESULTS

In this section, we begin by presenting some of the quantitative results of our questionnaire. Then, we analyse for each type of tasks the responses of teachers who say they never try to achieve it. Finally, we present a qualitative analysis of two didactical praxeologies based on the verbatims obtained from the questionnaire responses. For a sake of brevity, we have retained here two types of tasks for this deeper praxeological analysis: 'Link mathematics and other subjects' (T_{lmo}) and 'Teach basic mathematics' (T_{bm}).

Global quantitative analysis of the answers

We asked teachers how frequently they implemented approaches and activities aimed at each of the five types of tasks (Figure 1).

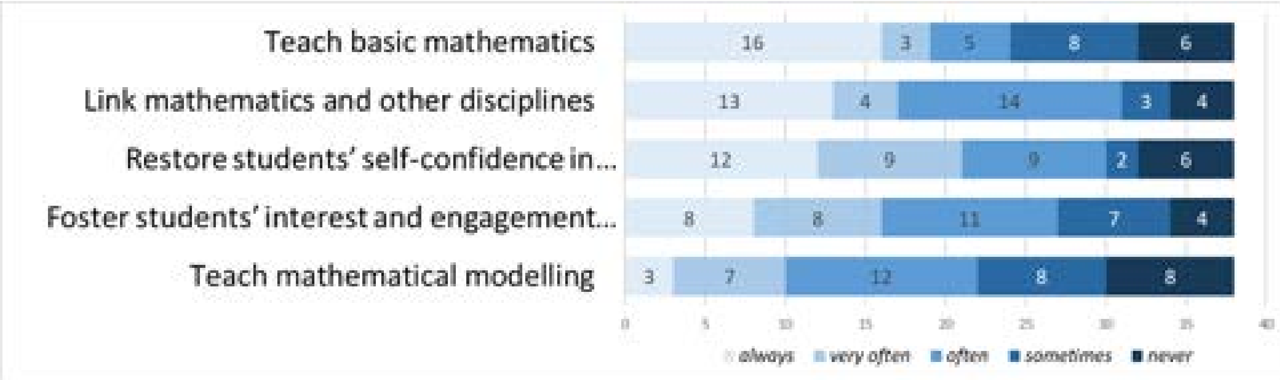

Figure 1: Occurrence of types of tasks in declared teaching practices

We would like to emphasise the large proportion of respondents who always "Teach basic mathematics" (T_{bm}, 16 respond., 42%), while only 8% of them always "Teach mathematical modelling" (T_{mm}, 3 respond.). Nevertheless, 22 respondents (58%) address T_{mm} always, very often or often and 63% achieve T_{bm}: the difference is minor. We also note the large number of teachers (31 respond., 82%) who "Link mathematics and other disciplines" always, very often or often. This can be a consequence of the concern of our respondents for teaching questions.

For respondents who address the different types of tasks, at least 84% explain how they do it, we considered it as element of technique. Between 34% ("Teach basic mathematics") and 78% ("Restore student's self-confidence in mathematics") of them explain why they do it this way (Table 1).

	T_{bm}	T_{lmo}	T_{scm}	T_{iem}	T_{mm}
addressing type of tasks	32	34	32	34	30
citing a technique	27	32	28	31	28
citing a technology	11	13	25	21	17

Table 1: Number of respondents tackling types of tasks and citing technique/technology

"I never do this!": explanations

The type of tasks with the largest selection of the "never" option is "Teach mathematical modelling (T_{mm})" with 8 selections. Then, "Restore student's self-confidence in mathematics (T_{scm})" and "Teach basic mathematics (T_{bm})" (6 responses). After, "Foster students' interest and engagement in mathematics (T_{iem})" and "Link mathematics and other disciplines (T_{lmo})" with 4 selections each. 14 of the 38 respondents selected "never" at least once.

Respondents could choose several reasons for their “never” answer. The option "I don't have time to do this in addition to the maths programme" was the most frequently chosen (13 times, 8 teachers), closely followed by "My students are not concerned by this problem" (12 times, 8 teachers). The third option was "It's not my responsibility" (6 times, 5 teachers). None of the respondents chose the reason "I would like to do it, but I don't know how" (Table 2).

	T_{bm}	T_{lmo}	T_{scm}	T_{iem}	T_{mm}	**Total**	*Nb respond.*
(i) It's not my responsibility	1	1	0	0	4	**6**	*5*
(ii) My students aren't concerned by this problem	4	0	3	3	2	**12**	*8*
(iii) I don't have the time to do this in addition to the…	2	2	4	2	3	**13**	*8*
(iv) I would like to do it, but I don't know how	0	0	0	0	0	**0**	*0*
(v) Other	1	2	1	0	1	**5**	*4*

Table 2: Reasons given for never performing types of tasks

These answers raise several issues, indicating the need for further investigations. We note the importance of the reference to insufficient time as an obstacle. This can be linked with the limited time devoted to mathematics, in the case of non-specialist students, compared to the curriculum to be covered. Nevertheless, only 8 of the respondents mentioned this reason; thus, some teachers did not feel that there was a lack of time, and the reason could be more the importance that these teachers attach to this type of tasks. Concerning the reason "My students are not concerned by this problem", which was also chosen by 8 teachers, it can correspond to a situation where the courses address high-achieving selected students. It can also mean that the teacher did not identify a problem of their students.

“Teach basic maths” (T_{bm}): praxeologies

The most frequently used technique for T_{bm} is "Reminders during lessons" ($\tau_{bm\_rem\_inclass}$, 20 answers). This technique is sometimes mentioned without any further details on how these reminders are given. In some cases, the respondents provide more details:

- Presentation of contents: "With numerous and frequent reminders on the blackboard (left written during the session)" (R4) (16 answers)
- Reminders in the form of exercises: "I insert small exercises from previous classes" (R7) (9 answers).
- Re-explain or reconstruct: "I re-explain as much as possible based on basic concepts" (R2) (4 answers).

Teachers who use this technique provide few explanations as to why they do so. Only six teachers suggest a technology linked to $\tau_{bm\_rem\_inclass}$ (30% out of 20). Three teachers note that providing reminders during lessons helps reassure students concerning their mastery of the mathematical basics. Two of the four colleagues who declare that they re-explain or reconstruct content during these reminders present an elaborate technology. They explain the benefits for understanding of revisiting certain content from previous years: "Students are often surprised to see that they actually know a result if we present it differently, or that a reasoning is finally not so difficult if we take it from the basics" (R2).

Another technique, cited by seven respondents, is "Offering external resources or remediation outside the classroom" ($\tau_{bm\_outofclass}$). This technique is not incompatible with the previous one: four colleagues provide reminders during lessons, and also offer support for students' personal work about secondary school content. The teachers using $\tau_{bm\_outofclass}$ provide students with online resources (5 answers) and/or refer to specific support measures (3 answers): “There is also the assistance room[1] during the whole year, and a support project with personalized courses, for those who don't have the basic knowledge of high school” (R38). Three colleagues offer a technology associated with the choice of $\tau_{bm\_outofclass}$: two say they do so because of a lack of time, and one considers that getting back to basics is the student's personal task.

Five teachers use a technique we call "responding to requests and needs" ($\tau_{bm\_requests\_needs}$). These colleagues do not systematically present reminders to the whole class, but do so in response to questions or according to their perception of the students' needs. With regard to the corresponding technology, two of these colleagues mention the heterogeneity of the students, and the need to differentiate their reminders; two others say they proceed in this way to "make elementary notions available" (R4).

We note that very few explanations of the technique seem to be linked with a reflection about the impact of teaching on students’ learning. The respondents who declare that they propose resources out-of-class because time is lacking for reminders in class do not argue that they choose resources actually helpful for the students. The respondents presenting reminders in class do not provide details about these reminders; their relevance may depend on the teachers’ knowledge of the secondary school curriculum, that is sometimes limited.

We note nevertheless that some teachers (5) teach basic mathematics according to the students’ needs or requests. Pinto and Koichu (2022) in their international survey of teachers views on the secondary-tertiary transition note that university teachers acknowledge the diversity of students and the need to take it into account in first-year courses; the technique $\tau_{bm\_requests\_needs}$ is directly linked with this diversity.

Most of the teachers replied without mentioning any specific mathematical content. However, six colleagues gave such examples: literal arithmetic (1 ans.), fractions (2

[1] The assistance room is a place opened two hours each week where students can go to ask questions.

ans.), percentages, proportions (1 ans.), triangle geometry (2 ans.). The topics mentioned are directly linked to the main disciplines studied by their students: percentages in economy-management, geometry in electrical engineering, for example.

"Link mathematics and other disciplines" (T_{lmo}): praxeologies

We identified four different techniques used (at least three times) to achieve "Link mathematics and other disciplines" (T_{lmo}).

The disciplines mentioned by respondents are physics (5 ans.), economics and management (6 ans.), chemistry (3 ans.), biology and computer science (2 ans.) and finally care study.

The most frequently used technique is "Proposing examples, exercises or applications linked to other disciplines" ($\tau_{lmo\_exercises}$), (18 answers): "I propose problem situations in the fields of biology, genetics, chemistry, physics, etc." (R28). Nine of them did not give any explanation justifying this technique. For those who gave an explanation for this way of doing things, they explain that proposing this kind of exercise, in the context of another discipline, makes it possible to show the existence of links between mathematics and other disciplines (7 answers): "it makes it possible to link different areas of knowledge" (R34), "With this course, [we] make links between maths notations and physics notations" (R3). It seems to us that this kind of explanation cannot be qualified as a technological element, as it refers to the type of task and not to the technique. For these respondents, proposing exercises in the context of the other discipline makes a link in a "natural" way that they struggle to justify.

Eight respondents used the technique of "Presenting mathematical concepts as a tool for solving a problem in another discipline" ($\tau_{lmo\_math\_as\_tool}$), for example: "by introducing each mathematical concept as a tool for solving a physical problem" (R11). Concerning the corresponding technology, three of them explained this technique makes it possible to support the students' ability to use mathematics in other disciplines: "a recurring problem is the students' inability to transfer the tools seen in maths to other disciplines" (R3); "The idea is not to replace the physics teachers but to have done a calculation "properly" once, in the maths course" (R9). One respondent justifies their way of doing things by making links between this technique and the future professional practice of these students: "The aim is to make applied engineers and not an expert in mathematics" (R15). Another teacher justifies that technique as it leads students to manipulate mathematical concepts and formulas (R11).

Six of them explain implementing a program's course that is essentially multidisciplinary, responding to institutional constraints: "The program itself provides for certain links between maths and computer science (encryption, encoding of numbers)" (R13). According to us, their technique is part of the implementation of a multi-disciplinary program (institutional curriculum) ($\tau_{lmo\_inst\_curriculum}$), like R21 who explains: "In the BUT GEA [3-year post-baccalaureate course] national program, the content of the mathematics program is explicitly linked to a management course". Only two of them gave an explanation for using this technique, the aim being to show the existence of links between mathematics and other disciplines: "it's time to create a link between mathematics and agronomic issues" (R32).

Three respondents made the link with the previous didactical type of tasks "Teach mathematical modelling" (T_{mm}) and the questions they had already answered in the online questionnaire. They explain linking mathematics and other disciplines through modelling activities or situations: "same answers as before... this is what reinforces students' interest in mathematics, and it's linked to the modelling problem" (R1). The type of tasks "Teach mathematical modelling" thus becomes a technique ($\tau_{lmo\_modelling}$) associated with the didactical type of tasks "Link mathematics and other disciplines" (T_{lmo}). These three teachers give no justification for using this technique.

CONCLUSION

To answer **RQ1**, a majority of the respondents tackle at least "often" each of the five didactical types of tasks in our list. The colleagues who answered our questionnaire are most probably concerned in teaching issues, we do not claim that this represents the practice of all mathematics teachers. The type of tasks "Teach mathematical modelling" (T_{mm}) has the highest proportion of 'never' or 'sometimes'. This is perhaps the most ambiguous type of tasks, since teachers can give different meanings to 'modelling'. In fact, some answers refer to 'real modelling', which according to some respondents is too difficult to teach in the first year. Finally, as we pointed out in the previous section, T_{mm} can be used as a technique to address T_{lmo}.

Concerning the reasons for "never doing this" (**RQ2**), the reason "it's not my responsibility" was rarely given. The answer most frequently given was linked to a lack of time. The reason "my students aren't concerned" requires further investigation to determine whether the students really aren't concerned (e.g., the course only recruits high-achieving students) or whether the teacher hasn't diagnosed an existing need.

The didactical praxeologies developed by the teachers who tackle the five types of tasks are quite diverse (**RQ3**). At least 84% of them cite at least one technique. Far fewer cited a technology, especially for T_{bm} and T_{lmo} (less than 38%). It can be linked with a bias in our questionnaire: there were more technologies cited for the two first types of tasks, the respondent perhaps found the questionnaire too long. Nevertheless, it can also suggest that the teachers do not provide themselves with the means to ascertain whether the techniques they use actually make it possible to accomplish the types of tasks. However, some answers do contain some in-depth reflections, for example on how to deal with the heterogeneity of students.

With regard to the perspectives to this research, we would like to continue analysing the data collected, to see whether there is a link between the techniques declared and the teaching fields or initial training of teachers questioned.

In our further work, we will firstly interview the teachers who gave their contact details, and observe their teaching. The observations in particular can shed light on the techniques they actually use and on the relevance of these techniques (for example proposing exercises in a kinematics context is not enough for making links between mathematics and mechanics, Hitier and González-Martín, 2022). We intend subsequently to design a refined questionnaire, drawing of the analysis of the interviews and observations, and to submit it to a larger population, with the aim of carrying out a comparison between Argentina and France (which is not currently possible due to the small size of the sample). Our study could contribute to the training of university teachers by raising their awareness about these types of tasks, the different possible techniques and the need to question the reasons justifying a technique.

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Analysing the implementation of study and research paths: the students' voice

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The implementation of study and research paths (SRPs), an inquiry-based instructional proposal elaborated within the Anthropological of the Didactic, has always been analysed from a qualitative perspective, to better understand the factors affecting their ecology, that is, the conditions that enable their management and the constraints hindering it. This paper addresses the SRPs' ecology through a quantitative analysis of the students' answers to the common final survey of eleven SRPs implemented from 2020 to 2023 in first courses of statistics and accounting for chemistry, biotechnology, engineering, and management degrees. Results show a remarkable uniformity of the students' positive reception of this new instructional format, despite the differences in their implementation modalities, length, generating questions and subjects. Some sensitive variables affecting the SRPs' ecology appear, confirming many expected outcomes but also refining others.

Keywords: novel approaches to teaching, teachers' and students' practices at university level, inquiry-based teaching, anthropological theory of the didactic.

INTRODUCTION

Despite the European Higher Education Area's recommendations to evolve towards more competence-driven and student-centred instruction ("Yerevan Communiqué," 2015), university education in Europe has trouble getting out of the lecturer-centred and content-driven instructional tradition. The European project PLATINUM is a clear symptom of how much work there is still to be done (Gómez-Chacón et al., 2021; Katz, 2023). In the Anthropological Theory of the Didactic (ATD), these resistances are interpreted in terms of difficulties in moving forward from the paradigm of visiting works to the paradigm of questioning the world (Chevallard, 2015). Study and Research Paths (SRPs) are instructional approaches proposed to transition towards the latter paradigm (Bosch, 2018). They are based on the inquiry of open questions that are valued by themselves, while knowledge and learning are nothing but a consequence of the process of elaborating answers to these questions. The research approach to SRPs consists of analysing their *ecology*, that is the conditions that enable their implementation and the constraints that hinder their dissemination and development beyond controlled local settings. Their design, implementation and analysis usually follow a Didactic Engineering (DE) methodology (Artigue, 2014; Barquero and Bosch, 2015).

As presented in Barquero et al. (2022), empirical studies on the implementation of SRPs in university education describe different modalities used to integrate SRPs in university courses of different subjects, like Mathematics for Business, Elasticity, Strength of Materials, Statistics for Engineering and Statistics for Business. Many of these SRPs have been implemented by ATD researchers acting as lecturers, designers, and analysts at the same time, while in others the researchers participated in a team of lecturers who were not ATD experts. More recently, some designs and implementations of SRPs have been carried out by lecturers who were not educational researchers and worked in close collaboration with ATD researchers (Fernández-Ruano et al., in press).

From the first implementations of SRPs and, especially, since the research work of Florensa (2018) about SRPs for engineering education, a survey of 35-39 questions was used at the end of each implementation for students to comment about their experience during the inquiry process. These surveys were analysed as part of the a posteriori analysis in the DE process and the results were used to make decisions about the next design and implementation round (see, for instance, Markulin et al., 2022). The fact that the similar surveys have been used in all cases facilitates the comparison of the data in search for similarities and specificities in the students' answers. The results obtained can complement and expand the previous analyses of individual SRPs. In Fernández et al. (in press), an SRP on statistics is introduced and discussed; it was well-received by the students and it positively affected the teacher's practice, although the lack of time was a concern. Martinez-Blasco et al. (in press) present and analyse an SRP used in an accounting class, concluding its value to enrich the curriculum content and foster skills like teamwork, self-criticism, and curiosity.
This is the aim of the research we present in this paper: the joint analysis of the survey data collected after the implementation of eleven different SRPs, all in the same university but in different subjects, in different degrees, and led by different lecturers. It is also the first comparative study of SRPs implemented in different subjects using a quantitative analysis. The research question we address with this analysis is what commonalities the students' perceptions of the work done in the different SRPs show, what specificities related to the concrete SRPs appear.

THE IMPLEMENTED INSTRUCTIONAL PROPOSALS

We are considering eleven SRPs all implemented in an engineering school and a management school belonging to the same Spanish university. Figure 1 provides an overview of the SRPs undertaken in various academic courses over the years. Each entry includes the SRP code, subject, level and degree, academic year, connection to the course, the generating question Q_0 and external contracts, and the hours of work under the teacher's supervision. For instance, in the statistics field, S_M_es1 corresponds to a Business Administration's first course of statistics implemented in 2020-2021 and organized during the three last weeks of the course. Similarly, the SRPs in statistics for Industrial Engineering, Chemistry, and Biotechnology (2020-2021, 2021-2022 and 2022-2023) were conducted in parallel with the respective courses, focusing on topics such as Air Quality and Covid-19 chronic disease. The Fundamentals of Accounting SRPs in Business Administration covered areas like taxes, sales, and purchase transactions, and were integrated in the middle or at the end of the course for 16 or 10 hours.

Notable differences in the amount of work conducted under the teacher's supervision appear, from the 15 hours of S_M_es1, contrasting with no supervised hours for S_I_es1 and S_C_es1. Additionally, S_B_es1 and S_C_es2 involved 9 hours of supervision, while a subsequent SRP (S_C_es3) increased the teacher's involvement to 11 hours. The level of work conducted under the teacher's supervision in the 4 SRPs in accounting presents a noteworthy pattern: the initial project (A_M_en1) in 2020-2021 required a substantial 16 hours of supervision, the subsequent projects consistently maintained this high level, but the last one reduced it to 10 hours because the course turned from annual to semestral. In all SRPs, the introduction of new knowledge tools was mainly carried out by the lecturers (before or during the SRP) but in all cases, some data, information, or pieces of knowledge were also searched by the students. In the SRPs of accounting, the introduction of new knowledge was organised at the students' request but not always exclusively performed by the lecturer.

SRP	Subject	Level and degree	Academic Year	Connection to the course	Q_0 and external contract	Work under the teacher's supervision (h)	Number of enrolled students
A_M_en1	Fundamentals of Accounting	1st year-Business Administration	20-21	In the middle of the course	Taxes and sales and purchase transactions	16	25
A_M_en2		1st year-Business Administration	21-22	In the middle of the course	Sales and purchases transactions	16	33
A_M_es1		1st year-Business Administration	21-22	Last weeks of the course	Taxes	16	47
A_M_es2		1st year-Business Administration	22-23	Last weeks of the course	Sales and purchases transactions	10	43
S_B_es1	Statistics	2nd year-Biotechnology	21-22	In parallel and part of the course	Covid-19 chronic disease	9	48
S_C_es1		2nd year-Chemistry	20-21	In parallel and part of the course	Air Quality	0	42
S_C_es2		2nd year-Chemistry	21-22	In parallel and part of the course	Air Quality and Covid-19	9	36
S_C_es3		2nd year-Chemistry	22-23	In parallel and part of the course	Air Quality and Low Emissions Zone	11	43
S_I_es1		1st year-Industrial Engineering	20-21	In parallel and part of the course	Air Quality	0	60
S_I_es2		1st year-Industrial Engineering	21-22	In parallel and part of the course	Energy comsumption and waste	6	63
S_M_es1		2nd year-Business Administration	20-21	Last weeks of the course	Best location for a cooperative supermarket in the city	15	114

Figure 1. Summary of SRPs characteristics

Despite the differences among the eleven SRPs, the fact that they were implemented in different subjects and degrees, and by different lecturers, they share some characteristics that let us consider them as belonging to the same type of instructional format. More concretely, all inquiry processes generated in the SRPs: (1) started with the consideration of a generating question – the same for all the class – proposed by the lecturers or by an external organisation (a client); (2) students worked in teams of 3-5 and were requested to elaborate a final answer to the generating question or to some partial aspects; (3) several derived questions were raised by the teams of students, discussed and addressed (or discarded); (4) new pieces of information, data and knowledge were required, searched, studied, test and newly elaborated to be used in the elaboration of the final answer; (5) students had to regularly submit logbooks or progress reports that were shared and discussed in class; (6) the answers were presented in a final report each team of students had to submit and sometimes defend orally; (7) the work done in the SRP weighted between 10% and 30% of the final degree of the subject.

THE SURVEY STRUCTURE AND ADMINISTRATION

In the application of the DE methodology to a course, we can distinguish four steps: 1) the preliminary analysis sets the stage for a transformative approach to teaching. DE introduces SRPs to shift from traditional teaching methods to a more student-centred model. This change encourages active inquiry from students, starting with a realistic and engaging question that serves as a focal point for their exploration. It also encounters important constraints that hinder its development. 2) The a priori analysis involves careful planning and creating a roadmap for potential questions and answers to give the lecturers a broader vision of the potential paths that could appear and help them guide students through the inquiry. 3) The in vivo analysis is practically implemented in the classroom, where the focus is on effective management, observation of students' engagement, and data collection to gauge the methodology's impact. 4) The a posteriori analysis then evaluates the inquiry dynamics, comparing expected and actual outcomes, assessing student responses, and reflecting on the overall development of the proposal. This iterative process ensures ongoing refinement and enhancement of the SRP design for its next implementation.

As part of the a posteriori analysis, a survey was designed and administered to the students at the end of each SRP. This survey serves as a critical tool to capture the students' perspectives and perceptions of the experience. The collected data play an important role in confirming or contesting the a posteriori qualitative analyses performed by lecturers and, sometimes, researchers who participate as observers. They are critical to shaping future iterations of the instructional design and fostering a continuous cycle of improvement. Data were collected anonymously using Google Forms. The first part of the questionnaire included a brief introduction explaining the purpose of the survey as well as the anonymous treatment of the obtained answers. The purpose of conducting the survey was to obtain results with the aim of identifying weaknesses and promoting improvement actions for next SRP's implementations.

Three questionnaires were conducted based on the subject and degree of implementation. Questionnaires included between 35 and 39 questions grouped into six categories: General aspects of the course, General aspects of the project, Contents of the project, Teamwork, Project management, and Open questions. Figure 2 presents the 26 common questions among all questionnaires to evaluate the SRP, comprising 23 closed-ended and three open-ended questions. The nine questions that were not shared across questionnaires assess specific tools of the course or aspects within the specific scope of the subject. The last section included three open questions that enabled participants to write down their thoughts and opinions. Questionnaires were sent to all students during the last session of the project. The average response rate obtained was 71% (from 33% to 100% depending on the SRP). At least 20 responses were collected per SRP.

Category	Questionnaire Items			Question	Common code
	S_M	S_I/C/B	A_M		
General aspects of the course (1)	Q01	Q01	Q01	The combination of methodologies used during the course (lectures, exercises and project) has eased learning	CA1
	Q02	Q02	Q02	The different didactic activities used (lectures, exercises and project) have complemented each other	CA2
	Q03	Q03	Q03	The sessions not related to the project have been useful for learning theoretical contents	CA3
	Q08	Q07	Q05	The instructors have been accessible during the project	CA4
General aspects of the project (1)	Q09	Q08	Q07	The aim of the project ([…]) was interesting	PA1
	Q10	Q09	Q08	The project gave the possibility to obtain useful information about […]	PA2
	Q11	Q10	Q09	I think the project was related with the contents of the subject	PA3
	Q12	Q11	Q10	The project has been useful to learn	PA4
	Q13	Q12	Q11	The project has changed my idea of Statistics/Accounting	PA5
	Q36	Q32	Q34	I think it is positive to have done this project	PA6
Contents of the project (2)	Q14	Q13	Q14	The practical content of the project has been	PC1
	Q15	Q14	Q15	The theoretical content of the project has been	PC2
	Q16	Q15	Q16	The difficulty of the project has been	PC3
	Q17	Q16	Q17	The sessions dedicated to the project have been	PC4
	Q19	Q18	Q18	The time dedicated to the project in class has been	PC5
	Q20	Q19	Q19	The time spent on the project outside of class has been	PC6
Team work (1)	Q21	Q20	Q20	It has been easy for me to work as a team	TW1
	Q22	Q21	Q21	It has been easy to distribute tasks within my project team	TW2
	Q23	Q22	Q22	The work environment in the team has been good	TW3
	Q24	Q23	Q24	I would have preferred to work alone	TW4
Project management (1)	Q28	Q27	Q28	The instructors guided us too much during the realization of the project	PM1
	Q29	Q28	Q29	It has been easy to adapt in order to work on the project	PM2
	Q32	Q31	Q32	The project organization (selection of the topic, meetings, pre-reports, freedom in the organization of tasks, final report ...) was appropriate	PM3
Open question	Q37	Q33	Q35	Indicate two positive aspects of the project	OQ1
	Q38	Q34	Q36	Indicate two negative aspects of the project	OQ2
	Q39	Q35	Q37	Add any additional comments you see fit	OQ3

Figure 2. Common questions of the final survey. Responses go (1) from Strongly disagree (1) to Strongly agree (5), or (2) from Too little (1) to Too much/many (2)

A 5-point rating scale, with only the endpoints labelled, was used for the closed questions of the questionnaire. Questions categories marked with (1) correspond to the answers “from strongly agree to strongly disagree” while those marked with (2) correspond to “from too little to too much”. The common codes state for: course general aspects (CA), project general aspects (PA), project contents (PC), teamwork (TW), project management (PM), and open question (OQ).

THE A POSTERIORI ANALYSIS OF STUDY AND RESEARCH PATHS

When considering the data collected from the surveys globally, the most surprising result is the general coincidence of students’ answers despite the variability of the SRPs formats, subjects, and lecturers. For example, the word clouds generated from the responses to the open questions are highly consistent. Students indicate teamwork, learning and class work as the main positive aspects, being workload, time, and homework some of the most negative. Figure 3 shows the word clouds of students’ answers to these questions for the SRPs implemented in Spanish; unigrams are presented in the first row and bigrams in the second one.

Figure 3. Answers to questions OQ1 and OQ2 for all SRPs in Spanish

To identify the specificities of each SRP, Kruskal-Wallis tests have been performed per each question with the SRPs as subsamples. Significant differences (at 5%, 1% and 0.1%) are indicated in the table of Figure 4, the numbers indicating the difference between groups (Conover-Iman post-hoc tests, with Bonferroni adjustment, at 5%). The colours show the sense of the comparison. For instance, questions PA5 and PC2 do not show any difference. In PA5 there is no change in the subject conception produced by the SRP and in PC2 the theoretical amount of content is considered adequate in all implementations. Questions with a few differences among SRPs are those corresponding to teamwork management (TW1, TW2, TW4) with three exceptions. More differences appear when the work environment is assessed (TW3) (Figure 5).

Question (KW significance)	A_M_en1	A_M_en2	A_M_es1	A_M_es2	S_B_es1	S_C_es1	S_C_es2	S_C_es3	S_I_es1	S_I_es2	S_M_es1
CA1 ***		6	2	5	1	2	2	2	2	2	
CA2 ***	1	7	2	7	2	4	2	2	2	2	1
CA3 ***		4		3		2		2	1		2
CA4 ***	1	1		1	1	1		1	1		7
PA1 ***		4		1		3	1	1	1	1	
PA2 ***	1	4		1	3	1			1		1
PA3 ***		5	1	2	1	2		1			2
PA4 ***		7	2	6	1	2		2	2	2	2
PA5											
PA6 ***	1	1	1	4	1	7	1	1	1	1	1
PM1 ***			3	1		1		1		2	
PM2 ***		2	1	8	2	2	1	1	1	1	1
PM3 **				3		1			1		1
TW1 **				2	2	1				1	
TW2 **				1	2	1					
TW3 **				3	3	1	1		1		1
TW4 *			2	1		1					
PC1 **						1			1	2	2
PC2											
PC3 **					1	1					2
PC4 ***	1	1	5	4	3	1	2	1	2	6	2
PC5 ***	3	7	7	5	3	4	2	3	4	5	3
PC6 ***	5	3	6	5	1	4		3	3	4	4

Numbers in table indicate the number of SRPs with a significantly different response in this question ($p \leq 0.05$).

Colors indicate the order of the mean responses for SRP that can be distinguished from others.

SRP with mean response closer to the positive extreme of the scale

SRP with mean response closer to the negative extreme of the scale

SRP with mean response closer to 'Too little'

SRP with mean response closer to 'Too much/many'

Figure 4. Significant differences among SRPs from a Kruskal-Wallis test

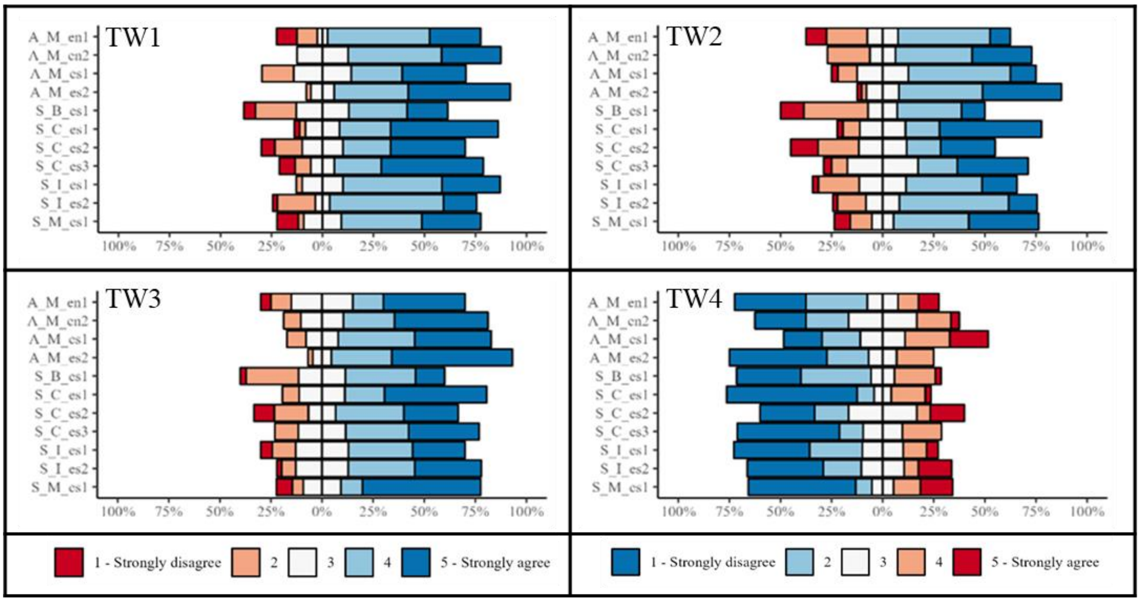

Figure 5. Responses for the common questions related to teamwork

In the case of the project content (PC), there are two main groups of responses: more unanimity in PC1, PC2 and PC3 (practical, theoretical and difficulty) with positive answers, and more diversity in PC4, PC5 and PC6 (time) and some negative perceptions, confirming the word cloud information (Figure 6). Specially relevant are time concerns expressed in PC5 and PC6.

In what concerns the SRP management (PM, Figure 7), the more pronounced difference appears in PM2 (adaptation to the new type of work), with a difference between the accounting and the statistics case (with an exception for A_M_es2). This can be explained by the diversity in the type of inquiry carried out and the management devices implemented by the accounting lecturer, where students had to prepare a list of requests every session to be covered by the lecturer and the students at the beginning of the following session.

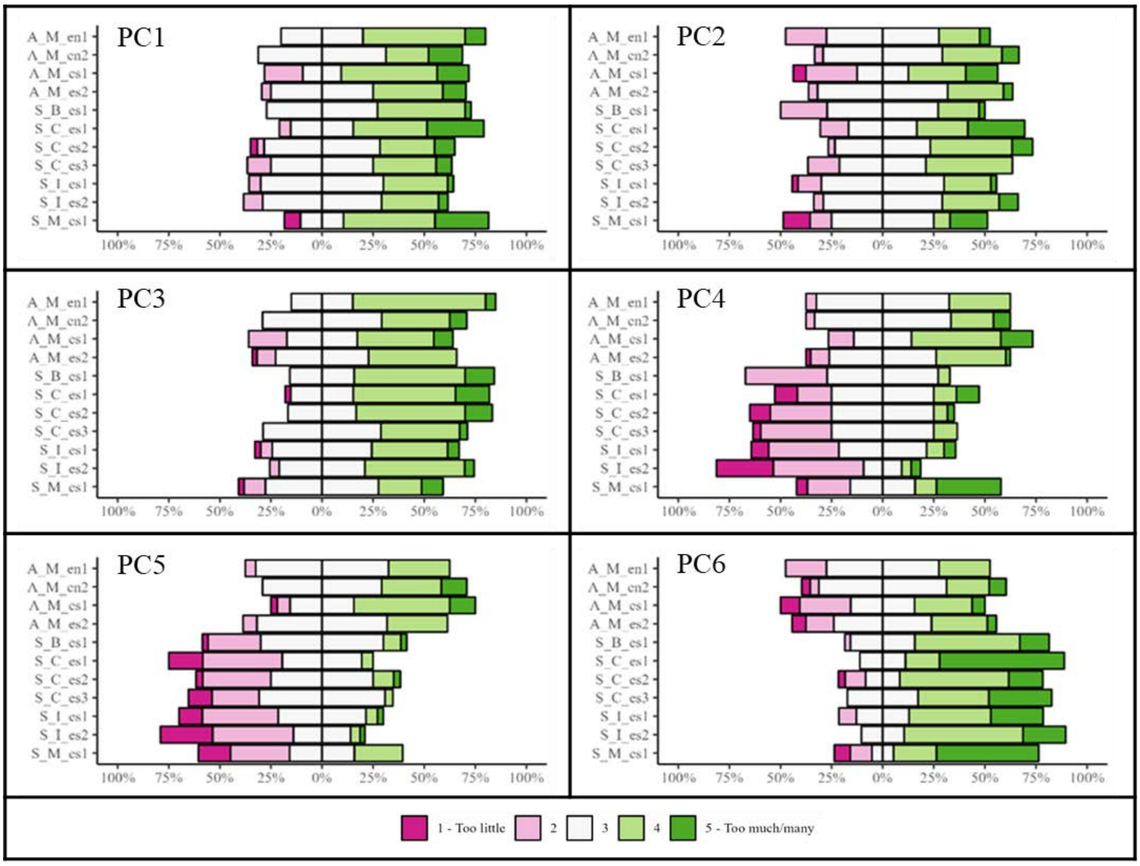

Figure 6. Responses for the common questions related to project contents

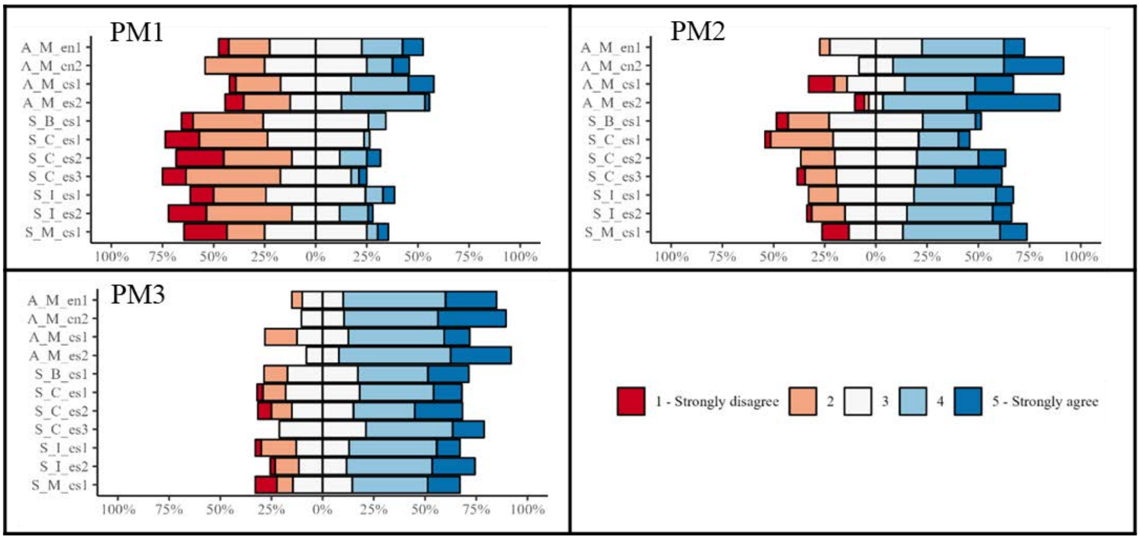

Figure 7. Responses for the common questions related to project management

Other more detailed results can be drawn from the analyses, but we are omitting them here for space reasons.

DISCUSSION AND CONCLUSION

The first finding of this study suggests that all conducted SRPs have consistently generated a positive perception among students, fostering both a sense of making the most out of the experience and facilitating learning. Despite the strong modification of the traditional learning format that SRPs represent, the data gathered supports the assertion that students perceive the activities as producing effective learning and consider it positive to have done the project (PA6). Second, and in addition to the positive impact on students' perceptions, another noteworthy conclusion is the effect of group motivation on the overall results. It also seems that students who participated in activities with a motivated group exhibited increased engagement, collaboration, and ultimately, more favourable survey results. Third, our results also indicate that students place considerable value on the allocation of classroom time on the SRPs activities and on the accessibility of their instructors. The significance of in-class support and guidance is highlighted as students express a preference for an environment where they feel accompanied in their learning. It is crucial to strike a balance, recognizing that overloading students with excessive workload may counteract the positive effects of supportive teaching practices. However, it is not clear how students distinguish between the workload that corresponds to the SRP and the one of the entire subject. The fourth conclusion emerges regarding the positive perception of teamwork. Students consistently express a favourable view of collaborative efforts, recognizing the benefits of working together towards common goals and mentioning almost no difficulties about it. While teamwork is valued, potential pitfalls such as unfavourable group dynamics seem to need specific management. Lastly, our research reveals that the level of difficulty of the SRPs, jointly with the guidance provided by instructors, is aligned with the principles of the zone of proximal development. When the difficulty of SRPs is carefully adjusted and helpful guidance from instructors is provided, students' perceptions are highly positive.

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THE PROBLEMATISATION PROCESS IN THE TEACHING OF STATISTICS THROUGH STUDY AND RESEARCH PATHS

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Inquiry-based mathematics education at the university level is getting much of the attention of researchers. In this paper, we present two instructional proposals in statistics in the form of study and research paths. We focus on the problematisation process carried out from the devolution of the problem to the selection of the question generating the inquiry. We analyse the share of responsibilities between teacher and students of this process, taking the specificity of statistics into account. The results raise the question whether the problematisation should remain the sole responsibility of the teacher or if this responsibility should be shared with the students.

Keywords: inquiry-based learning, anthropological theory of the didactic, study and research paths, statistics, didactic contract, problematisation.

PROBLEMATISATION AND RESPONSIBILITIES

The starting point of an inquiry process is usually a question or interrogation that a group of persons considers worthy of attention and decide to study. In the first steps of the inquiry, the question will undergo a lot of transformations and reformulations that are part of the process of problematisation of the initial issue. This article is devoted to the study of this very first step of inquiry processes in the context of statistics. We will focus, more precisely, on the sharing of responsibilities between the teacher and the students during the problematisation process.

Many research works on university inquiry-based teaching proposals agree upon the importance of transferring responsibilities to the students. Expressions like “student-centred”, and “learner’s responsibility to construct the knowledge, or for their own learning” appear in most of the definitions of inquiry-based learning or when describing activities (Artigue and Blomhøj, 2013; de Jong & van Joolingen, 1998; Dorier & Maaß, 2020; Jaworski et al., 2021). However, it is not easy to find what, how, or in what respect students are going to be responsible for. The Erasmus+ PLATINUM project, carried out by eight European universities in seven countries, whose aim is to develop an inquiry-based approach in teaching mathematics at university level, takes a clear position in this respect:

> In inquiry-based activity, it is the student’s responsibility to work on a task, asking questions, seeking patterns, making conjectures, proving or disproving. The teacher’s responsibility is to design tasks through which students’ activity may reveal key concepts and relationships to progress mathematical understanding (Jaworski et al., 2021, p. 19).

However, in the same project, if we have a look at the responsibilities that students assume in inquiry processes, we can only find general definitions. Little is said about the concrete responsibilities assumed or, more importantly, the strategies or resources used to transfer them and the difficulties found in this purpose.

RESEARCH QUESTIONS AND METHOD OF ANALYSIS

Our study is based on two consecutive experimentations of a *study and research paths* (SRP) and addresses the research question: What conditions can sustain implementing an SRP in statistics in which students take an active role in the problematisation process? What constraints hinder the implementation?

Our method of analysis to address the students' sharing of responsibilities regarding the problematisation process will rely on the Anthropological Theory of the Didactic and its description of inquiry in terms of inquiry dialectics (Chevallard, 2015). In this approach, the notion of *topogenesis* connected to the structure of the inquiry appears as a relevant model to address this question. The empirical data used for the analysis are all the resources and teacher's and students' productions of the course where the SRP was implemented.

DESCRIBING INQUIRY PROCESSES WITHIN THE ATD

Study and Research Paths

The ATD provides a general, but also concrete characterisation of inquiry processes through the model of *study and research paths* (SRPs) (Chevallard, 2015). Their dynamic is described in terms of three main dialectics: the media-milieu or *mesogenesis*, the questions-answers or *chronogenesis* and the individual-collective or *topogenesis* (Barquero & Bosch, 2015). The chronogenesis describes the possible pace for the inquiry, that is, the progress made through the consideration of an initial question Q_0 and the new questions derived from it, as well as the intermediate answers that are found or provided. The mesogenesis corresponds to the evolution of the inquiry *milieu*, the incorporation of new information and partial answers and their validation to transform them into new ready-to-use knowledge tools to proceed with the inquiry. The topogenesis refers to the evolution of the didactic contract (Brousseau, 2002), that is, how responsibilities will be shared between teachers and students during the different steps of the inquiry process.

In previously implemented SRPs at the university level, García et al. (2019) and Barquero et al. (2022) explain different modalities of SRPs and some commonalities and specificities in the way the three dialectics are organised. In all of them, the generating question is usually proposed by the teacher or an external organisation. But it is also mentioned that:

> The ideal situation of an SRP would be a shared assumption of responsibilities between teachers and students during the inquiry: deliveries, validation, and planning would be ideally shared by the whole community of study. (Barquero et al., 2022, p. 6)

The responsibilities assumed by the students are raising derived questions and discussing them, searching for answers and other information, reporting on their activity, managing the teamwork, exposing their intermediate answers, evaluating the other groups' ones, and jointly elaborating a final answer. In all cases, the decision of the generating question is left outside the equation.

The problematising gesture and the students' responsibility

In the first steps of the inquiry, the initial question will rapidly follow a lot of transformations and reformulations. The very triggering of the questioning by the realization that a situation is problematic is a determinant aspect of an inquiry process. And so is the choice of those aspects that are relevant and can be addressed throughout an accessible inquiry process. However, deciding on what to consider as the final question to be addressed from an initial situation or broad interrogation is not a trivial matter. What can be called the "problematisation" of reality (Rodríguez Zoya et al., 2019, p. 4) does not always find its place at school where answers tend to prevail in front of questions and questioning:

> At any rate, however, the non-existence of a legitimate space for problematisation, with a strong epistemological entrustment, can give rise to the ever-present propensity to deny problematicity, so that we are content with the answers we believe we have (Ladage et Chevallard, 2011, p. 22).

To highlight this tendency, the ATD brings out a set of five questioning "attitudes" students may develop (Chevallard, 2015, section 6.8) comprising the following three: the "problem finding attitude", which consists of "recognizing the *problematicity* of situations experienced or observed"; the "Herbartian attitude", which consists of "shirking no question as such (by dismissing, overlooking or repressing it)"; the "exoteric attitude", which consists of "always seeing ourselves [...] as having to study in order to learn more or, at least, to check what we think we know".

However, adopting these attitudes may be rendered quite difficult at school because of the distribution of responsibilities prevailing there. Indeed, not only should the student question herself, but also proceed to the study of this matter which only just appeared. This is what Brousseau (2002) calls the devolution. It is defined as the act in which the teacher makes the learner accept responsibility for a learning (adidactic) situation or problem, and the students accept the consequences of the transfer:

> Let us observe that it is not sufficient to "communicate" a problem to a student for this problem to become her problem and for her to feel solely responsible for solving it. Nor is it sufficient for the student to accept this responsibility in order for the problem she is solving to be, for her, a "universal" problem unattached to any subjective presuppositions. We use the term "devolution" to describe the activity by which the teacher seeks to obtain these two results. (Brousseau, 2002, p. 228).

THE SPECIFICITY OF STATISTICS EDUCATION

If we take the case of statistics education and the specificity of what is considered as "statistics inquiry", the problematisation process is explicitly mentioned. The Guidelines for Assessment and Instruction in Statistics Education (GAISE) College

Report proposes nine goals for the students to achieve in introductory statistics courses at tertiary level, and include, in Goal 2:

> Students should be able to recognize questions for which the investigative process in statistics would be useful and should be able to answer questions using the investigative process. [...] knowing how to obtain or generate data that are relevant to the goals of a study is crucial to providing useful information that supports decision-making [...]. (Gaise, 2016, p. 9)

Also, De Veaux and Velleman (2008) reiterate this approach in their suggestion that introductory statistics courses should involve students in the process of proposing questions, among others. It is also mentioned, in the GAISE report, that "students should practice formulating good questions and answering them appropriately based on how the data were produced and analyzed" (p. 17). Some investigative cycles can be found in the literature. We will focus on three of them: (1) PPDAC (Problem, Plan, Data, Analysis, Conclusions) (Wild & Pfannkuch, 1999), in which the problem is identified in the first place; (2) the statistical investigative cycle provided in the GAISE PreK-12 Report (Bargagliotti et al., 2020), with four stages: 1. Formulate questions, 2. Collect data, 3. Analyze data, 4. Interpret results, which starts formulating questions; and (3) the six-phased framework proposed by (González et al., 2020) when dealing with big data: (1) Assessing the quality of big data, (2) Patterns and relationships, (3) Questions**,** (4) Objectives, (5) Data mining, (6) Designing. Let us notice, in this case, that data explorations are proposed before posing questions, while the obtention of data is assumed at the starting point.

Some authors emphasize how the choice of a good statistical question is crucial to determining the quality of the posterior statistical analysis (Frischemeier et al., 2020; Leavy et al., 2016). An investigative question is defined as:

> [...] the statistical question or problem that needs to be answered or solved. In most instances, the investigative question starts from a wide-ranging or vague general question and then develops into a precise question. (Arnold et al., 2021, p. 124)

If we focus on the gradation of the distribution of responsibilities when posing statistical questions, Arnold et al. (2021) mention that: "In the earlier years of schooling, the teacher is likely to be leading the investigative question-posing [...]. In the later grades, the students are posing the investigative question." (p. 125). However, in most inquiry-based approaches to statistics (Farrell et al., 2019; González et al., 2010; Huang et al., 2022; , Marton et al., 2019; Solana et al., 2014), the responsibility for the problematisation process is left to the teacher.

As an exception, Arnold et al. (2021, 2022) propose two original approaches. The first one consists of letting students ask their statistical questions, which will then lead their inquiry. The students involved are middle school students. One 60-minute session is devoted to thinking and deciding the investigative questions based on the six criteria for what makes a good statistical investigative question (Arnold, 2013, p. 110–111). Beforehand, though, the students had carried out some prior work that consisted of asking questions about favourite places, bringing pictures of them, and classifying them. In the second approach, an activity of interrogating the data visualisation helps engage the students to raise questions and understand the data behind it, to design their survey and collect the data that is going to be analysed afterwards. It is also stated that working with the real data behind the visualisations is challenging and that it should be previously cleaned and prepared by the teacher.

TWO EXPERIENCES OF SRPS

Two statistics SRPs have been developed at the university level, in which the students have been involved in the problematisation process. The SRPs have been implemented in the second semester of two consecutive years (2021-22 and 2022-23) by two different groups of around 40 students of the first subject of Statistics of an Engineering in ICT Systems degree at the Escola Politècnica Superior d'Enginyeria de Manresa (EPSEM), a school of the Universitat Politècnica de Catalunya. The subject included four weekly sessions, two with the whole group and two with half of it. An important constraint faced in both implementations was that the SRP had to be implemented in parallel to the traditional lectures, but no additional time was given.

First implementation

The students were provided with the topic "water" and an open question "What worries you about water?" This topic was linked to the project AquaeSteam[1], which was implemented at the university to promote scientific culture and to build up resources to be used at all educational levels with an interdisciplinary approach in relation to the Sustainable Development Goals (SDG) of the UNESCO 2030 Agenda for Sustainable Development. The students were then asked to do some research to come up with a question and provide data that could answer that question. The whole process consisted of five phases:

(1) Devolution: topic presentation and teams' construction (session 1).

(2) Formulating questions about water: students searched information about water, shared it in a padlet and formulated questions (session 2); each team presented the findings and the project proposal (session 3).

(3) Linking data to questions: students searched data that could answer the proposed question and suggested possible variables of study (session 4).

(4) Analysing project proposals: first ideas of univariate descriptive analysis and use of R and R Commander provided by the teacher (sessions 5-7); each team analyse two proposals from other teams (session 8).

(5) Generating question: resubmission of a question to study taking the prior analysis into account.

In total, nine 50-minute sessions were devoted to the problematisation of the inquiry and to reviewing univariate descriptive statistics and learning the first steps of R and R Commander statistical software. This way of starting allowed the students to consider the type of data that was “statistically studiable” and how to link a question to some data (Freixanet et al., 2022, 2023).

Second implementation

In the second implementation, some changes were introduced taking into account the analysis made from the first implementation and its “a posteriori analysis”. A wide generating question was maintained, “How is the electrical consumption of the EPSEM?”, so that students could have an active role in the problematisation process. However, this time they were also provided with the dataset, to avoid the possible frustration of “non-studiable questions” and to improve the ecology of the SRP.

The construction of each team’s generating question was developed in four phases:

(1) Devolution of the problem: (a) presentation of the SRP by the teacher, (b) explanations about the dataset by the school’s maintenance manager (c) teams’ construction and variables distribution among the teams (session 1).

(2) First explorations: the teacher explained the first ideas of univariate descriptive analysis and the use of R and R Commander (sessions 2-4); students, in teams, carried out the first explorations of the variables they were assigned. A guide was provided by the teacher (Figure 1) (sessions 5-6); finish the explanation of how to carry out an univariate descriptive analysis. Publication of the results in a shared Google Slide (Figure 1) (sessions 7-8)

(3) Pooling of information: (a) pooling of information of the first explorations of all the teams (b) decisions about the lines of study, under the teacher’s guidance. The students had to hand in the presentation of the chosen study following a guideline (Figure 2) (session 9).

(4) Generating question: submission of the generating question, variables of study and data (Figure 2).

As in the first implementation, nine 50-minute sessions were devoted to the problematisation of the inquiry and to reviewing univariate and bivariate descriptive statistics and learning the first steps of R and R Commander.

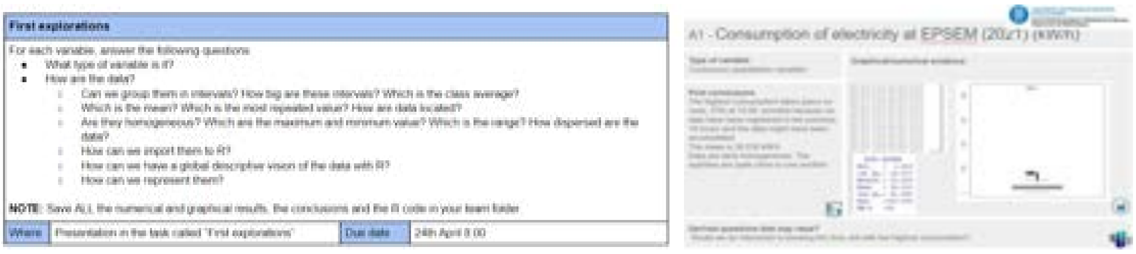

Figure 1: Task 1 – first explorations (teachers’ guide) and team A1 submission.

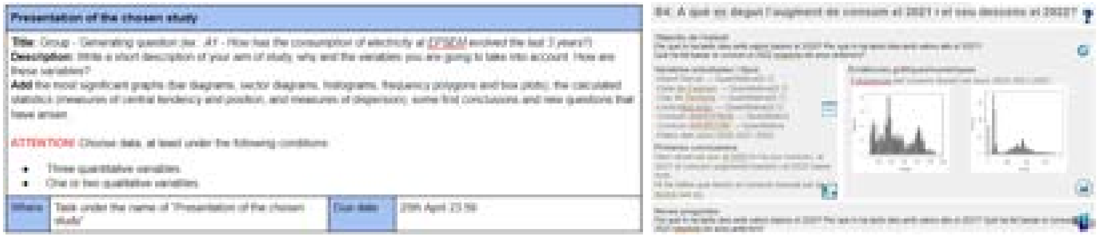

Figure 2: Task 2 – teachers' guide to study presentation and team B4 submission.

Types of questions raised in the second implementation

Students posed different types of questions after the first explorations of the dataset: (1) things they did not understand about the data or things they would like to know; and (2) the main generating question of their study. We made a classification of type (1) questions, and the categories that appeared are:

1. Questions about strange behaviour of data (most of them): "why are there time slots in which the consumption is well above average?", "why is the consumption from 8:00 to 11:00 similar in the lockdown and in 2019?", "How can consumption vary so markedly between seasons of the year?"
2. Data limitations and "cleaning" (bad quality of the data, missing or useless information, noise, etc.): "Should we discard the cumulative value?", "with the little data we have, can we consider it representative for 2019?"
3. Very general questions (not related to the data): "can climate change affect the consumption in the future?"
4. "Need more information" questions: "why are there solar panels in the school and not in the library?"
5. Possible generating questions: "if the consumption has been reduced remarkably in the last four years, can we reach a zero consumption?", "what causes the school having a higher consumption than the library?"

From type (1) to type (2) questions, there was a process of sharing the information with the class and discussing the questions. According to Arnold et al. (2013), the investigative question starts from a vague general question and then develops into a precise question, which is also in line with our experience.

DISCUSSION AND CONCLUSIONS

Our study contributes to research about SRPs within the ATD by giving more insight to the conditions arranged to favour students' participation in the problematisation process, as well as the constraints that hindered this participation.

Regarding conditions, we may list the following:

- Opening the questioning with rather "weak" or wide questions. The lecturer presented a topic (water) or a dataset (electricity consumption) and let the students formulate their own questions.
- Organizing questioning sessions at some steps of the process: initially (SRP1), after the first explorations (SRP1, SRP2), once variables have been identified

(SRP2). The progressively enriched milieu seems to favour the rise of new questions, fostering an exoteric attitude given the mass of new information.
- Discussing the "studiability" of the questions raised in the first implementation: what data are available, how to organise and treat them, etc. This considerably helped the devolution of the problem, since it transferred towards students the responsibility of going on with the inquiry. As Brousseau points out: "To accept responsibility for what happens to her, the student must consider what she is doing to be a choice selected from among various possibilities and must then envisage a causal relationship between the decisions she has taken and their results." (Brousseau, 2002, p. 33).

However, the experimentation of both SRPs also revealed some limitations:

- At different moments, teacher's interventions were needed to redirect the study, which could hinder the recent efforts made to increase students' topos.
- In SRP1, some teams had to reject their initial proposals and choose a new one. Although decisions were made, which goes along with enforcing their topos, it also created a perturbation of the SRP's chronogenesis. This could affect the problem-finding attitude, due to a perceived lack of time and interest in pursuing the inquiry.

Overall, the experimentations show that the problematisation process is neither easy nor spontaneous. It needs to be learnt and taught, and, above all, organised as part of the inquiry. In traditional inquiry-based learning, it tends to be under the responsibility of the teacher. We should question this "a priori topogenesis".

Concerning the specificity of statistics, the contribution of our study corresponds to real statistical inquiries. More and more, statistical studies start from data, not from questions (González et al., 2021). They also include the questioning of the data and the analysis of their quality. These first explorations are likely to include data cleaning, management, and exploration as an important step of the problematisation process. Even the experience from (Arnold, 2021), which did not start with data, contained prior work before raising statistical questions that could be interpreted as "problematisation". The second goal of the GAISE report also means that, in inquiry-based instructional proposals in statistics (or elsewhere), students should be more involved in the devolution of the inquiry.

We must therefore continue along this path. The formalisation proposed by the SRPs within the ATD offers a favourable context for questioning the problematisation process. Indeed, a first step could well be to reconsider as problematic both the origin of questions and who is entrusted with the task of studying them.

NOTES

1. https://www.convocatoria.fecyt.es/Repositorio/Proyecto-destacado.aspx?Id=1332

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Categoría reproducción de comportamientos: una alternativa para la enseñanza de la transformada de Laplace en ingeniería

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Este documento aborda la relación entre la enseñanza de la transformada de Laplace (TL) y sus usos en una situación de ingeniería. Esta relación no es clara en el sistema educativo, pero es necesario lograr el diálogo recíproco entre estos escenarios. Ante esta problemática, presentamos a la Categoría de Modelación Socioepistemológica, que relaciona de manera recíproca diversos escenarios de la ingeniería, entre ellos el profesional y escolar. Un ejemplo de esta categoría es la Reproducción de Comportamientos, la cual proponemos como una base epistemológica para la enseñanza de la TL. Esta epistemología está basada en los usos de la TL que emergen en una situación específica de ingeniería electrónica, los cuales resignifican a esta transformada como una instrucción que organiza comportamientos tendenciales con rapidez.

Palabras clave*: teaching and learning of mathematics in other disciplines, teaching and learning of analysis and calculus, differential equations, Laplace transform, mathematical modelling.*

INTRODUCCIÓN

La problemática principal que se aborda en este ensayo parte de reconocer que en la enseñanza de las matemáticas en las carreras de ingeniería no son claras las relaciones explícitas entre las matemáticas y sus usos en la realidad de la ingeniería, tanto en el escenario profesional como en los cursos especializados de su formación (p. ej., González-Martín et al., 2021; Hochmuth, 2020; Mendoza et al., 2018). Estos estudios identifican una falta de relación entre los cursos de matemáticas y sus usos en los cursos de especialización de otras disciplinas. Hochmuth (2020) indica que esta falta de relación provoca en los estudiantes una mezcla incoherente de definiciones y conceptos, y es causa de dificultades en su aprendizaje. Además, es posible que los estudiantes se cuestionen sobre la relevancia de los contenidos matemáticos para resolver problemas en sus cursos de especialización (Harris et al., 2015). Incluso es difícil encontrar estudios que reporten cómo los cursos de matemáticas ayudan al estudiante a entender mejor los cursos de especialización en sus programas de estudios (Winsløw y Rasmussen, 2020).

Sin duda las matemáticas son de vital importancia en la formación de los ingenieros y, por lo tanto, su enseñanza debe responder a la realidad de la ingeniería, tanto del escenario profesional como del escenario escolar. Sin embargo, el tratamiento de los cursos de matemáticas está alejado de esta realidad y, generalmente, centra su atención en objetos matemáticos. Por ejemplo, la enseñanza de las ecuaciones diferenciales usualmente se centra en métodos algorítmicos para encontrar la solución de la ecuación (Bissel y Dillon, 2000), pero se soslaya la funcionalidad que tienen en situaciones reales de la ingeniería (Cordero et al., 2022; Mendoza et al., 2018). Otro ejemplo es la transformada de Laplace (TL), un método muy utilizado para resolver ecuaciones diferenciales lineales, la cual se introduce en los cursos de ecuaciones diferenciales mediante la integral $\int_0^\infty e^{-st}f(t)dt$. Su tratamiento principal consiste en aplicar, de manera mecánica, fórmulas de esta transformada a funciones o a ecuaciones diferenciales. Esto privilegia solo una justificación algorítmica de la matemática, e ignora las argumentaciones funcionales, por ejemplo, los *comportamientos gráficos* de las ecuaciones diferenciales, que podría ser más cercano a la realidad de la ingeniería (Mendoza-Higuera et al., 2018; Mendoza-Higuera et al., 2022). Este enfoque centrado en el objeto matemático de la transformada de Laplace podría causar dificultades en los estudiantes; al respecto, coincidimos con Holmberg y Bernhard (2016) quienes afirman que "destacar la importancia de las aplicaciones de la transformada de Laplace a la hora de enseñarla, pero no vincular explícitamente los «mundos» de los «objetos» y los «sucesos», parece plantear obstáculos para el aprendizaje" (p. 13, traducción propia).

Dado el planteamiento anterior, reconocemos la importancia y necesidad de investigar acerca del uso de las matemáticas en la realidad de la ingeniería, incluida la realidad en sus cursos especializados en el escenario escolar. Esto es importante ya que hay escenarios en donde los ingenieros resuelven problemas de la ingeniería usando una matemática que no está en sus cursos de matemáticas tradicionales de su formación (Kent y Noss, 2002; González-Martín et al., 2021). Además, tal como Artigue (2016) señala, generalmente la investigación acerca de la enseñanza de las matemáticas del nivel universitario descuida lo que requieren realmente las disciplinas acerca de su uso de las matemáticas y, por el contrario, se centra implícitamente en aspectos relacionados con la estructura y el rigor de las matemáticas.

En este sentido, por ejemplo, Gainsburg (2007) reportó que en ingenieros estructurales las reflexiones más importantes se dan en situaciones donde los procedimientos matemáticos rutinarios no conducen a resultados suficientes. Por otra parte, Cordero et al. (2016, 2022) reportan investigaciones que dan cuenta de la funcionalidad de la matemática en la realidad de la ingeniería: los usos de la matemática de los ingenieros en las situaciones específicas de su realidad responden a lo que les es útil en su quehacer profesional, académico o escolar. Por ejemplo, en una ecuación diferencial $y'' + y' + y = f(t)$ los usos de los ingenieros se refieren a *comportamientos tendenciales*. La atención no está centrada en los procedimientos algorítmicos para obtener la solución, sino que se entiende a la ecuación diferencial como una *instrucción que organiza comportamientos*: la solución y tiende al comportamiento de $f(t)$ cuando $t \to \infty$ (Cordero et al., 2016). En este sentido, el uso de la ecuación diferencial en situaciones específicas de la ingeniería es *reproducir un comportamiento* deseado.

Estos estudios han revelado que los usos de las matemáticas que aparecen en las situaciones reales de la ingeniería requieren mayor atención. Se necesita más investigación ya que las prácticas y concepciones matemáticas relacionadas con otras disciplinas como la ingeniería no están suficientemente claras (Winsløw et al., 2018). De acuerdo con Huchmuth (2020) existe una dialéctica intrínseca entre las matemáticas y su uso en otras disciplinas. Nosotros consideramos que sí la hay, pero en los cursos de matemáticas esta relación no es clara; habrá que revelarla, habrá que encontrar las relaciones entre la matemática y sus usos en la ingeniería.

Por esta razón y dada la necesidad de estudios sobre el uso de las matemáticas en escenarios escolares y no escolares de la ingeniería, en este documento nos proponemos responder a la pregunta: ¿cuáles son los usos de la transformada de Laplace que emergen en una situación específica de un curso especializado de ingeniería electrónica? Como resultado, en este paper mostramos que estos usos conforman una estructura epistemológica que sería base para la conformación de actividades escolares para la enseñanza de la transformada de Laplace en la ingeniería. Esta epistemología está conformada por elementos funcionales propios de situaciones reales de la ingeniería, en donde emerge el uso del conocimiento matemático de los ingenieros.

MARCO TEÓRICO: CATEGORÍA DE MODELACIÓN

Varios estudios con enfoque de modelación matemática han expuesto la necesidad de investigación sobre los usos de las matemáticas en escenarios de disciplinas distintas a la matemática, por ejemplo, disciplinas STEM (p. ej., English, 2016; Maaß et al., 2019). Coincidimos con Borromeo Ferri (2018) quien indica que la modelación matemática es una herramienta muy fuerte y que desempeña un papel muy importante en la educación matemática, además consideramos que la modelación matemática podría ser un posible instrumento que contribuiría a relacionar las matemáticas con sus usos en la realidad de la ingeniería.

De acuerdo con Borromeo Ferri (2018), existe un fuerte consenso en que la modelación matemática puede describirse como una actividad que implica una transición de ida y vuelta entre la realidad y las matemáticas. Muchas de las perspectivas de modelación han desarrollado para su estudio un proceso cíclico compuesto de fases que pretenden vincular a las matemáticas con el mundo real. Kaiser y Sriraman (2006) han identificado diferentes perspectivas teóricas para estudiar la modelación matemática: realística, contextual, educativa, sociocrítica, epistemológica and cognitiva. La mayoría de estos enfoques de modelación siguen un principio P común: que concibe a las matemáticas y a la realidad como mundos separados, y mediante un ciclo de modelación se pretende vincularlos.

Por otra parte, en este documento tomamos un enfoque de modelación diferente, denominado *Categoría de Modelación* (Cordero et al., 2022), el cual se sustenta en la Teoría Socioepistemológica (Cantoral, 2019; Cantoral et al., 2018) la cual se interesa en estudiar la construcción social del conocimiento matemático en diferentes áreas del conocimiento (sea este científico, técnico o popular) y su difusión institucional. Para explicar la *Categoría de Modelación* conviene llevarla a la noción de *variedad,* que expresa la idea de crear una definición alternativa para diferenciarla de las definiciones tradicionales de modelación matemática, pero sin perder la *unidad* que consiste en relacionar la matemática con la realidad.

En los enfoques de modelación matemática tradicional identificamos un principio P (el ciclo que conecta el mundo real y la matemática), que asume la existencia de un conocimiento matemático M y la existencia de una realidad R. Entonces, dada la realidad R, existe una matemática M' que la "matematiza" y es una interpretación de esa realidad R. En cambio, en la *Categoría de Modelación* se formula un principio P' que es *la funcionalidad*[1] *de la relación recíproca y horizontal entre la matemática y la realidad*. En este principio P' la matemática no preexiste a la realidad, ni viceversa. Este principio deja de concebir a la realidad como un mundo alejado de las matemáticas, dado que la construcción del conocimiento matemático (sus usos y significados) obligatoriamente se lleva a cabo en diversas situaciones de la realidad. De esta manera, la construcción del conocimiento matemático transforma la realidad, y la realidad propicia la resignificación de la matemática. Este principio P' es el que genera la *Categoría de Modelación* $\zeta(Mod)$ (ver Figura 1), que pone en juego una epistemología E_r de usos del conocimiento matemático, U(CM). Esta epistemología de usos sucede en diversas situaciones S_{ij} y dominios D_j de conocimiento donde los usos se resignifican. De esta manera la *Categoría de Modelación* es la resignificación de usos del conocimiento matemático, *Re*(U(CM)), cuando sucede la transversalidad entre situaciones o dominios (Cordero et al., 2022).

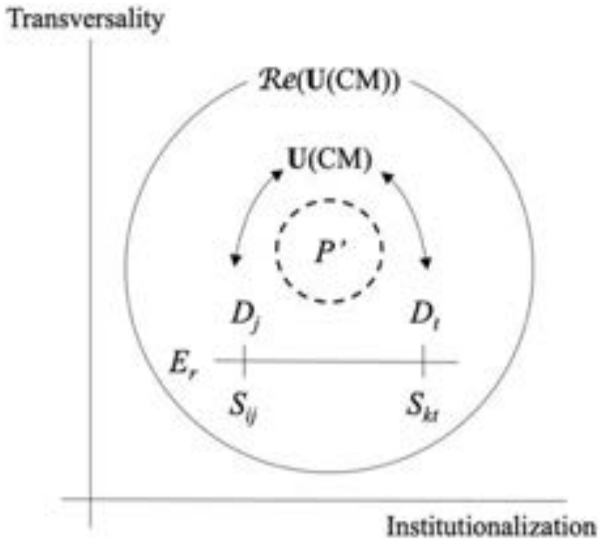

Figura 1. ***Marco del saber matemático de la*** $\boldsymbol{\zeta(Mod)}$ **(Cordero et al., 2022, p. 255)**

[1] La *funcionalidad*, en términos sencillos, significa un conocimiento útil de las personas en situaciones de su vida cotidiana y profesional (Arendt, 2005). En la Modelling Category, la funcionalidad es el resultado de la transversalidad del uso del conocimiento matemático de la gente en diferentes situaciones de la realidad donde se resignifica.

La realidad y la situación específica en la Categoría de Modelación

La *Categoría de Modelación*, como un enfoque socioepistemológico, considera a la realidad como lo habitual en todos los escenarios donde se expresan usos rutinarios, es decir, lo habitual en los escenarios profesional, académico y escolar (Mendoza-Higuera et al., 2018). Por ejemplo, la realidad de un estudiante de ingeniería se entiende como el cotidiano disciplinar de la ingeniería en el escenario escolar, como aquellas situaciones ingenieriles que estudia en su formación, donde el estudiante hace uso de su conocimiento matemático en situaciones específicas correspondientes a su disciplina.

Una situación específica es una estructura epistemológica que subyace a la caracterización de usos de las matemáticas en dominios de conocimiento, como la ingeniería. Los elementos que componen esta estructura son: *significaciones, instrumento, procedimientos* y *resignificación* (Cordero et al., 2022; Buendía & Cordero, 2005). Las ***significaciones*** se manifiestan en los argumentos que le dan sentido a la situación. El ***instrumento*** es un sistema de recursos sobre el cual se hacen ejecuciones para construir significados. Los ***procedimientos*** son las ejecuciones u operaciones inducidas por los significados, realizadas sobre el instrumento. Y la ***resignificación*** se refiere al conocimiento matemático funcional que emerge en la situación, la cual es la resignificación de los usos de la matemática.

LA REPRODUCCIÓN DE COMPORTAMIENTOS: UNA EPISTEMOLOGÍA FUNCIONAL DE LA TRANSFORMADA DE LAPLACE

Un ejemplo de la *Categoría de Modelación* en situaciones específicas donde se ponen en uso las ecuaciones diferenciales en la ingeniería corresponde a la *Reproducción de Comportamientos*. Se ha revelado la emergencia de esta epistemología en situaciones específicas de la ingeniería, cuando se hace uso del conocimiento matemático (ver Mendoza-Higuera et al., 2018; Mendoza-Higuera et al., 2022). Estos autores han encontrado que, en situaciones reales de la ingeniería, por ejemplo, una ecuación diferencial $ay' + y = f(t)$ no está centrada en los procedimientos algorítmicos para encontrarle la solución, sino en los *comportamientos tendenciales* que la ecuación describe; donde la solución y tiende al comportamiento de $f(t)$ cuando $t \to \infty$. En este sentido, la ecuación diferencial se resignifica como la *instrucción que organiza comportamientos* (Cordero et al., 2022; Mendoza-Higuera et al., 2022).

La *reproducción de comportamientos* emerge en situaciones específicas que están compuestas por ***significaciones*** como patrones de comportamientos gráficos y analíticos, que conllevan ***procedimientos*** de variación de parámetros; la función o ecuación diferencial es un ***instrumento*** tomado como *instrucción que organiza comportamientos*, de tal manera que los argumentos que emergen aluden a la búsqueda de tendencias al *reproducir un comportamiento conocido o comportamiento deseado, lo cual es la* ***resignificación*** *del uso del conocimiento matemático* (Cordero et al., 2022; Mendoza-Higuera et al., 2022).

La funcionalidad de la transformada de Laplace en una situación específica

La *categoría reproducción de comportamientos* podría contribuir a crear una relación entre los usos de la transformada de Laplace en la ingeniería y su enseñanza, pues responde a la realidad de diversas disciplinas, incluyendo la ingeniería. En esta categoría una ecuación diferencial es el *modelo de la reproducción de comportamientos deseables*; y, como se expondrá a continuación, la transformada de Laplace es una *instrucción que organiza comportamientos con tendencia*, dada ciertas *situaciones específicas* de la ingeniería donde esta transformada se resignifica.

El análisis de una situación de sistema de control —que es una situación específica de la realidad de varias ingenierías— nos ha revelado que la atención de los ingenieros electrónicos está centrada en el estudio del comportamiento de las señales del sistema, las cuales se desean controlar (Giacoleti-Castillo, 2020). Dada una señal de entrada (comportamiento deseado) se llevan a cabo ciertos procedimientos para obtener una señal de salida (comportamiento obtenido) (ver Figura 2). Entonces, el propósito de un sistema de control es que el dispositivo o proceso que se está controlando tenga las características deseadas en los momentos que se determine (Ogata, 2010). Es decir, dado que se tienen ciertos comportamientos deseados, se diseña y ejecuta el sistema para *reproducir* dichos *comportamientos*.

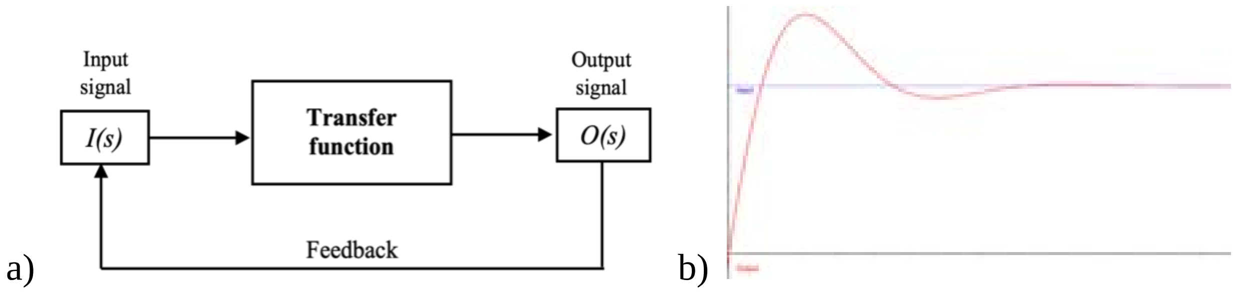

Figura 2. a) Diagrama de bloques de un sistema de control, b) Gráfica de las señales del sistema

En el diagrama de bloques de la figura 2 se observa que tanto la señal de entrada $I(s)$ y salida $O(s)$ aparecen con una variable s. Esta variable representa el dominio de Laplace, dado que en un sistema de control la transformada de Laplace es fundamental en la interpretación de las señales del sistema. De hecho, la función de transferencia, que es el componente principal de un sistema de control, se define como el cociente que relaciona la transformada de Laplace de la señal de salida con la transformada de Laplace de la señal de entrada, es decir la función de transferencia es $G(s) = \frac{O(s)}{I(s)}$. En la función de transferencia se llevan a cabo los procedimientos de realimentación necesarios para obtener el comportamiento deseado del sistema. Es decir, el uso de la transformada de Laplace ocurre cuando se llevan a cabo las modificaciones o procedimientos de control (realimentación), para que el comportamiento sea el deseado lo más rápido posible.

El análisis de esta situación de sistema de control nos ha permitido identificar elementos epistemológicos funcionales de la transformada de Laplace, relacionados con la *categoría reproducción de comportamientos* que caracteriza a esta situación específica (ver Tabla 1): en dicho sistema, lo que interesa es obtener en la señal de salida (O) un comportamiento que tienda con rapidez al comportamiento de la señal de entrada (I) (***significaciones***); para ello comparan las señales haciendo realimentación y ajuste de parámetros (***procedimientos***), esto se realiza sobre la *función de transferencia* (G) (definida mediante $G(s) = \frac{\mathcal{L}\{O\}}{\mathcal{L}\{I\}}$), la cual permite organizar el comportamiento de las señales del sistema con rapidez y lograr su estabilidad (***instrumento***). Todo lo anterior les permite a los ingenieros controlar los comportamientos del sistema. De esta manera identificamos en esta situación específica la emergencia de la *categoría reproducción de comportamientos* (***resignificación***).

Construcción de lo matemático	Situación de sistema de control
Significaciones	Comportamientos tendenciales en el tiempo Comportamientos tendenciales con rapidez
Procedimientos	Comparación de las señales del sistema (realimentación) (Ajuste de parámetros en la función de transferencia)
Instrumento	La TL organiza los comportamientos tendenciales con rapidez en las señales del sistema La TL logra la estabilidad del sistema
Resignificación	Reproducción de Comportamientos

Tabla 1. Epistemología funcional de la Transformada de Laplace en la situación

CONCLUSIONES

El análisis presentado en este documento muestra que, en estas situaciones específicas de la ingeniería, el centro de atención no es la aplicación de la transformada de Laplace para resolver ecuaciones diferenciales. El propósito principal es interpretar los comportamientos de las señales del sistema y realizar los procedimientos necesarios en la función de transferencia (definidas con la transformada de Laplace) para obtener los comportamientos deseados lo más rápido posible. A partir de lo mostrado, se puede concluir que los usos de la *transformada de Laplace* que emergen en esta situación están relacionados con la reproducción de comportamientos, de tal manera que esta transformada se resignifica como una *instrucción que organiza comportamientos tendenciales con rapidez*.

Desde la perspectiva socioepistemológica de la *Categoría de Modelación* se considera que es necesario poner atención a los usos y a la funcionalidad de la matemática en la realidad de la ingeniería y otras disciplinas (Cordero et al., 2022). Este podría ser un referente para la enseñanza y el aprendizaje de las matemáticas, en particular para la enseñanza de la transformada de Laplace en los cursos de ecuaciones diferenciales, ya que es una perspectiva que reconoce los usos del conocimiento matemático, los cuales son más cercanos a la realidad disciplinar de los estudiantes.

Para finalizar, consideramos que la enseñanza de las matemáticas plantea grandes retos, uno de ellos es cuestionar el enfoque de enseñanza centrado en los objetos matemáticos que descuida el entorno de usos que dichos objetos tienen en situaciones de la realidad de otras disciplinas. Esto deriva en la problemática planteada al inicio de este documento: una falta de relación entre la matemática escolar y la realidad de diversos dominios y escenarios, como el de la ingeniería. Al respecto, consideramos que la *categoría reproducción de comportamientos* (conformada de los elementos descritos anteriormente: *significaciones, procedimientos, instrumento, resignificación*) al implementarla en los cursos de matemáticas, podría coadyuvar en la creación de una relación recíproca entre la enseñanza de la matemática y sus usos en la realidad de la ingeniería. Con esta orientación, actualmente estamos desarrollando una investigación que toma como base esta categoría para la conformación de situaciones escolares. El propósito es estudiar episodios de implementación de estas actividades para la enseñanza de la transformada de Laplace en un curso de ecuaciones diferenciales en ingeniería electrónica. Esto nos permitirá caracterizar la resignificación de usos de esta transformada que emerjan en los estudiantes.

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Applications in Calculus courses in engineering: practices of teachers with different backgrounds

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In this paper, we investigate the use of applications in calculus courses for engineering students. We are interested in analysing how the type of applications selected, and the way there are used, relate to a teacher's academic and professional background. Since we are interested in situating knowledge and practices in different institutions, we use an ATD perspective. Our interviews with two calculus teachers—one with a background mostly in mathematics, the other with a background in physics, biology and engineering—reveal significant differences in the teachers' overall practices, in their selection of applications, and even in their views of what an "application" might be.

Keywords: teaching mathematics in other disciplines, teachers' practices at university level, applications in calculus, calculus in engineering, Anthropological Theory of the Didactic (ATD).

INTRODUCTION

A recent literature review carried out by Pepin et al. (2021) compiled an important amount of current international research focused on mathematics in engineering education, to provide a deeper understanding of the way the "characteristics of current teaching and learning practices […] can inform the design and implementation of future innovative practices" (p.164). The authors point out that mathematics courses are one of the main obstacles for engineering students early on in their studies. Along the same lines, Faulkner et al. (2019) discuss that the main reason why many students abandon engineering programmes is not because they failed a specific engineering course, but because they failed a mathematics course. Other studies also seem to agree with the idea that the challenge posed by mathematics courses is the biggest cause of students dropping out in their first year of engineering studies (Bigotte de Almeida et al., 2021; Ohland et al., 2004).

In a conventional curricular structure, the first year of an engineering programme is supposed to provide students with a solid foundation in science and mathematics. It is only after this theoretical basis has been established that students move on to study specific engineering disciplines. However, this structure can lead to a disconnect between mathematics and professional engineering courses (Ellis et al., 2021) and consequently contribute "to the gap between industry needs and the skills of engineering graduates" (Charosky et al., 2022, p.353). Traditionally, first-year mathematics courses (such as Calculus or Linear Algebra) in engineering programmes are taught by teachers with a background in mathematics (Pepin et al., 2021, p.164), which can explain why these courses are usually disconnected from actual engineering practices. This contradicts Harris et al.'s (2015, p.334) recommendation that "mathematics should be embedded with the engineering principles being taught. There [is] a danger that when mathematics becomes isolated from its use in engineering, the opportunity to foster a perception of its use-value in the wider sense [is] lost."

Research on the teaching and learning of calculus has spurred a growing interest in teachers' practices (Rasmussen et al., 2014). In the case of engineering education, calculus teachers may come from a variety of backgrounds, and there is scant research about how this influences their practices. Nathan et al. (2010) worked with teachers of engineering preparatory courses, comparing those with science or mathematics backgrounds to those with a technical background. Their study suggests that the first group tends to see academic excellence in mathematics courses as a kind of prerequisite for engineering, which can lead them to emphasise formalism and downplay practical applications. Our initial research focusing on two teachers with different backgrounds (González-Martín & Hernandes-Gomes, 2020) suggests that teachers in engineering programmes lean on their professional and academic experience to justify some of their teaching practices. It is still uncertain how teachers with different backgrounds tackle the same mathematical content, and what kind of applications relevant to engineering they may provide to their students. Our secondary analysis (González-Martín & Hernandes-Gomes, 2021) of data from a previous study involving five calculus teachers seems to indicate that teachers with different backgrounds may offer different applications when teaching calculus, with varying degrees of real-world usefulness. In this paper, we further explore this issue and investigate what types of applications teachers with different academic and professional backgrounds present in their calculus courses in engineering programmes, as well as how these applications relate to their professional and academic backgrounds.

THEORETICAL FRAMEWORK

We are interested in the practices employed by teachers in preparing and delivering calculus courses to engineering students. Moreover, we wish to establish connections between teachers' practices and their different academic and professional backgrounds. To study these phenomena, we believe an institutional approach is appropriate. We therefore use Chevallard's (1999) Anthropological theory of the didactic (ATD).

According to ATD, human activity can be modelled using the key notion of *praxeology*. A praxeology is defined by the types of tasks to carry out, the techniques that allow these tasks to be completed, a rationale (technology) that explains and justifies the techniques, and a theory that explains and justifies the rationales. According to ATD, learning is constrained and influenced by the institution in which it takes place. In other words, the types of tasks and techniques allowed or promoted by an institution—together with the rationales that justify these techniques—have an impact on the individuals operating within the institution. Moreover, individuals occupy positions in institutions, and these positions, as well as the institutions themselves, influence their learning. In our case, we are interested in teachers occupying the same position (teaching a calculus course in an engineering faculty).

However, since these teachers have previously occupied different positions in other institutions, they may approach the tasks imposed by their new institution and in their new position in different ways.

For instance, we consider the large task "prepare and teach a calculus course". This task is subdivided into different sub-tasks, each with their own techniques. Teachers may, for example, choose to prepare applications to use in their course. We are therefore interested in analysing how the particular sub-task of "choosing applications" is performed by teachers with different backgrounds. Our study examines the techniques put in place to accomplish this sub-task and, especially, the rationales behind teachers' choices. In particular, we seek to better understand how these rationales are connected to the teachers' previous experiences in other institutions. We believe that ATD offers an interesting lens through which to observe and analyse teachers' approaches to performing this sub-task, and to identify differences between teachers' praxeologies, which might explain their divergent practices and the various choices they make in preparing courses.

METHODOLOGY

In May 2023, we interviewed six university teachers with different academic backgrounds who teach calculus in engineering programs at a private university in Brazil. Two of the interviews were conducted in person and the other four were conducted via videoconference; all were recorded and transcribed. We developed a four-part interview questionnaire structured as follows: Part 1: demographic questions, which sought to obtain information about each teacher's academic and professional background; Part 2: general questions about the preparation of their calculus courses; Part 3: questions about chosen applications involving limits, derivatives and integrals in the teachers' courses, as well as the reasons for these choices; and, Part 4: questions about specific exercises from their reference book (Stewart, 2012). The profiles of the participants are displayed in Figure 1:

T1 Male	T2 Male	T3 Female	T4 Male	T5 Female	T6 Female
Mathematics (B)	Physics (B)	Physical & Biomol. Sciences (B)	Mathematics (B)	Mathematics (B)	Mathematics (B)
Mathematics Education (M)	Physics (M)	Biomolecular Physics (M)	Applied Mathematics (M)	Administration (M)	Space Engineering and Technology (M)
Mathematics Education (D)	Materials Engineering (D)(IP)	Mechanical Engineering (D)	Mathematics (D) (NF)	Administration (D)(IP)	Mechanical Engineering (D)

Figure 1: Profile of the six university teachers (B: bachelor, M: master, D: doctorate, IP: in progress; NF: not finished)

Our analyses are still ongoing. For this paper, we selected the interviews with teachers T1 and T3. Our choice was motivated by the sharp difference between their profiles, which is more likely to result in clear dissimilarities in their practices; this first analysis process can later be duplicated to compare profiles that are less different.

Both instructors teach Calculus I at the same university. This mandatory first-year course lasts one semester, and is included in the Basic Curricular Components, a set of courses common to all engineering programmes at the university. T1 has been a university teacher since 1977 (46 years) and has taught calculus for 44 years. He worked for one year as an accountant and four years as a data analyst at a private company. T3 has been teaching at the university level since 2013 (10 years) and has taught calculus courses for seven years. She also has experience as a data analyst in a hospital. Calculus I covers functions, limits and derivatives, and concludes with rate of change and optimisation problems.

The goal of the interviews was to better understand the participants' practices related to their use of applications aimed at engineering students in their calculus course. In our analysis, we paid special attention to the teachers' possible use of applications, their knowledge/repertoire of the applications they use, the reasons they use these applications (or not), and the difficulties they face in using applications. We endeavoured to connect these issues and the teachers' choices to their training and professional experience. All interviews took place in Portuguese, with a duration of between 60 and 90 minutes each. After the transcriptions were completed, the teachers' responses were coded by both authors in terms of tasks, techniques and rationales, allowing us to organise the data to develop our analysis. Excerpts were translated into English for this paper.

DATA ANALYSIS

At the beginning of the interviews, we asked the participants about the resources they use to prepare their course. We could immediately detect connections to their backgrounds in their responses:

T1: Today it's Stewart. But from the beginning I have used several [mathematics] textbooks. […] Now, here it is Stewart, it's the textbook. […] but I sometimes get exercises from other books too. Sometimes there are some interesting exercises and such, you fit them in too. But the base [book] is Stewart.

T3: Yes, we use Stewart, which is our textbook, and it is this one I use in my courses. But I always try to bring something external, because just the textbook today is really boring, and students don't pay as much attention and don't see as much value. Today, they really like the artificial intelligence part, so I always try to bring it to computing students. I always bring something, like a neural network. […] how you train the neuron […] this derivative thing here, you use it up front and I explain it not with all the concepts, because the concepts are much deeper, but the minimum that I can do. I notice that the students' eyes are already popping out.

We can see that, to tackle the large task of "prepare and teach a calculus course", both participants rely on a different resource system. On the one hand, T1 relies strongly on the course textbook and on other mathematics books to select "interesting" exercises; we believe this choice is linked to his long career as a mathematics teacher and little experience outside the classroom. On the other hand, T3 manages to use examples drawn from scientific applications of calculus, going beyond the content of the course textbook and making connections to different branches of engineering. T3 expressed her desire to "bring some application or something that they will use later, or something to focus on, even if it piques the slightest interest in them. I take any point, no matter how theoretical, and I try to find some application there for their daily lives." In particular, T3 drew connections to her training:

T3: For instance, with students in [Chemical Engineering]. To teach calculus in Chemical [Engineering], if you don't provide specific concepts, it gets very boring. Then, I even tried to change the questions. I have textbooks, since I took some courses in chemistry, right? Then, I always pick up something to make things different, [for instance] I used derivatives for molecules in crystal lattices and tried to use this minimally in the calculus course.

As in previous studies (González-Martín et al., 2018), we can see how the teachers' backgrounds influence their choice of resources to accomplish the task of preparing a course. When we asked how their training and experience influence their teaching of calculus to engineering students, both teachers agreed that this is an important resource:

T1: [My academic training and professional background] is everything […] My PhD was good, since it was a contact with education […] Then, we try to show applications, right? For calculus, for instance, moment of inertia, moment of an area, centre of mass, fluid pressure centre […] We use many examples, several exercises with these applications […] It's always trying to make a connection, whenever I can, take from other disciplines.

T3: I often have it easy, because as I have knowledge of a little bit of everything and knowledge, sometimes very in-depth, I can provide an example about everything. So, I was teaching a class on Differential Equations the other day, and then I brought up concepts of crystal lattices […] which, often those who do not have this training do not know what [they are]. If I didn't have this broad knowledge, I wouldn't be able to offer this to my students, right? So, I have the facility and agility to even be able to give examples, which sometimes I didn't even prepare, you know. […] I clearly mobilise the mathematical training that I've had, which was quite heavy, quite dense. I studied many physics concepts, the more specific disciplines themselves. So, for example, I can show them the quantum mechanics element which they love, because it involves calculus. The programming language element that also uses the calculus element. Yes, and I confess that my practical experience, which was not so great in industry, but which took place at *start-ups* and for which I needed to use calculus concepts. […] This is very valuable to them. And then, we start with a simple derivative question, but when you provide this storytelling, for them, it motivates them…

These responses show how their backgrounds influence their praxeologies. For T1, his training in mathematics education leads him to search for examples and applications.

He later stated that most of these applications come from exchanges with other engineering colleagues. For T3, we can see that she mobilises a large repertoire of applications that are deeply connected to her academic training and professional experience. In order to assess the types of applications used to introduce specific calculus content, we asked the teachers about the applications they use to teach limits, derivatives, and integrals. Regarding limits, their responses were as follows:

T1: In fact, the limit, I have always been a critic of the limit, because I think it is exaggerated for engineering. […] Why do I teach limits in engineering? For them to understand derivatives, for them to understand integrals, that‘s what I think […]

T3: I use limit applications […], bacteria growth rate, and I teach them to analyse at the limit… […] I also discuss some applications in neural networks, where we talk about how the network corrects itself, and then there is a point where the limit must tend to zero. Then, I explain why. And it's a very easy calculation, so, I put the application there and say: "this equation governs the back propagation of a neural network"…

There's even another one that I remember now, that I … there was a time […] when I worked with palaeontologists […] and we did the carbon dating of a Tapuiasaurus […] we did dating, but then, that wasn't a limit, it was a derivative and then I showed it to them too. This is from my experience. Neural networks are my … my research work, right. So, I also offer it to them, because of my research project.

We can see important differences in their practices for introducing limits. While T1 sees this chapter as a stepping stone to derivatives and cannot cite specific applications, T3 introduces the topic with a variety of real-world applications that are connected to her background. She clarifies that these applications are presented at the beginning of the chapter, "to introduce the concept" to "win over the audience", before moving on to "the boring part" and the exercises.

The teachers' responses concerning the applications of derivatives that they choose to present are as follows:

T1: Regarding derivatives, we show a lot of applications in maxima and minima […] The graph, study the graph to understand the meaning, always emphasising the meaning […] And all the applications you have of derivatives, you go until differential equations. […] These are always applications that we get from books. Now, I try to find some applications that are more elaborated […] Moment of inertia, moment of an area, mass, volume, surface of solids. These examples are always interesting […] To use applications, it's more towards the end, because they need to have the complete foundation […]

T3: About derivatives, there is the one I mentioned about dating, about the dinosaur that we calculate, you take a derivative in relation to time, so I know that every *x* million years I have a 50% decay of carbon-14 and then I can calculate how old the dinosaur is. […] I talk a little about my experience, that I worked with these palaeontologists […] I also use applications, for instance, the rate of change in the amount of charge over time in the current, they like it in electrical [engineering] […] I also use some examples from medicine too, you can determine if the blood vessel is obstructed or not, calculate using blood pressure itself, it is a derivative, so I bring up some application questions like that.

T3 adds that these examples are drawn from her experience and from books other than Stewart (2012). Moreover, contrary to T1, she starts the chapter with some easy applications, to address rate of change, and she then introduces the more elaborated content. She also uses these applications in different parts of the chapter, and not only at the end. We observed that T1's praxeology has a strong mathematical component in his selection of applied calculus exercises for engineering students. The main applications he cites concerning derivatives are basically mathematical applications (extrema, sketching graphs) and it is not clear how the other applications (moments, surface…) are used; it seems that while he mentions them in class, he does not present any actual activities that make use of them. He also holds the opinion that applications should be taught after the main mathematical tools are introduced, which is consistent with an "applicationist" point of view (Barquero et al., 2013).

This contrasts with T3's response, which clearly reveals practices that stem from her diverse academic and professional experience. As she puts it, her multidisciplinary background and experience allow her to introduce a variety of applications and even improvise some on the spot. We also notice that T3 manages to provide specific examples and to explain how she integrates them into her teaching, which we interpret as indicating that she has participated in praxeologies that involve these examples (at least, at the task and technique levels). Her position regarding applications is also different: applications are not to be left to the end, they rather can serve to motivate the study of a topic, and she uses them throughout the chapter.

Finally, regarding the applications they choose to use in the chapter on integrals, the teachers gave the following responses:

T1: So, for example, when I taught [at another university], we did more applications than we do here [at this university]. So, I did arc length, work, centre of mass, centroid, volume of a solid of revolution, the method of cylindrical shells, disk method […] There, we did all the axes, here we only do *x*-axis and *y*-axis. I do not know why. So, I feel that this part, that I think is more important, is more reduced. […] I think we could cut a little bit in limits and make more room for these applications, which for them is what matters most.

T3: For integrals, […] there is a very cool image, which depicts a ladder, a ramp, then the ladder is the sum, and the ramp is the integral. So, I'll start there.

> Then I start to explain the concept of integral, always bringing up the key problem, which is that if I want to calculate the area of a triangle, I know it, but for a crooked thing, oh… I don't know. So, what am I going to do? Putting the pieces together, it gets weird. And then I get that traditional area, right, under the curve. I have a little simulation here, it's automatic, it reduces the number of rectangles until the curve is adjusted, it's a GIF, and then they can see it. So, I use a lot of things from the book, because when they are starting out, the applications are the technical exercises, right? I bring a lot of simulation stuff so they can visualise it too.

As with the case of derivatives, we can see that T1 mostly turns to "applications in mathematics", which we see as a consequence of his lack of experience regarding professional engineering practices. Although he expresses the view that the way calculus is taught in a mathematics program must be different from the way it is taught in an engineering program, his techniques for choosing applications for integrals also seem to be influenced by his background in mathematics. He expresses the rationale that applications are very important for engineering students; however, his background leads him to propose applications that are closer to practices in mathematics and far from practices in engineering. On the other hand, T3 acknowledges having less experience with integrals; however, her background leads her to rely on animations and simulations. This is justified with another rationale: "You realise that you have to bring other examples to engineering, because your audience is quite demanding. They also demand a lot from the teacher."

Finally, it is worth mentioning that both teachers see time constraints as an important factor in their approach to using applications. One of T1's techniques for dealing with this constraint leads him to present applications at the end of chapters. He believes that by introducing definitions before theoretical content, the content will be better understood. We see again how his "applicationist" view plays an important role in justifying his techniques. Moreover, "applicationism" justifies his view that it is better to present applications at the end, but also that by doing so, it will be easier to manage time constraints. On the other hand, T3 sees applications as important for introducing concepts and she uses them throughout a chapter; however, when time is limited, she tackles the sub-task "adapt the course's implementation to time constraints" using a technique related to her experience with computers: "I already solved it, I put it to them on Moodle and they solve it, they [are curious] because it is something that is different".

FINAL CONSIDERATIONS

In our previous studies (González-Martín & Hernandes-Gomes, 2020, 2021), we observed that teachers with different backgrounds mobilised different techniques and rationales in their praxeologies. That said, our 2020 study focused on teachers supervising capstone projects, which provides a considerably large margin of manoeuvre for teachers to develop very different practices, and on general practices regarding calculus courses. Some of the data used in González-Martín & Hernandes-Gomes (2021) hinted at important differences in the use of applications, as well as differences in what teachers consider “an application” to be, but that study did not explore in depth the use of applications for teaching specific mathematical content.

In this study, we delve deeper into these questions, working with instructors teaching the same course and inquiring about practices related to the use of applications for teaching specific content in calculus. We believe that the choice of T1 and T3 as an initial comparison is useful, since their profiles are quite dissimilar and can point to practices at different ends of the spectrum of application use. T1’s conceptualisation of applications is extremely intra-mathematical; he seems to mention extra-mathematical applications to students without exploring actual activities around them. This approach seems clearly connected to his academic and professional background, which prevents him from importing practices learned elsewhere when using applications in his course. The contrast with T3 is clear; she has a vision of applications strongly connected to her academic and professional experience. Her previous practices in other institutions provide her with a rich repertoire that can be integrated into the techniques she uses to teach her course. We intend to analyse the remaining interviews and study how these differences in practices vary when teachers’ profiles are more similar. This analysis may be complemented with an examination of the teachers’ course materials.

Finally, we note that Pepin et al. (2021) state that current practices of instructors in engineering have not yet been comprehensively investigated, and that this could be an interesting direction for future research. We hope that our study contributes to this line of research, and we aim to create a repertoire of applications that teachers—no matter their background—can use to render their calculus courses more engaging and connected to practices in other scientific fields.

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Mathematics in mathematics courses and mathematics in physics courses: toward a comparison method using praxeologies

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Our aim in this paper is to develop a method to analyse the similarities and differences between mathematics in mathematics courses and mathematics in physics courses, in the first year of university. Referring to the anthropological theory of the didactic, we propose an initial method using the concept of praxeology, and we test it by analysing solved exercises from a mathematics and a physics textbook. We identify types of tasks that are present in both; sometimes these types of tasks intervene in physics as ingredients of techniques. We then compare the associated praxeologies in the two disciplines. We also uncover physics types of tasks containing elements where mathematics and physics are intertwined. Lastly, we discuss what we believe are necessary evolutions of the initial method.

Keywords: Teaching and learning of mathematics in other disciplines, Teaching and learning of analysis and calculus, Anthropological theory of the didactic, Mathematics in physics courses.

INTRODUCTION

The difficulties encountered by first-year 'non-specialists' students due to mathematics is an international issue, increasingly being studied by research in mathematics education (González-Martín et al., 2021). Some of these difficulties come from differences between the mathematics in mathematics courses and the mathematics in other disciplines courses (Taylor & Loverude, 2023). Identifying these differences is thus a crucial first step towards a better understanding of the causes of students' difficulties and the design of interventions. The aim of the study we present here is to design a method for a systematic investigation of such differences.

Adopting an institutional perspective, we refer to the anthropological theory of the didactic (Chevallard, 1999). In the next section, we present this framework as well as background literature related to our work. We propose an initial method for the analysis of the differences between mathematics in the mathematics courses and mathematics in the physics courses. We then test this method and discuss its affordances and potential areas of improvement. Concerning mathematics, we focus on three fundamental concepts of calculus: derivation, integration, and differential equations. Concerning physics, we focus on mechanics, as a domain where these concepts frequently intervene. This work is part of a broader study of the difficulties encountered by first-year physics students in mathematics and how to overcome them.

RELATED WORKS AND THEORETICAL FRAMEWORK

A first approach to related works

Mathematics and physics education research has evidenced that the mathematics in mathematics courses differ from the mathematics in physics courses. Karam et al. (2019) drew on the history of the two disciplines to demonstrate deep epistemological differences. They also showed that different conventions are used in the communities of physicists versus mathematicians. Redish and Kuo (2015) argue that there are "dramatic differences in how the disciplinary cultures of mathematics and physics use and interpret mathematical expressions" (p. 562). Taylor and Loverude (2023) showed that students at the university level perceive differences between mathematics in mathematics and physics courses, and that they cannot transfer to physics what they learned in mathematics. The authors gave the students a graph displaying, for a given object, its position relative to time; the students were asked to determine its velocity. They were not able to reinvest their calculus knowledge in this physics task. White Brahmia (2023), focusing on modelling in physics, observes that the 'physical world' and the 'mathematical world' are not separate. The activity of modelling involves hybrid knowledge, situated at the intersection, and not usually taught in either course.

The anthropological theory of the didactic (Chevallard, 1999) is a socio-cultural theory with a strong focus on epistemological aspects. It is thus relevant to identify differences between mathematics in mathematics and physics courses. Other authors already made a similar choice, we present their works after introducing our theoretical framework.

Theoretical framework

The Anthropological Theory of the Didactic (ATD, Chevallard, 1999) posits that knowledge is shaped by the institutions where it lives. According to the ATD, an institution is any legitimate social group; hence, the physics courses and the mathematics courses for first-year students can be considered as two different institutions. How knowledge is shaped in the institutions is analysed by the ATD with the concept of praxeology. A praxeology comprises four elements: a *type of tasks* T; a *technique* τ to perform this type of tasks; a *technology* θ which is a discourse explaining and justifying the technique; and a *theory* Θ which is a more general discourse, supporting the technology. The pair $[T, \tau]$ constitutes the praxis block, while the pair $[\theta, \Theta]$ is called the logos block.

A type of tasks gathers all the tasks with a similar aim, e.g., 'Solve a differential equation'. Following Chaachoua (2020), we consider a technique to be "a set of types of tasks called technique ingredients" (p. 110). A technique for solving differential equations can be composed of the types of tasks: 'Find a particular solution of the differential equation', and 'Solve a homogeneous linear equation', amongst others. While the concept of praxeology has been mostly used in mathematics education research so far, it can be applied to other disciplines (or even to any human activity). In physics courses, the knowledge is shaped as physical praxeologies, and we are interested in the mathematics present in these physical praxeologies.

Praxeological approaches of the gaps between mathematics and physics courses

Referring to the ATD, González-Martín (2021) studied how integrals were used in physics courses regarding bending moments and electric potentials. In both cases, integrals appeared in the logos block of the praxeologies in a very different way from mathematics courses. Indeed, elements of mathematics and engineering were mixed, and several properties of the integral were implicit. Hitier and González-Martín (2022) investigated the use of derivatives to study motion in five mechanics and five calculus textbooks. They compared the associated praxeologies and identified significant differences. For example, the definition of the derivative using a limit barely appeared in the techniques used in mechanics whereas it was present in about 50% of the tasks within a kinematics context in calculus. They also found that these tasks only dealt with velocity, whereas in mechanics, acceleration was also often present. The authors concluded from their textbook analysis that these inconsistencies were likely to "impact students' ability to connect derivatives with the notions of velocity and acceleration" (Hitier & González-Martín, 2022, p. 307).

In physics education, Caussarieu (2022) studied the differences between practices of mathematics in mathematics and physics exercises. Using the ATD in a non-systematic way, she found that these differences could be grouped into four categories: 1) *different notations*, as for the derivative: $\square'$ in mathematics and $\frac{\mathrm{d}\square}{\mathrm{dt}}$ or $\dot{\square}$ in mechanics; 2) *differences in the objects manipulated*, this category is drawn on Dray and Manogue's (1999) work, which highlights that physicists manipulate physical quantities whereas mathematicians manipulate functions; 3) *different techniques for a similar task*, for example in physics, when asked to find the minimum of a function, one is expected to find where a derivative is null whereas in mathematics one also has to study the sign of the derivative; 4) *different types of tasks performed using the same notion*, for example, the logarithm is often used in integration tasks in mathematics whereas, in physics, students often use it as the reciprocal of the 'power of ten' function.

Hitier and González-Martín (2022) demonstrated the feasibility of a systematic analysis of physics and mathematics textbooks to identify differences and similarities between praxeologies, but they limited their study to tasks involving derivatives and one-dimensional motion. The work done by Caussarieu (2022) suggests that these differences might be grouped in categories. In this paper, we would like to extend these works to get a more comprehensive and systematic view of differences between mathematics in physics and mathematics courses. Nevertheless, comparing praxeologies in mathematics and physics is complex; in particular, we need to find relevant criteria to decide what is similar and what is different. Thus, the first step in our broader study, which we present in this paper, is to establish a method for this comparison.

Our research question is: How can praxeologies in a physics course be compared to praxeologies in a mathematics course, with the aim to identify similarities and differences between the mathematics present in both courses?

PRESENTATION OF A PRELIMINARY METHOD

In this section, we present a method that uses praxeologies to analyse mathematics present in a mathematics course and a physics course proposed to the same students. In the next section, we carry out a test of this method to identify its affordances and limitations. Given textual resources which we generically call ‘the mathematics course’ and ‘the physics course’, the principles of this method can be described as follows.

Establishing the list of praxeologies present in the physics course.

We first build a list of praxeologies present in the physics course. To do so, we identify tasks from the physics course, which we then gather into types of tasks. We define a type of tasks by a verb (e.g., ‘Determine’) followed by a direct object (e.g., ‘the velocity’). Then, we study the techniques associated with these types of tasks and, when needed, determine the ingredients composing these techniques. Finally, we describe the technological discourse justifying these techniques. We recall that the ingredients are themselves types of tasks; and we determine their associated praxeologies. We obtain a list of praxeologies which we call physical praxeologies to refer to the fact that they were found in the physics course. For each type of tasks, we specify whether it appears directly (in which case we call it a primary type of tasks) or as an ingredient of techniques (in which case we call it secondary). A type of tasks can be both primary and secondary.

Identifying which physics praxeologies incorporate mathematics.

We then identify, among these physical praxeologies, the ones that incorporate mathematical elements, whether in the type of tasks, technique, or technology. To discern whether an element is, in fact, mathematical, we refer to a corresponding mathematics course’s syllabus. This step provides us with a list of physical praxeologies incorporating mathematics, and, for each praxeology, whether the type of tasks is primary or secondary.

Identifying common types of tasks and investigating the associated praxeologies.

In the third step, we determine the praxeologies present in the mathematics course using the same method as for the physical praxeologies. We compare the mathematical praxeologies with the physical praxeologies to identify which types of tasks are present in both. For these types of tasks, we determine the praxeologies appearing in the mathematics course (called mathematical praxeologies). We label each element of each praxeology with an M, P or MP corresponding to the course where we found the element (mathematics, physics, or both) and we call types of tasks that appear in both courses common types of tasks. This provides us with a second list of common types of tasks and their associated praxeologies. For a given common type of tasks, we analyse these praxeologies at the scale of the techniques and the technologies.

Analysing the types of tasks only present in the physics course.

In the final step, we go back to our first list and consider the remaining types of tasks. We investigate their features, trying to understand why they do not appear in the mathematics course.

TEST OF THE METHOD

We test our method on texts corresponding respectively to a mechanics course and a calculus course, and we limit our study to the following mathematical concepts: derivation, integration, and differential equations, as these are central concepts of calculus in the first year of university. We chose a first-year mathematics textbook (Boualem et al., 2013) and a first-year physics textbook (Brunel et al., 2015) from the same series. We analysed all exercises from the mechanics section of the physics textbook, and the derivation, integration, and differential equations chapters of the mathematics textbook. There is no prescribed textbook at the national or university scale in France. Therefore, our choice of textbooks was motivated by our want to ensure both a coherent editorial line between the textbooks and having a relatively high number of solved exercises and examples. Moreover, these books can be found in many French university libraries and, in the case of université Paris-Saclay, are relatively frequently borrowed. We analysed 101 calculus exercises and 52 physics solved exercises and worked examples, and our reference for mathematical content was the first-year calculus course summary provided at université Paris-Saclay for physics-chemistry-geoscience students.

Identifying types of tasks common to mathematics and physics

Our method identified multiple common types of tasks that students have to perform in both disciplines. Comparing the types of tasks identified at the level of the exercises questions and sub-questions leads to the identification of three primary types of tasks that are common to mathematics and physics. These are $T^{MP}_{\text{CompDerivative}}$: 'compute the derivative', $T^{MP}_{\text{StudyVariations}}$: 'study the variations', and $T^{MP}_{\text{ShowConstant}}$: 'show that a quantity or function is constant'. We present task examples for these primary types of tasks in Table 1 below.

Type of tasks	Task example in physics	Task example in maths
$T^{MP}_{\text{CompDerivative}}$	Compute the derivative of the angular momentum at O with respect to time.	Compute the derivative of f defined on R by $f(x) = 3x^2 + 7x - \sin x$.
$T^{MP}_{\text{StudyVariations}}$	Study the variations of the mechanical energy of a system.	Study the variations of f defined on R by $f(x) = x^4 + 2x^3 - 2x + 1$.
$T^{MP}_{\text{ShowConstant}}$	Show that the angular momentum is constant.	Show that any function verifying, for all x and y, $\lvert f(x) - f(y)\rvert \leq k\lvert x - y\rvert^{\alpha}$ for a given k and $\alpha > 1$ is constant.

Table 1: Common types of tasks appearing as primary types of tasks in physics

Only a minority of the primary types of tasks we identify in physics incorporate mathematics. Most often, the formulation of the types of tasks derived from the physics textbook does not explicitly contain mathematical elements. We contend this does not mean the corresponding praxeologies do not incorporate mathematics. Indeed, further analysis of the techniques associated with physical types of tasks leads to the identification of additional mathematical types of tasks appearing as ingredients of techniques. These are $T^{MP}_{CompAntiderivative}$: 'Compute the antiderivative', $T^{MP}_{SolveDiffEq}$: 'Solve a differential equation', and $T^{MP}_{CompIntegral}$: 'Compute an integral'. Moreover, two of the primary types of tasks we already identified also appear as ingredients of techniques. Table 2 presents them with the primary type of tasks for which they are found as ingredients of techniques.

Mathematical T	Types of tasks they appear in as an ingredient of technique
$T^{MP}_{CompAntiderivative}$	$T^{P}_{detPosition}$: 'Determine the position' $T^{P}_{detEnergy}$: 'Determine the energy of a system'
$T^{MP}_{SolveDiffEq}$	$T^{P}_{detPosition}$ $T^{P}_{detVelocity}$: 'Determine the velocity'
$T^{MP}_{CompDerivative}$	$T^{P}_{detVelocity}$ $T^{P}_{detAcceleration}$: 'Determine the acceleration'
$T^{MP}_{ShowConstant}$	$T^{P}_{showMoveUniform}$: 'Show that the movement is uniform' $T^{P}_{studyMoveForm}$: 'Study the form of movement'
$T^{MP}_{CompIntegral}$	$T^{P}_{detWork}$: 'Determine the work of a force'

Table 2: Mathematical types of tasks appearing as ingredients of techniques in physics

This shows that we would have missed several types of tasks common to physics and mathematics if we had not expanded the analysis to ingredients of techniques.

Investigating the techniques and technologies associated to common types of tasks

Our method allows us to question whether tasks are performed in mathematics and in physics using the same technique or not. We illustrate this through the example of $T^{MP}_{CompDerivative}$: 'Compute the derivative'.

Three techniques are observed in mathematics. They are $\tau^{M}_{rateOfChange}$: 'Compute the limit of the rate of change at that given point', $\tau^{MP}_{operations}$: 'Use the operations on derivatives to break the problem down to usual functions for which the derivative is known', and $\tau^{M}_{thmCalculus}$: 'Use the fundamental theorem of calculus'. In physics, we observe the technique $\tau^{MP}_{operations}$ as well as several more techniques. They are $\tau^{P}_{coordinates}$: 'Derivate each coordinate (using $\tau^{MP}_{operations}$) and use the formula $\frac{d\vec{v}}{dt} = \dot{x}(t)\overrightarrow{e_x} + \dot{y}(t)\overrightarrow{e_y}$ for $v = x(t)\overrightarrow{e_x} + y(t)\overrightarrow{e_y}$', τ^{P}_{slope}: 'Determine the slope of the function's graph at each point', and τ^{P}_{phyEq}: 'use a physical equation giving the derivative as a function of other quantities'. We note that τ^{P}_{phyEq} is the only technique appearing in relation to $T^{MP}_{CompDerivative}$ as a primary type of tasks.

Varied technologies appear in physics, for example in relation to τ^{P}_{phyEq}. One example is Newton's second law of motion, θ^{P}_{Newton}: 'The acceleration multiplied by the mass of an object is equal to the sum of the forces applied to it'. Another, in the case of the angular momentum, is the technology θ^{P}_{am}: 'The derivative of the angular momentum at a point O with respect to time is equal to the vectorial product of the position vector and the sum of the forces'. This allows us to point out that usage contexts are varied in physics in comparison to mathematics: praxeologies involving derivatives, integrals and differential equations are exclusively found over a real interval or an open subset of R in mathematics. In physics, an analysis of the technologies shows the question of the interval and whether the subset is open is never discussed, however, derivation praxeologies appear with both real-valued and vector-valued functions.

Praxeologies incorporating mathematics but only appearing in physics

Our analysis unearthed the existence of praxeologies incorporating mathematics that were present only in physics. These praxeologies have a physical primary type of tasks, and neither the associated technique nor the associated technology could be found in the mathematics textbook. However, the ingredients of techniques comprising the technique of these praxeologies used both mathematics and physics. We illustrate this through the example of exercise 18.7 (Brunel et al., 2015, pp. 373–374), which we propose a translated version of below:

> An object is moving along a straight line. Its acceleration is given by $a = -\omega^2 x$, where x represents the distance of the object to an origin point O.
>
> Determine the expressions of $x(t)$ and $v(t)$ knowing that the object is at O when $t = 0$ and has initial velocity $v_0 = 4\ \mathrm{m} \cdot \mathrm{s}^{-1}$.

The types of tasks identified through the wording of the exercise are $\mathrm{T}_{\mathrm{detPosition}}$: 'Determine the position' and $\mathrm{T}_{\mathrm{detVelocity}}$: 'Determine the velocity'. These are physical types of tasks. The solution reads as follows:

> The acceleration $a = -\omega^2 x$ is actually a second-order constant-coefficients differential equation, whose resolution is described by theorem 30.13: $\ddot{x} + \omega^2 x = 0$, where $\ddot{x} = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}$.The solution of such an equation is $x(t) = x_0 \cos(\omega t + \varphi)$, where x_0 and φ are constants to be determined with the initial conditions. (Brunel et al., 2015, pp. 773–774)

In this solution we identify the following technique: $\tau^{P}_{\mathrm{idSolve}}$: 'Recognise that a given equation is a differential equation and solve the equation'. This technique is made up of two ingredients of techniques, $\mathrm{T}^{MP}_{\mathrm{solveDiffEq}}$: 'Solve a differential equation', which is found both in physics and mathematics, and $\mathrm{T}^{P}_{\mathrm{recoDiffEq}}$: 'Recognise that a given equation is a differential equation'. $\mathrm{T}^{P}_{\mathrm{recoDiffEq}}$ and its associated technique only appear in the physics textbook, and the technique blends mathematical and physical concepts. Indeed, to recognise a differential equation one must both know what forms a differential equation may take, which is a mathematical concept, and know that $a = \ddot{x} = \frac{d^2 x(t)}{dt^2}$, which is a physical concept.

The fact that recognising a given equation is differential is not practised in the mathematics textbook may have an impact on students' ability to perform this type of tasks in physics.

DISCUSSION

A systematic comparison of the mathematics in the mathematics course and in the physics course using praxeologies is a challenging project. So far such comparisons have been limited to exercises in a kinematics context (Hitier & González-Martín, 2022), or made in a non-systematic way (Caussarieu, 2022). The test of our tentative method evidences that some of our choices are relevant, while others need to be questioned or revised.

We note the relevance of decomposing the techniques (in particular in the physics course) into ingredients of techniques. Indeed, if we had not decomposed the technique down to multiple ingredients of techniques, we would have missed several types of tasks incorporating mathematics that are present in physics.

Another significant choice in our method was to characterise a type of tasks by a verb followed by a complement, e.g., 'Compute the derivative'. This led us to describe types of tasks with a high level of generality and allowed us to find similar types of tasks in the mathematics course and in the physics course. If we had opted for a more precise definition of the type of tasks, 'Compute the derivative of a polynomial function' and 'Compute the derivative of a position' would have been two different types of tasks. In this case, we would have had to conclude that there are no types of tasks common to the physics and the mathematics course. As evoked above, one of the difficulties we face is to find relevant criteria to decide what is the same and what is different, when we compare the two courses. Saying that everything is different would not be productive for our final aim of supporting students transitioning between courses.

This test also enabled the identification of some limitations in the tentative method and perspectives of improvement. First, we found it sometimes difficult to build complete praxeologies in the physics course, particularly regarding technologies. We think that this is a limitation coming from our use of exercises and worked examples, and this can be overcome by adding other material, e.g., lecture notes. Second, the test of our tentative method suggests that techniques in physics tend to apply to a more specific subset of the tasks of a type of tasks whereas the techniques observed in mathematics tend to be more general. We could include what Castela (2008) describes as the efficiency domain of a technique in our framework to investigate this issue. Third, we observed, in physics, the presence of types of tasks intertwining mathematics and physics: this is a key output of the test of our tentative method. Nevertheless, we have probably missed other explanations for the presence of these types of tasks and how mathematics intervene in the associated praxeologies. We could complement the initial method by an analysis which starting point would be technologies in physics that incorporate mathematics (and using, e.g., lecture notes). Last, some important differences are not captured by our method, like differences of notations. This suggests the need for evolutions linked with concepts likely to complement the praxeologies, e.g., the comparison method could investigate ostensives (Bosch & Chevallard, 1999) appearing in mathematics courses versus physics courses.

CONCLUSION

Our research question was 'How can praxeologies in a physics course be compared to praxeologies in a mathematics course, with the aim to identify similarities and differences between the mathematics present in both courses?' To answer this question, we proposed a tentative method and tested it on exercises and worked examples from a mathematics and a physics textbook.

This test evidenced that some aspects of the tentative method are relevant. Incorporating elements of techniques in the description of physics praxeologies allows us to identify mathematical types of tasks present in the physics course, and then to analyse the associated techniques and technologies in each course.

It also evidenced that other aspects need to be refined or modified. We chose to consider broad types of tasks; the consequences of choosing more precise types of tasks must be further investigated. Moreover, we plan to complement the initial method with an analysis taking technologies as a starting point, to avoid overfocusing on the praxis block. This evolution should also be linked with the use of another material, e.g., more theoretical parts of textbooks, or lecture notes. Indeed, the test presented here produced results concerning mainly the praxis block, since the technological aspects were not often described in the solution of the physics exercises. Moreover, we also intend to complement the method by referring to other concepts: ostensives in particular.

We will continue to work on the design of the method, and at the same time use it for the identification of differences between the mathematics in a mathematics course and in a physics course. This should enable us in the next stages of our work to analyse student difficulties to design interventions taking these differences into account.

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Derivatives in calculus and in mechanics. Missed opportunities in the context of free-fall

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Free-fall contexts have been identified as appropriate for learning the basic contents related to derivatives. In this paper, we investigate how these contexts are used in calculus and mechanics courses. We summarise the results stemming from our analysis of textbooks, course observations, and interviews with students. The results indicate that in both courses, opportunities are missed to use free-fall as contexts for fostering a conceptual learning of derivatives. Moreover, students associate these contexts with typical mechanics tasks, which can be solved using simple equations and no covariational reasoning.

Keywords: teaching and learning of analysis and calculus, teaching and learning of mathematics in other disciplines, teaching and learning of mechanics, Anthropological Theory of the Didactics (ATD), college level.

INTRODUCTION

Mechanics and calculus have historically strong ties (e.g., Hitier & González-Martín, 2022a). In particular, Doorman and van Maanen (2008, p. 5) note that "questions about falling objects were essential for the development of calculus". Investigating these questions contributed to the formulation of fundamental concepts, including the key notion of derivative, which appears in the study of motion (kinematics), among other notions, as velocity and acceleration. Consequently, in post-secondary education, calculus is often a prerequisite or co-requisite to physics courses like mechanics. However, as highlighted by Biza et al. (2022) in a recent literature review, despite its potential as a scaffold for the development of reasoning in client disciplines, calculus often acts instead as a filter, leading to high failure and dropout rates in STEM (Sciences, Technology, Engineering and Mathematics) disciplines.

The teaching and learning of the derivative in calculus have been the focus of numerous studies. In particular, Jones and Watson (2017) observed that students with a comprehensive understanding in three of the "four *source* contexts"—graphical, symbolic (as limit of difference quotients), verbal (as rate of change), and physical (as velocity)—(p. 210), are more likely to "[extend] their derivative understanding in future contexts" (p. 204). In their recent literature review on calculus in mathematics education, Thompson and Harel (2021) identified covariational reasoning and rate of change as "foundational ideas for students to recognise the utility of calculus in scientific fields" (p. 512). In physics education, White Brahmia (2023) also described "constant rate of change and linear function, and changing rates of change, using covariation" as "calculus ideas that are particularly relevant in physics" (p. 79).

Furthermore, Karam and Krey (2015) pointed to the study of motion in general, and free-fall in particular, as an "extremely appropriate context to learn the basic concepts of derivative" (p. 672). This led Doorman et al. (2022), among others, to use the context of free-falling objects to build students' understanding of the derivative as rate of change, which seemed to help students develop reasoning about the relationship between the variables involved. Therefore, it seems that free-fall contexts have the potential to help build some basic ideas related to rates of change, in particular using the notion of covariation. However, to our knowledge, no study in mathematics education has investigated the practices related to free-fall contexts in a regular calculus course. For that reason, in this paper we address practices related to the use of derivatives and rates of change in free-fall contexts in calculus courses, and contrast them with practices in mechanics courses.

THEORETICAL FRAMEWORK

In this paper, we are interested in free-fall, namely, "object[s] moving under the influence of gravity only" (Knight, 2017, p. 50), in two different disciplines (calculus and mechanics). These disciplines represent two different *institutions* as defined by the Anthropological Theory of the Didactic (ATD–Chevallard et al., 2022). To facilitate the study of any human activity, ATD provides the key notion of praxeology $[T / \tau / \theta / \Theta]$, which consists of four components organised into two blocks. The first block, $[T / \tau]$ or *praxis*, includes a type of task, T, with a technique that can be used to solve this type of task. It answers the questions, "*What* do people in [a certain position in a given institution] do? [And] *How* do they do it?" The second block, $[\theta / \Theta]$ or *logos*, consists of a rationale about the technique (called technology, θ) encompassed by a wider theory Θ, and answers the question, "*Why* do they do it that way?" (p. 184).

One key postulate of ATD is that knowledge is institutionally situated. In other words, praxeologies depend on the institution in which they operate. They may also undergo changes by evolving within an institution or when moving from one institution to another. Therefore, we can expect praxeologies related to the study of free-fall to differ in calculus and mechanics courses; these differences may have an impact on students' learning and on their view of the use of derivatives in the context of free-fall. For this reason, we are interested in answering the question: *What praxeologies in the context of free-fall are present in mechanics and differential calculus courses, and how are derivatives as a rate of change used in these praxeologies?*

METHODS

In this paper, we continue our work on the practices around the notion of derivative in a kinematics context, both in calculus and mechanics courses, with an emphasis here on the context of free-fall. Our study took place in Quebec, Canada, at a large college, a post-secondary institution that Quebec students must attend before pursuing university studies. This work is part of the first author's ongoing PhD research project structured in three phases. The main results from the first phase, consisting of a praxeological analysis of five calculus textbooks (CP, HA, HH, StEC, and StC) and five mechanics textbooks (Ba, Kt, LP, Sa, SJ), were reported in Hitier and González-Martín (2022a) [1]. In a second phase, we turned to the instructors, interviewing four calculus teachers and three mechanics teachers. We also observed one calculus and one mechanics course led by two other teachers. This work allowed us to make conjectures about students' understanding of the derivative in both courses, as well as about their capacity to make connections between the two subjects (Hitier & González-Martín, 2022a). Those conjectures were tested in the third phase, through an online questionnaire and student interviews with four participants (for some results, see Hitier & González-Martín, 2022b, 2023). Our data collection took place during the COVID-19 pandemic, so class observations and interviews had to be conducted online.

In this paper, we use data from each of the three phases. Drawing on data from the first phase (the praxeological analysis of the textbooks), we focus on early sections of the calculus textbooks that address the derivative, which include the introduction and definition of the derivative, the derivative as a function, and the first formulas necessary for differentiating a polynomial. We exclude other sections on average rate of change, as well as more advanced rules such as the product and quotient rules, and the application of the derivative to curve sketching and related rates. As for the mechanics textbooks, we focus on the chapter addressing one-dimensional kinematics, starting with the definition of instantaneous velocity; being interested in instantaneous rates of change, we exclude any sections dedicated to average velocity or to motion with constant velocity. We focus our analysis on kinematic tasks (i.e., involving position, velocity, or acceleration), that is all the tasks in mechanics, but only part of the tasks in calculus. Each question is considered to be one task unless it explicitly consists of multiple subtasks. For instance, a question asking for the velocity and acceleration function would be split into two distinct tasks. We identified the main types of tasks and associated praxeologies. Each task was also classified according to the context in which it is situated (see Free-fall in textbooks below). Regarding the second phase, we present here a similar analysis of the tasks presented to the students in the calculus and mechanics courses that we observed during the Winter 2021 term. The relevant excerpts were recorded and transcribed. Examples of the teachers' work, such as screenshots from the video recording, were also saved to support the analysis. Finally, using data from the third phase, we focus here on one question from the students' problem-based interviews that took place in March 2021. During the Fall 2020 term, all four participants had already taken differential calculus and mechanics, their first college-level mathematics and physics courses. At the time of the interviews, they were enrolled in an integral calculus course.

ANALYSIS

Free-fall in textbooks

Figure 1 shows the proportion of tasks in a free-fall context among the kinematics tasks in the textbooks. We distinguished kinematics tasks presented in "realistic" contexts involving some kind of scenario (like the movement of a car) from tasks presented in a more "abstract" context, where no "real" scenario is involved (for instance, a moving particle). While the proportion of abstract tasks and tasks in realistic contexts differs greatly between the calculus and mechanics textbooks, the proportion of the kinematics tasks in a free-fall context is similar (23% on average in calculus vs. 22% on average in mechanics).

In all the mechanics textbooks, free-fall is presented in a dedicated section that appears right after the section on uniformly accelerated motion (UAM), that is, motion with constant acceleration. We see this as an indication that mechanics books consider free-fall as a context of application for UAM. In calculus, free-fall is used to introduce the notion of instantaneous velocity in four out of the five textbooks (HA, HH, StC, StEC), and even to motivate the use of limits in three of them (HA, StC, StEC). Therefore, in calculus, free-fall appears to play more of a conceptualisation role.

	Free-fall context		"Realistic" contexts (free-fall + other)	"Abstract" contexts (no "real" scenario)
	% of all kinematics tasks	(% of Contexts)	% of all kinematics tasks	% of all kinematics tasks
Cal	(on average) **23.0%**	(57.0%)	(on average) **39.1%**	(on average) **60.9%**
CP	45.5%	(69.0%)	65.9%	34.1%
HA	22.2%	(61.5%)	36.1%	63.9%
HH	20.8%	(47.8%)	43.4%	56.6%
StC	14.7%	(58.6%)	25.0%	75.0%
StEC	12.0%	(47.8%)	25.0%	75.0%
Mec	(on average) **22.0%**	(33.8%)	(on average) **64.5%**	(on average) **35.5%**
Ba	26.6%	(39.9%)	66.8%	33.2%
Kt	12.1%	(18.2%)	66.1%	33.9%
LP	27.1%	(43.7%)	62.1%	37.9%
Sa	14.3%	(26.7%)	53.5%	46.5%
SJ	29.9%	(40.5%)	74.0%	26.0%

Figure 1: Proportion of tasks in kinematics contexts.

In this paper, we elaborate on the praxeological analysis from Hitier and González-Martín (2022a) in which the large number of tasks were organised into broader groups based on type (for instance, "finding values of motion functions"). The tasks were further categorised. For calculus, these categories were based on the role of the kinematics context in the techniques and the technology, resulting in three categories: precalculus praxeologies (P), praxeologies relying on the definitions presented in the textbooks (C), and techniques requiring kinematics interpretations beyond applying mathematics formulas or definitions (K). In mechanics, our categorisation was mostly based on the four representations (Jones & Watson, 2017) of the motion functions (velocity and acceleration): slope (S), rate of change (R), derivative (D), and limit of difference quotients (L). We created two additional categories, one for purely mathematical (non-calculus) techniques (M) and the other for tasks where the technique uses one or more of the kinematics equations (E). A complete description of the categories can be found in Hitier and González-Martín (2022a) [2].

Figure 2 presents all the free-fall tasks for calculus and mechanics distributed across the above categories (the total adding to 100%), as well as under broader groupings based on types of tasks (e.g., "Finding the time when the object is at a given position"). We include, in brackets, the percentage represented by the free-fall tasks compared to all kinematics tasks grouped under the same type of task and within the same category.

For instance, in calculus, all free-fall tasks of the type "Finding the time when the object is at a given position" are in the P category; they represent 10.7% of all free-fall tasks in calculus, and 75.3% of all kinematics tasks in that category. We note that the tasks related to extrema represent the largest proportion of free-fall tasks among all kinematics tasks grouped under the same type of task, with all tasks (100%) in the P and K categories addressing free-fall. However, this type of task represents only 10.9% of the tasks in a kinematics context presented in the calculus textbooks.

Calculus				Mechanics							
P	C	K	Total	Total	M	E	D	L	S	R	O
50.5% (42.8%)	34.2% (17.0%)	15.3% (20.7%)	100% (23.0%)	100% (22.0%)	5.2% (15.1%)	73.9% (33.7%)	0.7% (15%)	0% (0%)	2.3% (2.8%)	15.0% (26.9%)	18.7% (20.6%)
Finding the value of kinematics functions: distance/position; velocity/speed; acceleration											
34.1% (49.2%)	19.3% (32.6%)	0% (0%)	53.4% (36.4%)	31.0%) (24.0%)	0% (0%)	29.7% (38.1%)	0% (0%)	0% (0%)	0% (0%)	0% (0%)	2.0% (9.7%)
Finding the time: when the object is at a given position; other											
10.7% (75.3%)	0% (0%)	0% (0%)	10.7% (58.3%)	12.1% (31.1%)	0% (0%)	11.8% (38.5%)	0.2 (25%)		0% (0%)	0% (0%)	1.6% (36.1%)
Extrema: finding time of maximum height; finding maximum height; other tasks related to finding extrema											
1.8% (100%)	0% (0%)	9.1% (100%)	10.9% (95.0%)	13.2% (59.4%)	0% (0%)	12.4% (87.1%)	0.3% (6.7%)		0% (0%)	12.6% (80.0%)	0.2% (33.3%)
Finding an instantaneous rate: velocity, acceleration											
	9.8% (26.6%)		9.8% (26.6%)	3.4% (11.1%)	0.3% (50.0%)	2.9% (12.2%)	0.3% (18%)				0% (0%)
Other "kinematics" task: involving one moving body; involving more than one moving body											
	0% (0%)	0% (0%)	0% (0%)	13.2% (22.2%)	1.6% (35%)	11.0% (28.0%)	0% (0%)		0.2% (16.7%)	0.4% (16.7%)	0.7% (3.7%)
Remaining tasks											
3.8% (23.6%)	5.1% (4.8%)	6.2% (12.7%)	15.2% (9.4%)	27.1% (15.8%)	3.4% (19.6%)	6.1% (14.9%)	0% (0%)		2.1% (2.2%)	2.1% (12.4%)	14.2% (27.8%)

Legend: on average, proportion of tasks in all free-fall tasks (proportion of free-fall tasks in all kinematics tasks from the same group).

Figure 2: Proportion of tasks in free-fall per category.

Figure 3 presents the praxeological analysis of a typical example of such a task in calculus and mechanics. For the calculus task, the velocity function had been determined in part a) using differentiation, and the equation $v(t) = 0$ had been solved in c). The two tasks illustrate the following observations from our analysis:

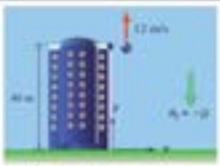
	Calculus: CP, example 1 p. 215	Mechanics: Ba, example 3.16 p. 76–77
Type of task	Determine the maximum height of an object in free-fall.	
Task (our translation)	From the top of a bridge, a stone is thrown vertically upwards. The position x of the stone above the river, as a function of time t, is given by $x(t) = 58.8 + 19.6t - 4.9t^2$, where t is in seconds and x(t), in meters [...] d) Let's determine the maximum height the stone will reach.	A ball is launched upwards with an initial velocity of 12 m from a roof 40 m above the ground. Find [...] (c) its maximum height. 
Proposed solution (our translation)	The object is at its maximum height when $v(t) = 0$, i.e., after 2 s (see c)). $x(2) = 58.8 + 19.6(2) - 4.9(2)^2 = 78.4$. Hence the maximum height is 78.4 m.	(c) At the point of maximum height, $v_y = 0$, so that the equation for the square of the velocity gives $0 = 12^2 - 2(9.8)(y - 40)$. So $y = 47.3$ m.
Technique	(Interpret where the maximum is located) Solve $v(t) = 0$ for t and substitute in the position function $x(t)$.	(Interpret where the maximum is located) Solve for the position in the equation of the square of the velocity using the given data and $v = 0$.
Logos	(Interpretation of zeros of the derivative as maxima or minima, and relation to the sign of the derivative with the variations of the function); Solving equations.	(The velocity at maximum height is 0); Kinematic equations; The acceleration due to gravity is $-g$. Solving equations.

Figure 3: Praxeological analysis of a typical "maximum height" task in calculus and mechanics (brackets indicate an implicit element).

- Mathematically, the technique used in both cases is mainly based on solving equations. In mechanics, it uses one of the kinematics equations; in calculus, it uses an equation obtained through differentiation (sometimes using the limit definition, a technique rarely used in mechanics and never in a free-fall context).
- These tasks are mostly in categories K (for calculus) and R (for mechanics), since although in principle these praxeologies can be tied to a covariational reasoning, those elements are left implicit both in mechanics and calculus.

Free-fall in the (online) courses

Due to the online setting, the calculus class was organised weekly as follows: at the start of the week, video(s) of the teacher's lecture (content videos), were made available and a list of recommended problems (RP) selected from the textbook (StEC) by the mathematics department was distributed; one of the two 90-minute synchronous online sessions was dedicated to answering students' questions, while during the second one, the teacher went over some of the RPs. In a departure from the textbook used in class, the free-fall context was not used to motivate the notion of limit nor to introduce the notion of derivative (although the teacher started with a "conceptual example" of a moving car). Figure 4 shows the distribution of the content during the two weeks when the derivative and the basic differentiation formulas were introduced.

Tasks	Percentage of tasks (in free-fall) in all kinematics tasks: M	C	K	Percentage of free-fall tasks in all kinematics tasks
selected by the department	39.3% (14.3%)	53.6% (14.3%)	7.1% (3.6%)	32.1%
selected by the teacher (RP)	40.0% (0.0%)	60.0% (20.0%)		20.0%
discussed (CV & synchronous)		100.0% (25.0%)		25.0%
Total	**35.5%** (12.9%)	**58.1%** (12.9%)	**6.5%** (3.2%)	**29.0%**

Figure 4: Distribution of the content in the observed calculus class.

In the tasks in a kinematics context, free-fall tasks represent a proportion close to the average proportion in all textbooks (Figure 1). But none of the two tasks selected by the teacher gave students the opportunity to go beyond the definition of velocity and acceleration as derivatives, or to use their representation as a rate of change in a meaningful way. For one of the two tasks, a student asked during a question session:

> "Why do we need to use derivatives for that kind of formula? Because at least from what I remember from physics, […] I learned how to calculate the velocity of stuff and I never used derivatives […], but […] because of the context I guess I need to use derivatives."

The student identified that similar tasks were solved in physics using a praxeology based on kinematics equations, while the calculus context prompted him to use derivatives. The teacher replied that the velocity is the rate of change of the position, i.e., the derivative of the position, and that acceleration can be obtained the same way from the velocity, adding that in calculus, there is no need to memorise the formula:

> "Given the information in this problem, you don't need to remember any physics formulas. You just need to use the derivative. That's the beauty of it all."

He also emphasised the historical role of calculus in elaborating the physics formulas:

“Calculus is at the heart of most physics, both modern and classical, starting with Newton. […] Physicists use shorthand and don’t actually bother calculating derivatives when in practice they just need to calculate the impact velocity of a ball hitting the ground being dropped from a plane. They’re not going to bother re-deriving all of their nicely discovered formulas, but how were these formulas derived? Well, precisely with calculus.”

In mechanics, online classes consisted of traditional lectures and the examples solved were taken from the textbook (SJ). Contrary to calculus, no exercises were selected for the students to focus on. Aside from using the case of a ball thrown upward suggested by a student as an example for a “nonzero acceleration typical case” (left column in Figure 5), as in the textbooks, free-fall was first studied after the section on UAM in a context where the kinematics equations could be used (right column in Figure 5).

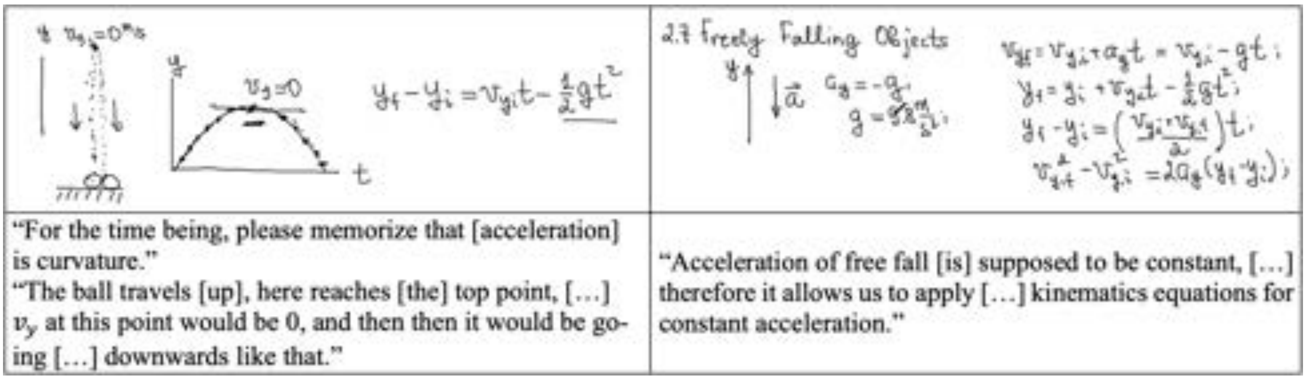


Figure 5: Free-fall in the mechanics class.

As in calculus, rates of changes were absent from the discussion of free-falling objects in the mechanics class: all three free-fall exercises solved in class (four tasks in total) relied heavily on kinematics equations (category E [3]), ignoring the fact that the velocity at maximum height is 0 m/s. We note that one of the tasks was different from those found in the calculus textbooks, as it is a “two-stage problem”, that is, a situation where the position function would be piecewise defined. For this specific exercise, a position function was provided for the first stage, while in the second stage, the object was in free-fall. The teacher focused more on the first stage of the motion, as this was the only example presented in class where the velocity was obtained as the derivative of a position function. Instead of using differentiation formulas as in the textbook’s solution manual, the teacher used the limit definition.

To find the maximum height in free-fall

In the problem-based interviews, students were asked questions clearly situated in one institution (see, for instance, the pairs of similar questions presented in Hitier & González-Martín, 2023), as well as more open questions in a kinematics context, such as the following one in the context of free-fall:

How would you proceed to determine the maximum height of an object thrown upward?

The student participants could ask for any information they might need, other than that related to the maximum height itself. All four students spontaneously placed the question in a mechanics context, which led them to base their answer on a typical mechanics technique. They all asked for punctual data that would allow them to use one or more of the kinematics equations.

> "I would need the initial velocity […and…] the time it reaches the same level […] we divide it by two […] and I get the time it reaches the maximum height. [Then] we would use the […] equation […] so that one with the x_f or Δx equals $v_i t + a$ or half at" [4] (S1)

We see that this technique uses ready-to-use formulas. It does not require students to use covariational thinking or even to situate the context of the question. Only S2 considered techniques from calculus as well:

> "I think I could solve it two ways. Either if I had three out of the five kinematics variables, then I could use the kinematics equations to determine the maximum height, or if I had the position function of this object, then I could use the first derivative test to determine the maximum of the function." (S2)

We conjecture that this may be due to the fact that S2 was enrolled in a paired differential calculus and mechanics class were the teachers made more explicit links between the topics and explained that some problems could be solved using either a calculus or a mechanics approach.

DISCUSSION AND CONCLUSION

Free-fall has been identified as providing a relevant context in which to learn the concept of derivative (e.g., Karam & Krey, 2015), and conceiving the derivative as rate of change is the most useful approach in other disciplines, such as physics (e.g., Thompson & Harel, 2021; White Brahmia, 2023). Tasks in a free-fall context are presented in all the textbooks we analysed (both calculus and mechanics). In our praxeological analysis of textbooks, we identified that tasks linked to finding the maximum height of an object in free-fall could lead to using a covariational reasoning in order to understand why the velocity has to be 0 when the object is at its highest point. We therefore see this context as favourable to developing an understanding of the derivative both as rate of change and velocity—that is, two of the four *source* contexts in Jones and Watson (2017). However, we found the praxeologies in both the calculus and mechanics textbooks to be deficient, the derivative as rate of change being only implicitly present in the technique and technology.

Our observation of mechanics and calculus courses confirms that the rate of change aspect of the derivative in free-fall is neglected. Only tasks based on definitions and formulas were presented in this context to the students of both courses. In contrast to the textbooks used in class, the calculus teacher did not use the context of free-fall to motivate the notion of limit or to introduce the notion of derivative. In the mechanics class, as in the textbooks, free-fall was mainly an application of UAM. Although the teacher used the example of free-fall immediately after defining the instantaneous acceleration, the logos is deficient: the teacher identified the acceleration with the curvature of the graph of the position function, stating that the velocity at the maximum had to be zero; however, he never quite explained why that is so, recommending instead that the students memorise the facts.

In line with the question asked by one student in the calculus class, the four interview participants could easily identify the free-fall task as a mechanics task. We conjecture that this is due to two factors. First, the study of motion is an important part of the mechanics curriculum. Second, as illustrated in the student's exchange with the calculus teacher, using the notion of derivative can be more tedious than solving simple equations. We also think that this discussion in the calculus class, as well as the student interviews, highlight the students' compartmentalisation of the "calculus" and "mechanics" praxeologies, which was also observed by Taylor and Loverude (2023).

Our study shows that the potential for using the context of free-fall to enhance students' understanding of the derivative as rate of change is not exploited. The use of the same types of task both in calculus and mechanics could help students understand the role of the derivative in kinematics. But this context in both courses also allows us to see the compartmentalisation of the praxeologies. Further studies would be needed to explore how using a free-fall context could foster students' understanding of the derivative as a rate of change and, more generally, to investigate how to leverage free-fall contexts to enhance students' understanding of the derivative as velocity and rate of change. Furthermore, we call for further studies on the institutional constraints that have fostered the current situation, characterised by missed opportunities to provide meaning to the derivative, and on possible approaches that would encourage a more explicit use of the derivative as rate of change in free-fall contexts.

NOTES

1. Complete references to and more details about the textbooks can be found in Hitier and González-Martín (2022a).
2. For this paper, we have grouped category AI (tasks involving integral calculus techniques) with category O (other).
3. One task, a two-stage problem with only the second stage in free-fall, was classified as DE, as differentiation of the position function was required to find the velocity at the end of the first stage.
4. Note that the student makes a mistake; the correct equation would be: $\Delta x = v_i t + ½at^2$.

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Bridging the gap: curriculum development addressing the transition into mathematics in economics education

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This paper concerns the transition issue of economics students' diverse mathematics school backgrounds. The study explores curriculum discontinuities between school and university mathematics-for-economics education in the secondary-tertiary transition phase. A two-step content analysis reveals six key mathematics areas that many of the first-year economics students previously had not had the opportunity to study, despite their relevance for the mathematics-for-economics course. This paper contributes to the limited educational research on mathematics in economics education and addresses how curriculum misalignments impact students transitioning into economics courses, by identifying critical gaps in mathematical learning opportunities essential for success in economics education.

Keywords: Transition to university mathematics, Teaching and learning of mathematics in other disciplines.

INTRODUCTION

Mathematical knowledge and reasoning are depicted as essential for grasping fundamental economic principles and therefore, mathematics takes up a large part of first-year economics education (Dawson, 2014; Monteiro & Lopes, 2007). First-year mathematics courses typically cover basic algebra, linear functions, logarithms, exponentials, financial mathematics, differentiation, partial differentiation, optimization, integrations, matrices, and difference equations (Dawson, 2014; Voßkamp, 2023). The objective of this mathematics education is to scaffold students' understanding and reasoning of economics at the transition phase (from school to the use of mathematics in economics courses). However, internationally, there is a problem of high failure rate in the mathematics-for-economists courses in economics education (Büchele, 2020).

This paper is part of the author's larger PhD study which addresses the problem of high failure rate by investigating the secondary-tertiary transition from school to university level mathematics in economics education. In the literature review of Landgärds-Tarvoll (2024), three secondary-tertiary transition issues specific for transition to university mathematics in economics education were identified. The current paper concerns one of the issues identified: the issue of the first-year economics student group being heterogenous in terms of school mathematics background.

It addresses the issue from an institutional perspective by examining and contrasting student's previous learning opportunities in school with the assumed mathematical preparation in the university mathematics-for-economists course in Norway. The aim of the research is to inform the curriculum design for a pre-course intervention at the University of Agder (UiA). The pre-course intervention has the objective of mitigating the issue of mathematical background heterogeneity in the student group and facilitating the inclusivity of all economics students in the mathematics course.

The research was conducted in two phases. Initially the curricula of the courses were analysed. Subsequently, the investigation focused on reviewing textbooks to explore opportunities for mathematical learning related to the mathematical topics identified during the initial phase. The research questions addressed in the respective phases are:

RQ1: "What opportunities to study the different mathematics sub-areas of the mathematics-for-economics course curriculum are displayed in the curriculum of practical (P), social science (S) and natural science (R) mathematics respectively?"

RQ2: "To what extent do the textbooks present learning opportunities in the domains of Algebraic Operations and Single-Variable Functions?"

THEORETICAL BACKGROUND OF THE STUDY

The process of teaching and learning always include some content or piece of knowledge to be taught and learnt. The larger PhD study, which this study is part of, posits that knowledge is shaped through the interaction between learners and educational institutions, and therefore, curricula and textbooks include the knowledge to be taught as it "lives" in the different mathematics courses (Bosch & Gascón, 2014). The crucial role of mathematics textbooks in the teaching and learning process is well-established in the case of economics and mathematics (Feudel, 2023; Mkhatshwa, 2023). In particular, this study draws on textbooks being the "potentially implemented curriculum" and thereby representing students' learning opportunities (Houang & Schmidt, 2008). Reys *et al.* (2004, p. 61) write: "the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn."

Curricular discontinuity in the transition phase is a common transition issue across educational systems. This issue has been addressed in several studies at previous INDRUM conferences (Hochmuth *et al.*, 2021). However, attention to the transition specifically into service mathematics courses has been limited and the transition to university economics mathematics has not been addressed (Hausberger & Strømskag, 2022; Hochmuth *et al.*, 2021).

In the case of economics education, the curricular discontinuity is pronounced, largely due to the diverse school mathematics backgrounds of students enrolled. Economics and business programmes typically have minimal, if any, mathematics prerequisites for admission (Asian-Chaves *et al.*, 2021; Darlington & Bowyer, 2017; Dawson, 2014; Opstad, 2021) and consequently, the student group is heterogenous in terms of mathematics background. However, economics and business study programmes expect their students to have a certain level of mathematics knowledge, which means that not all students, although admitted to higher education economics studies, are mathematically prepared for such studies (Opstad, 2021; Büchele, 2020).

Based in the Norwegian context, this study adds to the limited research on curriculum and transition in economics education as it addresses the curricular discontinuity between school and university mathematics in economics education.

THE CONTEXT OF THE STUDY

Admission to economics studies in Norway is based on Norwegian Higher Education Entrance Qualification (The Norwegian Association of Higher Education Institutions, 2011) which includes a minimum of two years of mathematics studies in upper secondary school. Students choose between mathematics courses pitched at three levels, practical (P), social science (S) and natural science (R) mathematics. These courses are different with respect to the level of mathematics involved, hours of study and type of treatment (Utdanningsdirektoratet, 2006a, 2006b, 2006c, 2006d)[1]. Having chosen the more theoretical path gives the student the possibility to study mathematics also in the third and last upper secondary school year (that is, a third school year of S2 or R2 mathematics). The mathematics-for-economists course at university assumes that students have studied to a level of R1 or S2, meaning that the relevant school mathematics curriculum for economics within these two school courses comprises 2 years of natural science mathematics or 3 years of social science mathematics (The Norwegian Association of Higher Education Institutions, 2011; Utdanningsdirektoratet, 2006).

At UiA, in 2017, approximately 40% of students with P-mathematics (about 50% of the student cohort) failed the mathematics-for-economists course[2]. Similar failure rates are reported for other universities in Norway as well, e.g., NTNU (Busch et al., 2017). Still, the broad admission requirements are motivated by both national and international studies reporting several reasons (non-academic and academic), for students interested in economics studies not choosing the recommended mathematics school path (Opstad & Årethun, 2019; Rylands & Shearman, 2018; Sikora & Pitt, 2019). The high portion of students with P-mathematics failing the mathematics-for-economists course is a serious problem as economics education should seek to be inclusive and students admitted to a university economics programme should all be given equal opportunities to succeed in their studies.

The Norwegian Association of Higher Education Institutions (2011) addresses the curricular issue by recommending that institutions provide additional mathematics instruction for students with insufficient mathematics proficiency from upper secondary school. However, the specifics of what this supplementary instruction should include, and its implementation are not detailed or further discussed in the report. Therefore, with the motivation of inclusivity of all students and building on the

[1] This discussion refers to the curriculum as it was established in 2006, the curriculum was updated in 2021.

[2] Course description: https://www.uia.no/en/studieplaner/topic/MA-138-1

hypothesis that the high failure rate indicates unsuccessful transition from school to university economics mathematics due to the heterogeneity in mathematics backgrounds, this study explores the curricular gap between the different school courses and the assumed mathematical knowledge level of the university mathematics course.

The ultimate goal for the study is to inform the curriculum design of new pre-course intervention consisting of a diagnostic test in combination with a bridging course. The design of such an intervention is elaborated in Landgärds (2019) and Landgärds (2021).

METHODOLOGY: METHOD AND RESULTS

To design a pre-course intervention with the aim to facilitate students transition from school to university mathematics in economics education, it is necessary to understand what mathematical learning opportunities students have previously had in school and what mathematical knowledge is assumed by the university mathematics course. To achieve this, a two-phased case-oriented content analysis (Kuckartz, 2019) of curricula and textbooks was carried out.

Case-oriented content analysis

To address RQ1 (above) a content analysis was carried out, in which the school mathematics curriculums (Utdanningsdirektoratet, 2006a, 2006b, 2006c, 2006d) and the curriculum of the mathematics for economics course (The Norwegian Association of Higher Education Institutions, 2011) were examined and compared. The mathematics for economics course curriculum's six sub-areas: Algebraic operations, Single-variable functions, Multivariable calculus, Arithmetic and geometric series, Financial mathematics and Integral Calculus, were defined to be the thematic categories to identify in the curriculum of the three levels of school mathematics (i.e., three cases: P-, S- and R-mathematics). Text segments for the different cases were coded with the categories. The analysis was made in Norwegian, and the codes presented in Figure 1 are translations. Figure 1 illustrates a summary of the coding frame. An empty box means there was no learning opportunities on the category within the school course.

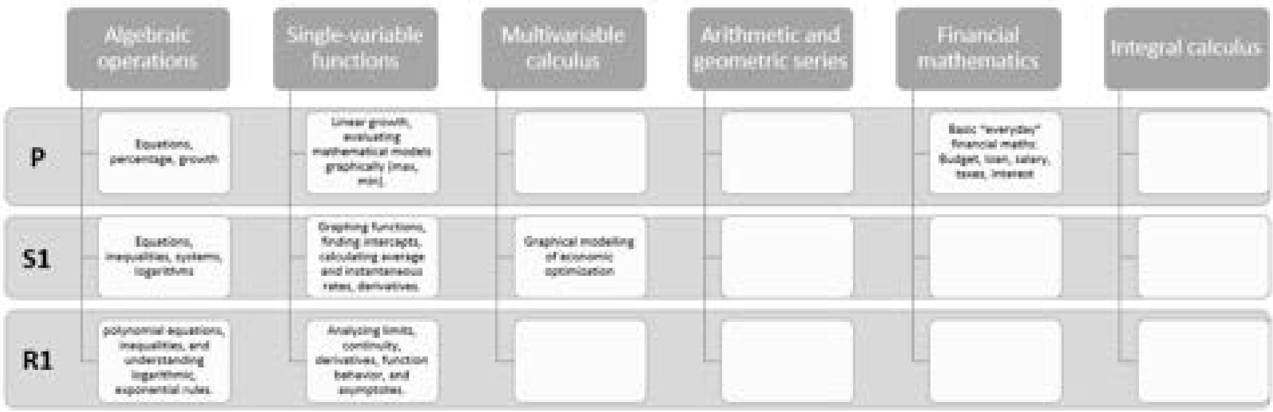

Figure 1: First phase of the data analysis on curriculum comparison.

To address RQ2 (above), school mathematics textbooks, the Sinus Series which constitute Oldervoll *et al.* (2013a, 2013b, 2014a, 2014b, 2014c) and the university mathematics textbook by Dovland and Pettersen (2015) were examined. The aim was to create a descriptive overview of the curriculum in the school textbooks related to the two categories identified in the first step of the analysis. The Sinus Series textbooks were chosen for a focused analysis avoiding the potential inconsistencies and gaps that might arise from mixing content between textbooks from different publishers. The service mathematics textbook served as a reference point to elucidate the categories on mathematical content encompassed by "Algebraic Operations" and "Single-Variable Functions," and the textbooks for the different school mathematics levels were considered cases. The two topics were covered in Chapters 1–5 of the university mathematics book. Therefore, all subsections of these chapters (37 sections) constituted the categories for the analysis. Each case in the categories was coded fully covered, partly covered, or not covered. From this step the main differences emerged. For example, the category on "linear equation systems" was marked fully covered by the S and R mathematics textbooks but only partly by the P textbook. As a final step, the categories where such differences between the cases were identified, the narratives and exercises were detailed further. Continuing the previous example then, the analysis made it visible that students who followed the P-mathematics course, as opposed to S- and R-mathematics, had not had the opportunity to study how to solve linear equation systems algebraically, they only learned how to find solutions graphically.

Results

In the second step analysis, six mathematical shortfall areas in which the P-mathematics curriculum lacked content that was included in R-mathematics relevant to the mathematics-for-economics course were identified. These are presented in Table 1 below. These six shortfall areas were decided to constitute the bridging course curriculum of the new pre-course intervention.

Target student groups	Topics of the six mathematical areas relevant to the mathematics for economics course, which the P-mathematics curriculum lacked compared to R-mathematics.	
Mostly repetition for all students, with some new topics for students with P-mathematics background.	**Fractions and percentage calculation** • Order of operations • Addition and subtraction of fractions • Multiplication and division of fractions • Basic percentage calculations • Growth factor • Power equations	**Linear functions, equations, and system of linear equations** • Solving linear equations by calculations and graphically • Domain of functions • Slope of a line and intersection with axes • Substitution method • Simple inequalities

New topics for students with P-mathematics background.	**Quadratic functions and basic algebra**
	• Multiplication with parentheses
	• Powers of exponents
	• Exponents of type: $\frac{1}{n}$, square roots and $n-$roots.
	• Quadratic equations
	    o Zero product rule
	    o Quadratic formula
	    o Discriminant
	• Factorization
	• Simplification of algebraic expressions
	• The quadratic function
	• Finding max/min using symmetry
	• Domain and range
	• Quadratic inequalities
		Exponential functions and logarithms
		• Practical situations of value increase/decrease
		• Expression of exponential function
		• Logarithms with the base 10
		• Simple logarithm rules for solving equations (e.g., $3^x = 28$)

New topics for students with P-mathematics background. Mostly new topics for students with S1-mathematics background.	**Limits of functions and the definition of the derivative** • Limits as x approaches infinity or a number a • Calculations with limits • One-sided limits • Continuity • Introduction to the derivative through average rate, tangency • The limit-definition of the derivative	**Derivatives, differentiation rules and graph analysis** • Simple differentiation rules, special differentiation rules ($e^x, a^x, \ln(x)$), the chain rule, derivatives of products and quotients • Increasing and decreasing function analysis • Finding max/min • Sign diagram

Table 1: The six short-fall areas of school mathematics (P-mathematics compared to S- and R-mathematics) relevant for the mathematics-for-economists course.

IMPLICATION OF THE RESULTS

At UiA, as at most other universities, the mathematics-for-economists course had until 2018 been offered in the first year's first semester with the intention that students needed to obtain the mathematical 'tools' for their economics studies as early as possible. However, the results above indicated a need to address the transition phase concerning the curricular gap. This led the Business School at UiA to reschedule the mathematics course to the second semester of the first year and implement a pre-course intervention, with the curriculum covering the six short-fall areas as outlined in Table 1, in the first semester.

All first-year economics students are required to take a diagnostic test, divided into the six corresponding mathematical shortfall areas of the bridging course (Table 1). After completing each part of the test, students receive a recommendation on whether to participate in the corresponding part of the bridging course or not. Students are to decide whether they want to follow the recommendation themselves. The goal of such a design is to enable students to assess their own knowledge and understand the expected knowledge level at the beginning of the mathematics for economics course to mitigate the transition issue of student underestimating the demand and level of mathematics in economics studies, a second transition issue identified by Landgärds-Tarvoll (2024). And, in fact, the pre-course intervention is now running for its sixth year and students express appreciation for the chance to be aware of the curricular gap, both in terms of the possibility to gauge their current mathematical proficiency and align it with anticipated academic standards in the yearly student course evaluation.

DISCUSSION

Hochmuth et al (2021, p. 200) summarizes: "Overall, the work presented at INDRUM sees the transition to the study of mathematics at university as a multifaceted process that requires a shift in the way students think mathematically." This study contributes to the discussion by addressing the under-research area of mathematics in economics education. Landgärds-Tarvoll (2024) found that students' diverse school mathematics background was one of the main issues in the transition to university mathematics in economics as opposed to the transition issue of shift in thinking mathematically for students transitioning to specialist mathematics courses. Thus, this study adds to the previous INDRUM studies by adopting an institutional perspective, asserting that textbooks and curricula represent the knowledge to be taught in the different courses. Consequently, the analysis of them operationalised the identification of curricular discontinuities in the transition phase.

The six mathematical shortfall areas identified in this study align with Büchele's (2020) observation that many first-year economics students seem to lack proficiency in basic algebra and secondary-school mathematics. The results from the study also explain the unequal distribution of students with P-mathematics background constituting the largest part of the failure rate in the mathematics for economist course as found by Busch et al. (2017) and Landgärds (2019).

In the economics education literature, support measures' effectiveness is a disputed topic (Büchele ,2020; Vosskamp, 2017). However, Lawson et al. (2020) emphasize the general evolving conversation about mathematics support delivery strategies and highlight the change from research merely justifying their existence to exploring various delivery strategies. This evolution in research perspective signals a broader recognition of the need to not only validate the necessity of support measures but also to refine and optimize how they are provided to students. Along these lines, this study underscores the importance of a detailed examination of the curricular gap as a crucial preliminary step to comprehending the potential transition challenges students may encounter.

Furthermore, the research detailed in this paper, while rooted in the Norwegian educational landscape, addresses the international issue of high failure rate in the mathematics-for-economists course(s). This shared concern underscores the relevance of the research methodology and results to a global audience, bridging local insights with global challenges. Consequently, the research process and the results of the study should hold significance for an international audience as well. They may encourage universities to conduct comprehensive assessments of student preparedness, especially in terms of previous school mathematics study opportunities. And thus, they might lead universities to develop and implement innovative support strategies, enriching the academic literature and practices focusing on the secondary-tertiary transition to university mathematics in economics education.

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How the Bologna Process changed the teaching of mathematics for engineering: the case of Croatia and Spain

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This study examines the impact of the Bologna Process on mathematics education for electrical engineers in Croatian and Spanish institutions. Through praxeological analysis of exams, we observe a reduction in task complexity post-Bologna, emphasising algorithmic approaches. Despite institutional differences, both countries show convergence in content changes, raising questions about the transferability of skills and knowledge. The study highlights a shift from explicit to implicit justifications, pointing to a shift towards praxis that requires little development of strategies or steps for resolution, which seems to conflict with the objectives set by the Bologna Process to promote competence-based instruction.

Keywords: Teaching and learning of mathematics in other disciplines, Mathematics for engineers, Anthropological theory of the didactic, Curricular and institutional issues concerning the teaching of mathematics at the university level.

INTRODUCTION

The culmination of the Bologna Process during the first years of the 21st century represented a significant change in European higher education institutions. The Bologna (1999) and Prague (2001) declarations fixed six main objectives: (1) the adoption of a system of easily readable and comparable degrees, (2) the adoption of a system essentially based on two main cycles (undergraduate and graduate), (3) the establishment of a system of credits (ECTS), (4) the promotion of mobility, (5) the promotion of European cooperation in quality assurance and (6) the promotion of the European dimensions in higher education.

To facilitate the achievement of objectives 1 and 3, the Bologna Working Group on Qualifications Frameworks (BWGQF) published the Framework of Qualifications for the European Higher Education Area (Bologna Working Group on Qualifications Frameworks, 2005). This document establishes the requirements of the elements of the national frameworks of qualifications and establishes that they must be formulated in terms of learning outcomes, including competencies. Learning outcomes are defined as “statements of what a learner is expected to know, understand and/or be able to do at the end of a period of learning” (Bologna Working Group on Qualifications Frameworks, 2005, p. 14), and they “are expressed in terms of the level of competence to be obtained by the learner” (OECD, 2011 p. 8). Analysis of documents on European higher education reveals that the concept of competence is not defined precisely and uniquely, nor is it systematically and consistently used (Davies, 2017).

The document from 2005, brought by the BWGQF, highlights that the change in the way knowledge should be described was supposed to have implications for the curriculum. It was stated in these terms:

> They are thus likely to form an important part of 21st century approaches to higher education (and, indeed, to education and training generally) and the reconsideration of such vital questions as to what, whom, how, where and when we teach and assess. The very nature and role of education is being questioned, now more than ever before. (Bologna Working Group on Qualifications Frameworks, 2005, p. 38)

As we explore the complexity of this transformative period in European higher education, our focus narrows to the mathematical education for engineers. There is a shared agreement that the knowledge and understanding of mathematical principles, as well as applying them, are general learning outcomes in the first cycle of engineering education (OECD, 2011).

Diverse studies have pinpointed the stability of the contents of the mathematics courses in engineering degrees. Romo-Vazquez (2009) states that this stability is related to the epistemological conception of mathematics in the institutions in charge of the mathematical training of engineers where “mathematics are considered as autonomous and as a pre-requisite of other courses” (p. 299). This stability was also explored by analysing exams from mathematics courses for engineers at an institution in Spain from 1994 to 2020. In this study, Florensa et al. (2023) observed that the implementation of the competence-based model did not stimulate new activities in mathematics courses related to modelling and solving engineering problems. In this paper, we aim to examine whether these conclusions are specific to the observed institution or if this phenomenon is wider and similar in another engineering school in Croatia.

THEORETICAL FRAMEWORK AND METHODS

We frame our research within the Anthropological Theory of the Didactic (ATD). In the transition from content-based curriculums from the pre-Bologna period to competence-based curriculums in the post-Bologna implementation period, we focus on the institutional assessment of the knowledge that is involved in mathematics courses within engineering degrees.

In ATD, any human activity, especially those related to knowledge generation, transmission, and dissemination, can be described in terms of *praxeologies* (Chevallard, 1994), with its dual but complementary nature of *praxis* and *logos*. *Praxis block* consists of a *type of tasks* (T) and the *techniques* (τ) that are mobilised for a specific type of tasks (T), while *technology* θ and *theory* Θ form a *logos block* and justify the techniques used in solving the task. Consequently, the knowledge at stake in our study is modelled in terms of praxeologies.

Another ATD theoretical construct that is mobilised in this study is the theory of *didactic transposition* (Chevallard, 1985), which models the institutional relativity of knowledge. In this study, we take as an object of study, the so-called, *taught knowledge* which refers to “the concrete practices and bodies of knowledge proposed to be learned at school” (Chevallard & Bosch, 2020, p. 214).

Finally, the third theoretical device that we mobilise is the *levels of didactic co-determinacy* (Chevallard, 2002), which enable us to model the ecology of didactical systems. In our study, we aim to identify the extent to which changes in taught knowledge occur at the upper (Societies ↔ Schools ↔ Pedagogies ↔) or lower (↔ Disciplines ↔ Domains ↔ Sectors ↔ Themes ↔ Topics) levels of the scale.

It is important to highlight that even if the notion of competence and learning outcome is widely used in the EHEA documents regarding the Bologna Process we do not assume them as an object of study: according to Gascón (2011), most of the competence-based models assume a dual conception of mathematics with a clear separation between procedural and declarative knowledge which is clearly contradictory with the praxeological model.

Our research questions are:

RQ1: To what extent is there evidence of changes in the taught knowledge in mathematics courses for electrical engineering following the implementation of the Bologna Process and the adoption of competence-based curricula?

RQ2: What are the similarities and differences in the changes that occurred in Croatia and Spain?

To model the evolution of taught knowledge in engineering schools in Croatia and Spain we analysed exams for Electrical Engineering Degrees from Croatian and Spanish engineering schools. Two exam samples were selected for each mathematics course (excluding statistics) from both pre-Bologna and post-Bologna periods. Through praxeological analysis, we identified over 200 types of tasks and their respective subtasks, coding them based on mathematics content (domains) as determined by course syllabi. The results of the praxeological analysis, which constitutes a qualitative analysis, formed the basis for subsequent quantitative analysis. Using the statistical software R, a quantitative analysis was performed on the coded types of tasks and subtasks, comparing the exams based on syllabi (pre-Bologna and post-Bologna) and by country. By integrating the insights gained from the praxeological analysis, a systematic examination was conducted on the differences in the content and structure of mathematics exams across different time periods and educational systems.

RESULTS AND DISCUSSION

Impact of higher levels of didactic co-determination on (taught) mathematics

Croatia signed the Bologna Declaration in 2001, leading to the enrolment of the first generation of students in study programs aligned with the Declaration's objectives in the academic year 2005/2006. The period, from the Declaration's signing to the formal integration of its directives, marks a transitional phase characterised by the phasing out of the old higher education system and the gradual implementation of the Declaration's principles. The institution in Croatia, under observation for changes induced by the Bologna Process in this study, educates engineers in five fields, including electrical engineering. The study of electrical engineering, which in the pre-Bologna period lasted eight semesters (excluding the thesis) as a one-cycle program, with the implementation of the Bologna process was reconstructed into a bachelor's program lasting six semesters and a master's program lasting four semesters. Both in the one-cycle program and in the undergraduate program the fundamental mathematics courses (Mathematics I – MI, Mathematics II – MII, and Mathematics III – MIII) are distributed in the first three semesters of those studies.

Even if Spain signed the Bologna declaration two years before Croatia, in 1999, the implementation of the new degrees did not start until the academic year 2008-2009. Another important difference with the Croatian context is that Spain opted to implement engineering degrees of eight semesters and master's degrees of two or four semesters while the pre-Bologna degrees lasted for six semesters. In both countries, one semester is equal to 30 ECTS credits.

For the present study, we have selected the exams of the mathematics courses for both pre-Bologna and post-Bologna degrees of an engineering school in Barcelona offering five engineering degrees. The pre-Bologna degree comprised three mathematics courses (Calculus - C, Algebra - A, and Mathematics - M), while the post-Bologna degree includes only two mathematics courses (Mathematics and Calculus).

Country	Pre-Bologna mathematics courses		Post-Bologna mathematics courses	
	Course code, hours/week, semester	Mathematical content	Course code, hours/week, semester, ECTS	Mathematical content
Croatia	MI, 4+4, 1st	N, LA, AG, DL, SE	MI, 3+3, 1st, 7	N, LA, AG, DL, SE
	MII, 4+4, 2nd	SIN, MDC, MIN, LIN, SUIN, VA	MII, 3+3, 2nd, 7	SIN, MDC, MIN, DE
	MIII, 4+4, 3rd	FA, DE, CA, LT	MIII, 2+2, 3rd, 5	VA, LIN, SUIN, FA, LT
Spain	C, 4, 1st	SE, DL, SIN, MIN, MDC	M, 5, 1st, 7	N, LA, SE, VA, SIN
	A, 3, 1st	N, LA, AG, VA	C, 5, 2nd, 8	DL, SIN, MDC, MIN, SUIN, DE, FA, LT
	M, 4, 2nd	DE, FA, LT		

Table 1: Comparison of courses across countries and in relation to the implementation of the Bologna Process

By analysing the syllabi of courses from both countries from the pre-Bologna and post-Bologna period, mathematical contents were identified and grouped into the following domains: numbers (N), linear algebra (LA), analytical geometry (AG), differential calculus of one-variable functions (DL), sequences and series (SE), integral calculus of one-variable functions (SIN), multivariable differential calculus (MDC), multiple integrals (MIN), differential equations (DE), vector analysis (VA), line integrals (LIN), surface integrals (SUIN), complex analysis (CA), Fourier analysis (FA), and Laplace transform (LT). The distribution of content across courses is provided in Table 1. Such a selection of knowledge is in line with the outcomes prescribed for the first cycle of engineering education (OECD, 2011), with the evident observation that in both countries, a larger proportion of knowledge to be taught is transposed from the domain of mathematical analysis rather than from the domain of algebra (as a part of the discipline of mathematics).

One of the conditions established by the Bologna Process, by the quality teaching systems, and that we can place at the societal level, is the establishment of quotas and study success rates as quality indicators. This requirement, as well as other requirements of the Bologna Process, induced further conditions for pedagogies in educational institutions, particularly in the manner of knowledge assessment.

In the pre-Bologna Croatian institution, exams consisted of both written and oral components, requiring a minimum of 50% on the written exam and subsequently passing the oral exam for successful completion. The written exam consisted of problems that not only tested the praxis blocks of the student's praxeologies, as has been the case since the introduction of the Bologna Process but also the logos blocks (solving the task without providing the deductive arguments was graded with zero points). With the implementation of the Bologna Process, oral exams were eliminated, and theoretical knowledge is now assessed through written means. The possibility of passing the course by parts is introduced through midterm exams, final exams, and remedial colloquiums. In the pre-Bologna Spanish institution, exams consisted of written theory and exercise's part, requiring a pass on the theory part to be able to pass the exercise part. The oral component of the exams was never imparted in the Spanish institution of reference or was traditional in Spanish higher education. With the implementation of the Bologna Process, the written theory part was eliminated and only exercise type of tasks remained in the exams. Like the Croatian case, passing the course by parts the curriculum is imposed through midterm exams and final exams, together with the introduction of other evaluative tasks along the course.

Passing the course in parts is a condition to be considered when interpreting the results of statistical analysis. For example, in Croatia, pre-Bologna course material was examined through a single exam with an average of five tasks, but with the introduction of evaluation of the course by parts, the cumulative number of tasks used to examine the material increased (8-14). The exam structure also influenced the complexity of the tasks. For instance, in the pre-Bologna period, tasks on multiple integrals rarely appeared as standalone types of tasks. Instead, this knowledge was assessed through types of tasks related to surface integrals, of which they were subtasks. For the pre-Bologna period in both countries, a significant reduction in the number of subtasks per type of tasks has been observed. The number of subtasks for a given task indicates the complexity of the observed task. In Table 2, there are two tasks of the same type along with corresponding subtasks. To solve a task of type T_i from the pre-Bologna period, it is necessary to solve 9 subtasks, of which 8 are of different types; whereas for the task of the same type T_i from the post-Bologna period, it is necessary to solve 4 subtasks, of which 3 are of different types.

<table>
<tr><td>Type of tasks T_i:</td><td colspan="2">Solve the matrix equation.</td></tr>
<tr><td>Period:</td><td>Pre-Bologna</td><td>Post-Bologna</td></tr>
<tr><td>Task of the type T_i:</td><td>Let $A = \begin{bmatrix} \alpha & \alpha+1 & \alpha-1 \\ \alpha+1 & \alpha-1 & \alpha \\ \alpha-1 & \alpha & \alpha+1 \end{bmatrix}$,
$B = \begin{bmatrix} \sin 2\alpha & -\cos 2\alpha & 1 \\ sin\alpha & -\cos\alpha & \cos\alpha \\ cos\alpha & \sin\alpha & \sin\alpha \end{bmatrix}$, and
$C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be matrices. Solve the matrix equation $(B-2I)^{-1}X = C - AX$, for a parameter α for which matrices A and B are equivalent</td><td>Let $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$,
$B = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, and
$C = \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix}$ be matrices.
Find A^{-1} and B^{-1}, and solve matrix equation $AXB = C$.</td></tr>
<tr><td>Subtasks:</td><td>T_{i_1}: determine the rank of the matrix depending on the parameter. T_{i_2}: calculate the determinant of the matrix. T_{i_3}: apply the characterisation of regular matrices.
T_{i_4}: apply the characterisation of equivalent matrices. T_{i_5}: apply properties of matrix operations (additive inverse, distributivity of multiplication over addition, multiplicative inverse, non-commutativity of multiplication). T_{i_6}: find the inverse of the matrix. T_{i_7}: add matrices. T_{i_6}: find the inverse of the matrix. T_{i_8}: multiply matrices.</td><td>T_{i_6}: find the inverse of the matrix (using the formula for 2x2 matrices).
T_{i_6}: find the inverse of the matrix (using the formula for 2x2 matrices).
T_{i_5}: apply properties of matrix operations (non-commutativity of multiplication).
T_{i_8}: multiply matrices.</td></tr>
</table>

Table 2: Analysis of examples of LA exam tasks pre-Bologna and post-Bologna in Croatia in terms of subtasks

Lower levels of didactic co-determination

The complexity of the taught knowledge also depends on instances of mathematical objects, for which specific type of tasks needs to be performed. In task of type T_i from the pre-Bologna period (Table 2), matrices A and B depend on a parameter, while in task from the post-Bologna period, the inverses of given matrices can be calculated directly using a formula for the inverse of a second-order matrix. The types of tasks for the analysis and sketching of function graph are some of the most frequent types of tasks in both periods. Pre-Bologna, these tasks were implemented for various classes and instances of functions[1], requiring the mobilisation of different techniques to solve the subtasks (e.g., solving different types of equations to determine the roots and stationary points of a function). Post-Bologna, there is a reduction of the classes of

[1] Examples: $f(x) = (\ln x + 8)e^{\frac{x-1}{x^2-4}}$, $f(x) = 2arctgx + arcsin\left(\frac{2x}{1+x^2}\right)$, $f(x) = \left(\frac{x+1}{x-2}\right)^{\frac{x+1}{x-2}}$, $f(x) = \tanh((x+4)e^{\frac{1}{x+2}})$.

functions (rational functions most commonly appear), as well as the complete elimination of types of tasks related to the examination of injectivity, surjectivity, continuity, differentiability, and smoothness of functions.

Type of tasks T_j:	Solve the first order differential equation	
Period:	Pre-Bologna	Post-Bologna
Task of the type T_j:	$xy' = y + \sqrt{x^2 - y^2}$	$y' + \dfrac{y}{x+1} = e^x$
Subtasks:	T_{j_1}: write the differential equation in canonical form. T_{j_2}: determine a suitable substitution. T_{j_3}: separate variables. T_{j_4}: solve the indefinite integral. T_{j_4}: solve the indefinite integral. T_{j_5}: return the substitution. T_{j_6}: express y as a function in terms of x.	T_{j_7}: apply the formula for solving a linear differential equation. T_{j_4}: solve the indefinite integral. T_{j_4}: solve the indefinite integral.

Table 3: Analysis of examples of DE exam tasks pre-Bologna and post-Bologna in Spain in terms of subtasks.

In pre-Bologna exams, the technique for solving tasks was rarely explicitly stated; instead, the choice of technique was part of the student's assessment. Table 3 illustrates how the differential equation in task from the post-Bologna period is given in canonical form, suggesting the technique, and reducing the number of subtasks required to solve the equation. As shown in Tables 2 and 3, post-Bologna tasks often entail the direct application of formulas, leading to a higher degree of *algorithmisation* of techniques compared to the pre-Bologna period.

The reduction in the diversity of types of tasks and subtasks indicates the fragmentation and atomisation of content transposed from domains of analysis and algebra after the introduction of the Bologna Process in Croatia and Spain. Topics related to linear operators (including eigenvalues and eigenspaces) have been marginalised in Spain since the post-Bologna period, and in Croatia, they have never been included in the taught knowledge. The marginalisation of analytical geometry can be observed in both countries. All of the aforementioned suggests a convergence of content between Croatia and Spain in the post-Bologna period.

The complex numbers tasks classified as types of tasks (e.g., calculating with complex numbers, solving the equation within the set of complex numbers) in Spanish exams, appear as subtasks of types of tasks (e.g., solving the system of complex equations, solving the system of complex inequalities) in Croatian exams. This could be a consequence of differences between the Croatian and Spanish secondary school curricula, i.e., the first Klein discontinuity. Namely, complex numbers are not part of the content in the Spanish secondary curriculum, whereas they are included in the

Croatian secondary curriculum. The content of complex analysis (continuity, differentiability, and integration of complex functions, Cauchy's integral formula, entire, and analytical functions) has been removed from the undergraduate program in Croatia after the introduction of the Bologna Process (another indicator of convergence of content between countries in the post-Bologna period). On the other hand, in the content related to Fourier analysis (especially Fourier transform[2]), which has been retained even after the introduction of the Bologna Process, the knowledge component may include complex functions. However, upon reviewing post-Bologna exams, it has been observed that complex functions do not appear as knowledge objects in the types of tasks related to the Fourier analysis. Considering the absence of content in complex analysis, which can be crucial for electrical engineering professionals (e.g., electromagnetics, signal processing), the question arises about second discontinuity in engineering education (Florensa et al., 2022), namely the transition from mathematics courses for engineers to engineering courses. This means that certain mathematical notions (e.g., poles for the transfer function in content about electronic filters in signal processing), which were not part of mathematics courses, need to be developed exclusively in engineering courses. Hochmuth and Peters (2020) investigated how the specialist course on system and signal theory complements and develops praxeologies acquired through introductory mathematical courses designed for engineers. The development of mathematical notions in engineering courses generally differs from the development of the same notions in mathematics courses. Therefore, the absence of content on complex analysis in mathematical courses may lead to phenomena not considered in this study.

CONCLUSIONS

The praxeological analysis of the types of tasks and subtasks of the mathematics exams, allowed us to detect certain phenomena referring to the complexity, frequency and focus of the types of tasks pre-Bologna and post-Bologna, both in Croatian and in Spanish institutions for university education of electrical engineers.

With regards to our first research question concerning the implementation of competence-based curriculums and the provoked changes in the taught knowledge in mathematics courses for electrical engineering, we observed a decrease in the number and variety of the types of subtasks that correspond to each type of tasks. This can be seen as an ampliation of the Florensa et al. (2013) results on the tendency of *algorithmisation* of the tasks in the post-Bologna period. The same conclusion arises from the analysis of the mathematical objects involved in certain types of tasks. Instances of mathematical objects undergo a significant simplification and reduction in variety, enabling the application of specific algorithms rather than the more complex process of justification and execution of different techniques seen in the pre-Bologna period. In ATD terminology, we can say that the praxis block gains more prominence

[2]The function $\hat{f}$ defined by $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda\xi} f(\xi)d\xi$ is called Fourier transform of the function f.

while the logos, which justifies the adequacy, limitations, and economy of the praxis block, tends to become more implicit.

In the pre-Bologna period in Croatia, the mathematical education of engineers was much closer to the model of education for professional mathematicians (Gascón & Nicolás, 2022) than was the case in Spain. With the introduction of Bologna, both countries experienced a reduction in content that could be justified by a focus on mathematics relevant to engineers. However, the results of our analysis show that it is not entirely clear what kind of mathematics is needed for engineers of this profile. The research suggests that it should include differential and integral calculus, emphasising types of tasks such as the application of differential calculus to analyse the graph of a function and the evaluation of integrals. Within the framework of mathematical courses for engineers, the purpose of differential and integral calculus is not to achieve competence in solving mathematical problems in engineering, as claimed in the post-Bologna course syllabi of both countries, but rather to solve types of tasks that involve more advanced mathematics (such as solving differential equations, surface integrals, etc.).

Gascón (2011) interprets the development of competencies by stating that: “The transfer of skills and knowledge between different contexts and their mobilisation in complex situations is essential for the development of competences.” [our translation] (Gascón, 2011, p. 14). However, we have not detected different engineering contexts and the mobilisation of such skills and knowledge that would be essential for engineering being fostered in the analysed mathematics courses. That goes in line with the second Klein’s discontinuity in engineering education and the transition between the outcomes, from “learning mathematics outside the engineering context” towards “applying mathematics within the engineering contexts”. Further research might corroborate our results in different contexts and analysing the mathematical object of study might give a more detailed view into the *algorithmisation* of the tasks.

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The university perspective on modelling: An exploratory study

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This paper explores the prevailing conception of mathematical modelling in university education through the lens of university textbooks. The study identifies two major conceptions of modelling: application-oriented modelling, emphasising models as tools for application, and integration-oriented modelling, emphasising interdisciplinary and recursive aspects of modelling. The diversity in textbooks reflects the evolving nature of the field and highlights the need for a more standardised discourse in mathematical modelling education. The findings indicate a lack of unified terminology and approach in teaching modelling, suggesting the need for further research to bridge the gap between mathematical and didactic perspectives, and to enhance the teaching and learning of this vital discipline.

Keywords: curricular and institutional issues concerning the teaching of mathematics at university level, didactic transposition, modelling, teaching and learning of specific topics in university mathematics, textbook.

INTRODUCTION

In recent years, there has been a significant rise in the emphasis on mathematical modelling within undergraduate university curricula. Bosch et al. (2021) present evidence for the growing centrality of mathematical modelling in university mathematics programmes, driven by *external didactic transposition* processes, by which scholarly knowledge is selected, adapted, and organised for the purpose of being taught in an educational institution. The authors identify two primary justifications for this trend: firstly, the curricular content across universities in various countries has begun to incorporate modelling as a fundamental competency, signalling a recognition of its value in both academic and professional realms. Secondly, the evolution of programme structures and the accounts provided by academic staff participating in curriculum development reveal a concerted effort to align educational objectives with societal and professional demands, which increasingly prioritise the application of mathematical concepts to complex, real-life scenarios. Complementing this perspective on the broader educational trends, Pepin et al. (2021) focus specifically on engineering education. They present studies that illustrate the positive outcomes resulting from the integration of mathematical modelling, thereby shedding light on its practical efficacy.

The growing recognition of mathematical modelling's significance is further evidenced by trends in scholarly publishing. Figure 1 illustrates the number of books in each of the subjects mathematical modelling, linear algebra, and calculus, published by Springer from 1989 to 2023 (the last two subjects chosen only for comparison). The dataset was compiled from searches on SpringerLink using the distinct keywords “mathematical modelling” or “mathematical modeling”, “linear algebra”, and “calculus.” From 1989 to 2008, the number of books on modelling shows a linear increase. Since 2009, this trend has shifted dramatically, evidenced by a rise from 1,716 books in the period 2014–2018 to 19,170 in the following period of 2019–2023. This represents an overall increase of approximately 1020% between the mentioned intervals. In contrast, linear algebra and calculus display a steadier climb over the same span. Although they too see a notable increase from 2009 to 2023, it does not match the exceptional growth seen in mathematical modelling, possibly due to its growing prominence in a wide range of disciplines.

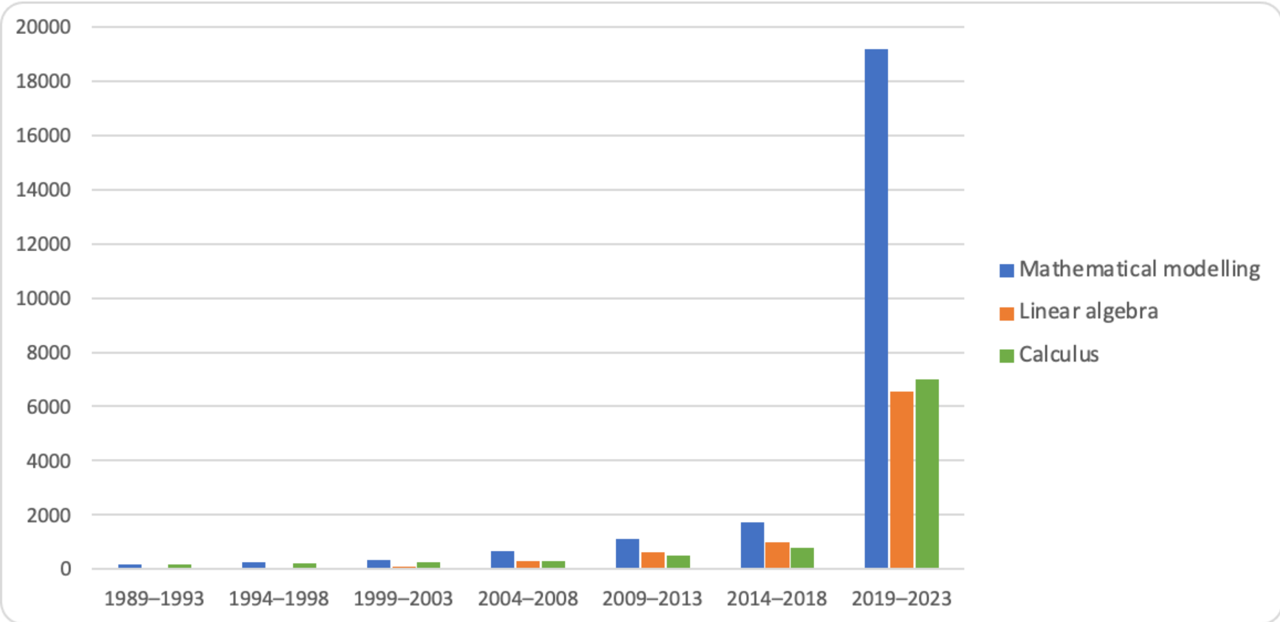

Figure 1. Springer books on mathematical modelling, linear algebra, and calculus

THEORETICAL TOOLS AND METHODICAL APPROACH

The research presented in this paper is developed within the Anthropological Theory of the Didactic (ATD). The notion of didactic transposition (Chevallard, 1985; Chevallard & Bosch, 2020) points to the fact that knowledge objects—like modelling—exist in different institutions where they are created, conceived, used, elaborated, made to evolve, etc., adopting different shapes. In every given institution, these objects get organized around tasks of a certain type, techniques to perform them, and discourses around those techniques—what is called *praxeologies* in the ATD. *Even* if the object is labelled under the same term (“modelling”), the way it is considered and used might differ from one institution to another. The process of didactic transposition refers to the changes made to an object from the institution that creates it—the *scholarly* institution—to the institution where it is taught and studied—the *educational* institution—, especially those elaborated to transform it into a body of “knowledge to be taught” (what is called the *external* didactic transposition). As stated by Bosch et al. (2021), the process of didactic transposition involves the interaction of three institutions: first, the scholarly institution of scientists who generate and utilise knowledge; second, there is the educational establishment, such as a university or school, where scholarly knowledge is transposed for teaching; and third, there is an intermediary layer of the noosphere, where knowledge is managed and organised within the educational institution (with members such as curriculum designers and policymakers). Within the context of university mathematics education, there is a significant overlap among these institutions, although their agents may occupy different positions in such institutions.

There are few studies to date about the didactic transposition of modelling, with some exceptions in secondary education (Cabassut & Ferrando, 2013; Jessen & Kjeldsen, 2021; Gjelstad, 2023). The aim of our research is to describe the prevailing conception of modelling at the university when it is proposed as an object to be taught. We are approaching it by considering the explicit discourse that has appeared in university textbooks about mathematical modelling in the past 20 years. Therefore, our research question can be stated as: What is the dominant conception of modelling at the university according to university textbooks? In other words, we are studying what textbooks about modelling tell us about the university conceptions of modelling and what diversity appears.

As suggested by Bosch and Gascón (2005), didactic transposition analysis requires researchers to elaborate their own conception of the object or body of knowledge at stake. In our case, we are adopting the vision of modelling introduced by Chevallard (1989) and recently described in Barquero (in press). The first elements are those of *system* and *model*, which must be understood as roles assumed by objects in a process of study: the system is the entity to be studied and the model is the tool used to produce knowledge about the system. Different steps intervene in the modelling process, namely the delimitation of the system, the construction of the model and the work within the model, and the work with the model to produce knowledge about the system. Defining systems and models more as roles than entities enables considering *recursive* modelling processes (when a model is considered as a system for further modelling processes) and *reversible* modelling (when what was considered as a system acts as a model of the model built to study it). Therefore, the dialectic between systems and models can be developed in multiple and complex ways, without reducing systems to extra-mathematical entities and including the consideration of intra-mathematical modelling processes.

The exploratory method used in the research is based on selecting a sample of nine university mathematical textbooks that include modelling in their title and searching within the information presented in the first chapters of the book (including the preface) how the authors propose answers to the following questions:

1. What is mathematical modelling?
2. How is the modelling process described?
3. How is the body of knowledge on modelling structured and what types of problems, models or modelling processes are distinguished?

4. What specificities about the teaching of modelling can be drawn from the textbooks?

The sample analysed in this study comprises nine textbooks on mathematical modelling, spanning from 2003 to 2020. These were selected based on their accessibility and relevance to the field. For ease of reference within the text, these books are referred to by a coding system as B1 through B9, arranged chronologically by their year of publication. The same coding is used in the reference list for quick identification. The results obtained by the analysis will lead to some general hypotheses to be tested through a more representative sample and other sources of information, like interviews with researchers in modelling, classroom teaching resources, etc. This second step of the research is still in progress.

RESULTS

We first note the diversity of discourses. This contrasts with the content derived from more stable mathematical fields such as “differential equations” (B1, p. ix), “Markov processes,” or “linear regression” (B1, p. xi) used in these books. This contrast makes it unclear whether we can talk about a mathematical field, which may be a symptom of its youth. Despite the explosive number of books titled “Mathematical modelling,” there is not a standardised structure of the knowledge organisation around modelling, nor a clear common discourse about what modelling is and which are its main elements. We can nonetheless note that they seem to all converge towards viewing modelling as a simplification process of the system considered, so that it can be “studied more cheaply and safely than in the real world” (B5, p.2). This simplification enables for a better understanding of the system allowing for “finding new insights that are impossible to obtain by other scientific methods” (B4, p. 2).

Definition of modelling

We can broadly categorise textbooks into two major categories based on their primary approach to modelling. However, some textbooks display characteristics of both categories. Our objective is not to rigidly “categorise” these books, but rather to systematically organise the different conceptions of modelling that are evident in them. The first category aligns with an *application-oriented* modelling approach. This approach predominantly centres on the mathematical models themselves, treating systems primarily as mediums for their application. Here, the emphasis is on applying these models to real-world problems, but there exists a certain independence between the mathematical and the extra-mathematical world. The exact fit of the model to the real-world system is not always a central concern, and there is typically no commitment to exploring the system once the modelling process is finished.

In this category, modelling processes are defined as “a bridge between the study of mathematics and the applications of mathematics to various fields” (B1, p. ix) or “the description of phenomena from nature, technology, or economy by means of mathematical structures” (B7, p. viii). The same authors comment that “with (mathematical) modeling we denote the translation of a specific problem from the natural sciences (experimental physics, chemistry, biology, geosciences) or the social sciences, or from technology, into a well-defined mathematical problem."

This is how B3 conceptualises the steps in a modelling process (p. 1):

> The process of developing a mathematical model involves several basic steps. The first step is to present the problem as simple as possible, for example by transformations of data. The second step is to use modeling concepts to derive various reasonable models. The third step is to evaluate the models obtained in order to identify the optimal model. ... The fourth step is to demonstrate the advantage of the model development by deriving valuable conclusions that are not directly given by the observations.

The second category of textbooks proposes what we can consider a more *integration-oriented* perspective on modelling. The process of model building is more developed, the interdisciplinary approach emphasised and the recursivity perspective between systems and models more present. In some cases, the focus extends beyond the system by analysing the scope of the models used.

For instance, the process of modelling includes identifying "the most important parts of the system" and determining "the amount of mathematical manipulation which is worthwhile" (B2, p. 1). Regarding the system, the same authors indicate that "it is important that all assumptions are stated clearly and concisely. This allows us to return to them later to assess their appropriateness" (B2, p. 5). Other authors emphasise the process of model building and how it can be differently elaborated:

> Being "adequate" sometimes suggests having a minimal level of quality, but in the context of modelling it describes equations that are good enough to provide sufficiently accurate predictions of the properties of interest in the system without being too difficult to evaluate. (B6, p. 7)

Description of the modelling process

The variety found in the definitions of modelling also apply in the elements of the modelling process the authors point at. Books more application-oriented tend to talk about the "translation" of problems into mathematical model, as if the models were already there. For instance, B9 lists the following steps:

1. Detect the aspects that are most relevant and meaningful;
2. Translate them into a mathematical model;
3. Compute the solution of the model;
4. Verify its soundness, by comparing the solution thus found to the observed (and quantified) phenomenon. (B9, p. 6)

Other authors consider the process in terms of "simplification" and "interpretation" in a specific modelling cycle, as shown in Figure 2.

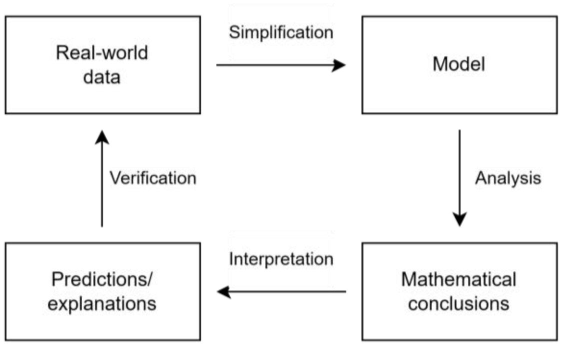

Figure 2. The modelling process as illustrated in B1 (p. 1)

As contrast, books with a more integrative conception of modelling are more explicit about the different steps of the modelling process:

> First, we must *identify the problem* that we are trying to solve by modelling. Next, it is important to understand *which building blocks have to be included in the model*. … Once the mathematical model has been built, it can be applied for *studying the modelled phenomenon*. ... The results gained from the simulation are then *compared to actual measurement data related* to the phenomenon. (B5, p. 1, our emphases)

Let us note here that terms like "identify the problem", "building blocks" or "simulation" are not often used by other authors. Some authors (B3, p. VII) also introduce specific terminology when putting a lot of emphasis on the "hierarchical structure of models" and their "range of applicability" that we can relate to the idea of model recursivity proposed by the ATD.

Types of modelling and structure of the body of knowledge

The diversity of the textbooks' seems to reveal that there is not a standard organisation of the modelling field or body of knowledge. Each author proposes an ad-hoc structure, sometimes based on the type of mathematical models, sometimes on the type of systems that are modelled. This variety can already be seen in the books' tables of contents, with examples shown in Table 1. We can see how B2 focuses on the modelling process, B3 on the types of models to build, B7 on the mathematical contents or domains, and B4 on the types of systems to be modelled:

B2	B3	B7	B4
I. Intro [Generalities about modelling] II. Building Models III. Studying Models IV. Testing	1. Deterministic Analysis Observations 2. Stochastic Analysis Observations 3. Deterministic States 4. Stochastic States 5. Deterministic Changes 6. Stochastic Changes	1. Modeling, Change and Simulations 2. Derivative and Integral 3. Equilibrium Behavior 4. Nonequilibrium	Steps of Mathematical modeling I. Mathematical Models in Economics II. Models in Ecology and

Models V. Using Models VI. Discussion [Description of a model, when to stop]	7. Deterministic Evolution 8. Stochastic Evolution 9. Deterministic Multivariable Evolution 10. Stochastic Multivariable Evolution	Dynamics: Oscillation 5. Chaos 6. Linear Algebra 7. Multivariable System	Environment III. Models of Economic Environmental Systems

Table 1. A panel of different tables of contents

For example, as we can see in the first column of B2, the content of the textbook is organised following the steps of the modelling process: each chapter corresponds to a different step, apart from the first one and the last one, which dwell on more general matters. As for B3, the organisation shows that the diverse aspects of the systems are what structure the content: there is an alternance of chapters dealing with deterministic and stochastic aspects, as well as an evolution towards the phenomena to be studied. In B7 the focus is more directed towards a separation of distinct mathematical contents, with chapter titles such as “derivative and integral” or “linear algebra.” In the case of B4, even if only the big sections are shown in Table 1, each chapter corresponds to one type of problem or system: for instance, modelling of technological change, models with air pollution propagation, or modelling non-renewable resources.

Furthermore, the model classification itself can be very diverse. Distinctions can be made between deterministic and stochastic as shown above from B2, but other types of distinctions are taken into account: some treats discrete and continuous models in different chapters, like B1 introducing first the models requiring only precalculus and then the models requiring calculus.

Teaching perspectives about modelling

It is clear that the authors’ perspective about modelling permeates the teaching strategy proposed. The epistemological and didactic proposals are closely related. Therefore, books more application-oriented propose teaching strategies where mathematical models must be studied prior to their use to analyse systems. Suggestions about teaching in B1 are particularly revealing to this respect, where the authors propose to put the “emphasis on using mathematics already known by the students” and let “the modeling course ... motivate students to study the more advanced courses” (B1, p. x). Here the study of mathematics is separated from its utility in modelling, and the construction of models is not part of the enterprise.

The authors who put the focus on the distinction between deterministic and stochastic methods will of course link the book structure to a teaching strategy: “The consideration of stochastic methods enables a comprehensive understanding, for example, of the basis of optimal deterministic models and how closed deterministic equations can be obtained" (B3, p. vii). However, the book is centred in the study of mathematical models and their use; the fact that a system is considered (or modelled as) deterministic or stochastic remains largely unquestioned (see Figure 3).

1. Determ. Analysis Observations *How can we develop models that describe the trend of observations?*	**2. Stoch. Analysis Observations** *How can we find optimal models that account for data randomness?*
3. Determ. States *What are characteristic properties of a deterministic variable?*	**4. Stoch. States** *What are characteristic properties of a random variable?*
5. Determ. Changes *What are characteristic changes of a deterministic variable?*	**6. Stoch. Changes** *What are characteristic changes of a random variable?*
7. Determ. Evolution *What are the laws of the evolution of one deterministic variable?*	**8. Stoch. Evolution** *What are the laws of the evolution of one random variable?*
9. Determ. Multivariate Evolution *What are the laws of the evolution of several deterministic variables?*	**10. Stoch. Multivariate Evolution** *What are the laws of the evolution of several random variables?*

Figure 3. B3's explication about the organisation of its contents (p. VII)

As for B5, the teaching strategy they suggest is nothing but an explicit description of their own conception of modelling:

> When teaching mathematical modelling, the main challenge is to familiarize students with the modelling process itself. It might start from a very imprecise set of questions riddled with an insufficiently defined terminology of the subject of study, and it ends with the created model, numerical solutions, the evaluation of the model's accuracy and the implementation of the results in the application field. Students of modelling should be encouraged to "get their hands dirty" by starting out with partial, experimental or tentative models. Also, students should be reminded that it is usually impossible (or at least hopeless) to find a perfect final solution when it comes to applications. We have to be satisfied with sufficiently good solutions. (B5, p. 3)

This looks like an enquiry-based process, starting with a question and then trying to build a model to answer the question. Students are encouraged to create the model and evaluate its efficiency, and possibly enhance it, by getting "their hands dirty."

DISCUSSION AND CONCLUSION

The diversity found in university textbooks on mathematical modelling reveals an underdeveloped didactic transposition process. Its main effect is the lack of standardised discourse and terminology regarding mathematical modelling, which might affect the mathematical and didactic resources available for teaching at the university level and also at secondary schools. In other words, the *logos* component of the "modelling praxeologies" that appear in the textbooks is highly author-dependent, revealing a weak institutionalised process in the community of scholars. This phenomenon may be attributed to the relative youth of the field.

In our analysis, we discerned two primary categories: application-oriented and integration-oriented approaches. The application-oriented approach focuses on mathematical models as tools for application, often with less emphasis on the adequacy of the model to the system. Textbooks in this category propose strategies where mathematical models must be studied prior to their use in analysing systems. Conversely, the integration-oriented approach delves deeper into the model-building process, emphasising the interdisciplinary and recursive nature of modelling. Textbooks with this approach suggest a process that begins with a question and then builds a model to answer it. This approach encourages students to engage in creating and evaluating models, a process resonating with the didactic principles of the ATD.

The disparities between these approaches underscore the need for an ecological understanding of mathematical education, as suggested by Barquero (in press), where institutional and curricular factors critically influence the teaching and learning of modelling. The recursive nature of models, central to the integration-oriented approach, aligns with Barquero's emphasis on the epistemological dimension that considers the adaptability and refinement of models in the educational process. Further research should investigate how academic conceptions of modelling impact teacher education and secondary school instruction, a component of the didactic transposition process. Our role as teacher educators has highlighted a disconnect between the mathematical and didactic perspectives on modelling. Addressing this gap is an aim of our ongoing research.

In conclusion, the varying conceptions and approaches to mathematical modelling in university textbooks reflect the evolving nature of the field. This diversity presents challenges in formulating a unified and comprehensive educational strategy. Therefore, establishing a more standardised discourse on mathematical modelling is imperative to enhance the teaching and learning (i.e., didactic praxeologies) in this vital discipline.

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A pilot *study and research path* in statistics at the university level

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This paper reports on a university teaching experience in statistics based on a study and research path (SRP), an inquiry-based instructional proposal elaborated within the Anthropological Theory of the Didactic (ATD). The SRP corresponds to a pilot case for an ongoing dissemination project aiming at implementing instructional proposals close to the paradigm of questioning the world in the university context. Its description illustrates the ATD characterisation of SRPs in terms of the Herbartian schema and its dialectics. It particularly focuses on the didactic infrastructure created by the lecturer and the elements of the instructional strategy that could be disseminated to other lecturers in other university settings.

Keywords: study and research paths, Herbartian schema, dialectics, inquiry-based teaching, statistics education.

INTRODUCTION

Nowadays, university education receives a lot of pressure to shift from a teacher-centred pedagogy based on the frontal study of pieces of knowledge towards more student-centred and inquiry-based approaches to teaching and learning. However, pedagogies of the latter type require the implementation of conditions that are sometimes difficult to reach (Markulin et al., 2021). As a result, a lot of inquiry-based teaching proposals do not overcome the experimental stage. When implemented under different institutional conditions, they tend to vanish or take on forms more consistent with the old "frontal" pedagogy (Markulin et al., 2022).

This issue may be characterised in the framework of the Anthropological Theory of the Didactic (ATD) (Chevallard & Bosch, 2020), which puts it in terms of a shift in paradigm from the *paradigm of visiting works* to the emerging *paradigm of questioning the world* (Chevallard, 2015). Theorizing on the latter paradigm has allowed ATD-researchers to develop the pedagogy of *study and research paths* (SRPs) (Bosch, 2018; Chevallard, 2015). The implementation and analysis of SRPs in different institutional contexts brought important local results about the conditions needed for SRPs to be integrated into educational institutions and the constraints hindering their development and dissemination (Barquero et al., 2022). For the conditions needed, three levels may be distinguished (Markulin et al., 2022): the epistemological related to the structure and organisation of the content that is to be taught and learnt, the didactic one related to the way this content is managed during the teaching and learning processes, and the pedagogical related to the strategies and devices that are general to different content, domains, and disciplines. In this paper, we focus on the didactic level.

LABINQUIRY (Lombard et al., this issue) is a project to transfer research results about SRPs to the secondary school and university levels. It aims at providing the necessary didactic infrastructure for the design and implementation of SRPs. It consists of the development of a website as well as a set of online didactic modules adaptable to learning platforms (such as Moodle and Google Classroom), together with the creation of a community of lecturers all implementing SRPs. LABINQUIRY will feature prototypical SRPs with their potential generating question, a priori analysis, collections of structured resources and analyses from previous experiences, links to external resources and potential experts, etc. In this context, the LABINQUIRY research team selected two SRPs that have been developed and tested throughout the years, as pilot studies for the transfer procedure. The first one is about combinatorics implemented many times at secondary schools (Vásquez et al., 2021) and the second one is an SRP in statistics for bachelor's degrees in chemistry, chemical engineering, and industrial technologies engineering (Fernández-Ruano et al., 2024).

In the present paper, we report on the university SRP, which is taught by the second author of this paper, and whom we will call P_1 in the following. The SRP taught by P_1 was implemented over the last four years adopting different forms. Each implementation is analysed and used to improve the next one. Therefore, the last version of the SRP is considered to be the greatest adapted to its institutional setting. In the following, we will describe how we observed it to give it the role of a pilot for the LABINQUIRY project. More precisely: *What didactic infrastructure is created for the SRP and how is it used or activated by the lecturer and the students in class?*

THEORETICAL FRAMEWORK

SRPs are long teaching and learning processes that start with the consideration of an open-generating question students address under the guidance of the teachers. The generating question is expected to be open enough to require the search for information from different sources and the study of this information to elaborate and validate a final answer, obtained collectively by the students under the guidance of the teacher(s). An important aspect of SRPs is the fact that the question approached should always remain the main goal of the inquiry, instead of being addressed as a pretext to introduce new concepts, knowledge organisations or tools.

SRPs can be described through the so-called *Herbartian schema*: $S(X; Y; Q_0) \rightarrow A^{\heartsuit}$, where a group of students X, helped by a group of teachers Y, form a didactic system S to address an initial question Q_0 and provide a final answer $A^{\heartsuit}$. In the process from Q_0 to the collective elaboration of $A^{\heartsuit}$, the *didactic system* $S(X; Y; Q_0)$ displays Q_0 into derived questions Q_i, searches already available "labelled" answers $A_j^{\diamond}$, elaborates and adapts them to Q_i, finds new questions during the process which, in turn, call for new answers, and so on. Bosch (2018) pointed out the importance of the *questions and answers* (*Q-A*) *dialectic* to ensure the dynamics of SRPs. The Q-A dialectic provides visible proof of the progress of the inquiry and contributes to the overall process management. To elaborate $A^{\heartsuit}$, the didactic system creates a didactic *milieu M*: $[S(X; Y; Q) \rightarrow M] \hookrightarrow A^{\heartsuit}$. This *milieu* is composed of derived questions Q_i, “ready-made” answers $A^{\lozenge}_j$ that seem helpful to answer Q_i, any kind of works W_k (knowledge or material), and the sets of data D_m of all natures gathered during the inquiry. The extended *Herbartian schema* is symbolised as:

$$[S(X; Y; Q) \rightarrow \{Q_1, Q_2, ..., Q_i, A^{\lozenge}_1, A^{\lozenge}_2, ..., A^{\lozenge}_j, W_1, W_2, ..., W_k, D_1, D_2, ..., D_m\}] \hookrightarrow A^{\heartsuit}$$

The *media-milieu* (*Me-Mi*) *dialectics* becomes crucial during the whole SRP. To analyse this dialectic, we look at where external information, data and answers come from, and how their access is managed (*media*). We also ask how they are validated and transformed; and with what materials the final or intermediate answers are developed (*milieu*). Finally, an SRP is a collective inquiry process during which small groups X_i are formed and individual work is also carried out. X_i and Y_j should organize themselves to work together. Hence a necessary share of responsibilities must be constantly established, as to what questions should be studied, what strategy should the class (as a group) adopt, what answers are considered valid, and so forth.

So, when analysing an SRP, one may pay close attention to three principal aspects (Barquero & Bosch, 2015). First, the *chronogenesis* of how the teacher monitors the questions-answers dialectic; then the *mesogenesis* of how the teacher stages the media-milieu dialectic. Finally, the *topogenesis* corresponds to the position (*topos*) assumed by the teacher and the students and their sharing of responsibilities. In this paper, we focus on the analysis of these three aspects in the pilot SRP.

METHODOLOGY TO DESCRIBE THE PILOT SRP

This paper presents the analysis of the last version of the SRP (course 2023-2024), out of four implementations. The SRP is carried out in the first statistics subject of a bachelor’s degree in chemical engineering, a subject taking place in the first semester of the second year. It corresponds to 6 ECTS credits and students have 39 hours of classes together. The first and last authors of this paper conducted non-interference observations of the SRP’s sessions and in some of the lectures to observe the P_1’s management and the students’ activity. Then, they interviewed P_1 to shed light on aspects of the preparation and the management of the SRP we could not grasp through our sole observations. We also asked P_1 about the changes made since her first edition of the SRP (2020-2021) until the last version in progress.

The data gathered consisted of our written records of the non-interference observations in the SRP sessions, the transcribed interview with P_1 and the materials made available by P_1 on the SRPs previously implemented. With these data, we particularly focus on P_1’s strategy to manage the SRP’s *chronogenesis, mesogenesis*, and *topogenesis*. So, we described these three aspects in terms of the Herbartian schema, the Q-A and the media-milieu dialectics. We also described the general organization of the SRP and the role of an external instance as a possible element in promoting changes in the didactic contract in the current SRP. These two moments were articulated with P_1’s *topos*, which has an essential position in both the running of the SRP and the development of the LABINQUIRY project.

These pieces of evidence will enable the dissemination procedure of the SRP didactic infrastructure as a "model device" that will be available in the LABINQUIRY project to lecturers who want to develop it in their practice.

THE GENESIS OF THE SRP AND ITS INTEGRATION INTO THE SUBJECT

P_1 is a lecturer with a long experience and who in recent years dedicated her teaching to topics of statistics in chemical engineering courses in a private university in Barcelona. In 2020, she learnt about the SRP device through an ATD researcher and began to design and implement it in her statistics courses.

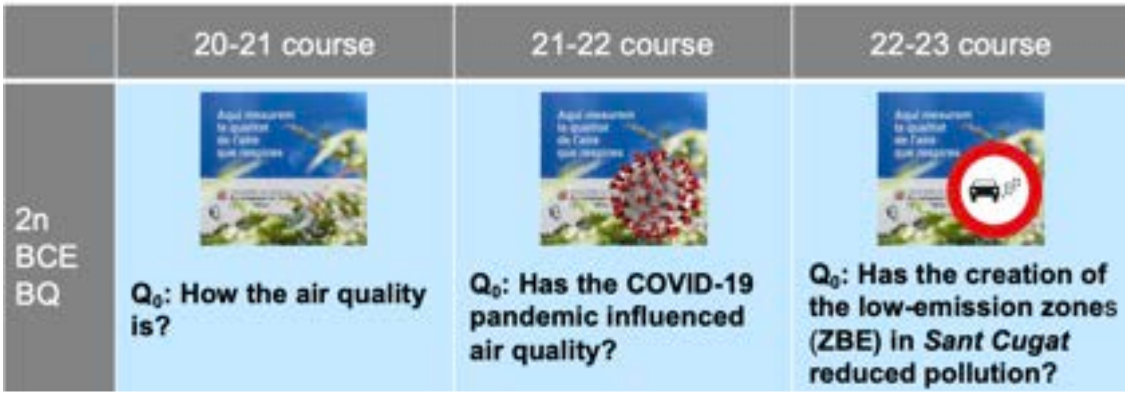

Figure 1: Generative questions of the previous editions of the SRP designed by *P_1*

In the first edition (2020-2021), P_1 proposed an SRP as an extra class work. Previously, a data set about the concentration of air contaminants in Catalonia was found by her, and she aimed her students to work on these data. She gave the students the data set and requested them, organized by groups, to raise questions about the air quality and try to give an answer after organizing, representing, and analysing the data (Fernández-Ruano et al., 2024). In the second edition (2021-2022), the students had to work with the same data set but this time the SRP was part of the statistics subject, and P_1 organised the sessions to introduce statistics knowledge and sessions dedicated to the development of the SRP. In the third edition (2022-2023), P_1 structured the SRP to run in parallel to the lectures. All the generating questions are presented in Figure 1. In all these editions, the common point was the questioning about the air quality and the accessibility of a data set regularly updated by the Catalan government about the information collected at several observation points in the territory. Evolutions over time appear because each edition of the SRP provided elements of the *a priori analysis* of the next one leading P_1 to make changes not only to the generating question but also to the didactic infrastructure and its management.

In the last edition in 2023-2024, P_1 implemented the SRP in the subject of statistics for a second-year bachelor's degree in chemical engineering. This subject focuses on the fundamental statistical principles of the field of chemical engineering. Its syllabus highlighted the development of abilities to: recognize, create, and resolve chemical engineering problems that call for the application of statistical approaches; disseminate knowledge, concepts, issues, and solutions to both specialized and general audiences; and solve computational issues and statistical analysis. The general organisation of the entire subject is presented in Table 1. It is important to highlight the relationship between the statistics lectures and the SRP and how this last one gives a raison d'être to the subject of statistics, in line with the proposal by Barquero et al. (2018) about linking transmission with inquiry at the university level.

Week	Brief description of the lectures	Brief description of the SRP
1	Presentation of the subject: What is statistics? Objectives of statistics. Uncertainty of the data and core idea.	P0. Presentation of Q_0 by the external instance. P1a. Viewing the dataset about air contaminants. *Autonomous work: Organization of the groups. Derived questions raised by groups in the diary.*
2	Exploratory data analysis. Construction and interpretation of statistics, tables, and charts.	P1b. Organizing the derived questions and discussion to select the relevant ones. *Autonomous work: search for "ready-made" answers $A^{\Diamond}_j$ in different media; report in the diary.*
3	Distributions: from sample to population. Definitions and models of frequent use.	P2. Revision of answers $A^{\Diamond}_j$ and new questions. *Autonomous work: continue to search for information on the assigned questions, and report in the shared document and in the team diary.*
4	Inference: introduction and parameter estimation. Inference techniques. Sample distributions. Interval estimation of the mean and variance. Sample size.	P2. General discussion of the selection of variables and first organization of data. *Autonomous work: continue to search for information on the assigned questions, and report in the shared document and in the team diary.*
5	Hypothesis testing for one and two samples. Parametric and non-parametric tests.	Bank holiday (no work in the classroom). *Autonomous work: Submission of pre-report 1 by teams (introduction and first data representation).*
6	Hypothesis testing for one and two samples. Parametric and non-parametric tests.	P2. Review of pre-report 1: general discussion about agreements and pending issues *Autonomous work: Integrate the feedback provided and fill in the team diary.*
7	Hypothesis testing for one and two samples. Parametric and non-parametric tests.	P3. Representation and description of the organized data. *Autonomous work: Propose questions to the external instance. Submit 2nd pre-report.*
8	Frequency tables. X^2 test for independence. Goodness of fit test.	Partial exam.
9	Analysis of Variance. Sources of variation. One-way ANOVA.	P3. Final data selection: general agreements. *Autonomous work: teams make the assigned basic graphs and one of them unifies the graphs.*
10	Analysis of Variance. Sources of variation. One-way ANOVA.	P4. Teams work on internal answers: graphs and data description, inferential analysis, conclusions. *Autonomous work: Groups work on pre-report 3 and on part of the final report.*
11	Analysis of Variance. Post-hoc analysis. Two-way ANOVA. Non-	P4. Preparing posters.

	parametric models. Introduction to Design of experiments.	
12	Non-parametric models. Introduction to Design of experiments. Regression models.	P4 & P5. Presentation of internal answers (posters).
13	Regression models. Correlation as a measure of linearity. Least-squares method. Diagnostic graphics. Adequacy of the regression. Interpolation.	Bank holiday.
14	Regression models. Least-squares method. Diagnostic. Adequacy of the regression. Interpolation.	P5. Presentation of the final answer to the external instance.
15	Finalization of the subject. Final exam.	

Table 1: General organisation of the lectures and SRP sessions by weeks

In the previous editions, P_1 recorded the progress of the SRP at each session, but without the need for a well-defined terminology to demarcate the different inquiry stages. In this latest edition, it appeared the need to co-create a didactic infrastructure that will later be made available to other teachers through LABINQUIRY. We then saw the need to distinguish between the different stages of the inquiry that can occur during the implementation of the SRP. These stages are distinguished in terms of phases (P0, P1a, P1b, P2, ..., P5) and have been highlighted in Table 1.

THE PILOT SRP AND ITS MANAGEMENT

Chronogenesis

The generating question of the SRP is Q_0*: Are the low emission zones (ZBE) correctly dimensioned and to what extent do their dimension affect their efficiency?* During phases P0-P2, the students raised derived questions, grouped the questions (location, legislation, temporal, pollutant, and working with data), and discussed and selected the relevant ones for the study. They then selected the data to work during phases 2-4. The derived questions studied during the different phases were:

Phases 0-2

Q1.1: Which areas of Catalonia are ZBE, where exactly are they located?

Q1.2: In the pollutant emission zones, are emissions measured in 2D or 3D? How is the extent and geographical radius of pollution defined?

Q1.3: Has there really been a decrease in the level of pollutants in the established ZBE?

Q1.4: Should all settlements with more than 5000 inhabitants be taken into account?

Q1.5: What percentage of zones are ZBE compared to the total number of zones in Catalonia?

Q1.6: Which population to select? Which stations should be selected?

Q2: Since when do the ZBE function (are sanctions imposed)?

Q3: What is the optimal time period to evaluate the effectiveness of the ZBE and should several years of data be considered?

Q3.1: Which hours, days, months, and years to select? How to select the data to obtain this information?
Q4: Which are the most important parameters (NOx, SOx, microparticles, ...) which affect the delimitation of the ZBE?
Q4.1: Are polluting gases differentiated by the degree of pollution they produce or are they treated equally?
Q4.2: Which pollutants are related to traffic?
Q4.3: Does meteorology affect the mobilisation of pollution?
Q4.4: Which pollutants to select?
Q4.5: What are the differences between primary and secondary pollutants?
Phases 3-4
Q5: How do you start working with the data provided at the statistical level?
Q5.1: How do you measure effectiveness?
Q5.2: What criteria will we use to choose the data? How should the data in the table and its analysis be related to the initial question?
Q5.3: How are the data taken from the available database?
Q5.4: How often are data on air emissions collected?
Q5.5: What variables are to be considered?
Q5.6: What would be the next step to move forward with the work?

During almost every session, we could see P_1 arranging some time to discuss a map with the questions addressed during the SRP with the students. Especially, she repeatedly asked about the status of the proposed answers to each question: whether they should be taken for granted, be further investigated (rather trusting or rather disproving them) or be dismissed. Then, the decision was on the students' side (see below). An important aspect regarding the animation of the dialectic of questions and answers was also the availability of data, which is a critical point in statistics.

Mesogenesis

During all the phases of the SRP development, students were guided by P_1 to consult numerous types of media. At the same time, the data accessed provided a rich *milieu* for the evolution of the inquiry. The resources available mainly consisted of existing $A^{\lozenge}_j$ answers found on the Internet and incorporated by the groups into their *milieu*, D_m data on air pollutants made available by P_1 (which, in fact, is the database used in all previous editions of this SRP), a summary of the derived questions grouped and selected collectively, and knowledge tools W_k provided by P_1 throughout her classes in statistics. Once again, the relationship between the statistics lectures and the development of the SRP is highlighted when P_1, in her lectures, provides students with the study of data in different contexts and indications of how to work with *Excel* for the organisation, representation and analysis of these data. Therefore, the new knowledge necessary to carry out the SRP is progressively introduced by her during the lectures.

In general, the validation of the answers is done through collective discussions about the answers elaborated by the groups. P_1 organises SRP sessions to work in groups (for the elaboration of partial answers) so that collective discussion sessions may happen afterwards to make decisions on the next steps to follow. The following excerpt can illustrate this type of discussion:

> P_1: The other day I asked x_1 "Are you not going to discuss anything about this? Because he had been saying for three or two months that "meteorology had to be taken into account"... He wrote an impressive report on why it had to be taken into account and when the day came to ask "shall we eliminate it?", he kept quiet and I said "x_1, now you're not going to tell anything about this, you're not going to discuss it?", and he said "no".

All the groups' productions are presented in diaries and pre-reports. Many of the productions are shared in cloud storage files, a collective means created by P_1 so that everyone has access to the groups' productions. So, P_1 facilitates working sessions in a variety of places. This expansion goes with a clear increase in the possible milieus of study, some being directly animated by the lecturer, others not. This is how she describes the infrastructure used:

> P_1: This year, the crucial difference is that we are combining work in groups in the tutorial room, with collective work in the lecture room, together with homework. So we use these three spaces. […]

Topogenesis

An important aspect of the topogenesis, besides the new responsibilities students must assume (raising questions, searching for information, elaborating answers, etc.), is related to the use of students' intermediate answers. Every week, P_1 gathered the answers $A_i^\heartsuit$ provided by the groups in their reports. These answers then became new pieces of work $A_i^\lozenge$ for the general inquiry. We see here how the questions and answers dialectic is entangled with media-milieu dialectic, as students are considered (first by P_1 but also by themselves) as legitimate media whose productions deserve to be discussed. This enforces their topos, because the students' results are not only intended to the lecturer but to the entire inquiry community.

Another important reinforcement of the topogenesis has been produced by the intervention of an external agent to propose the generating question. In this edition, it was a representative of the Environmental Department of the Government of Catalonia who was invited to present the generating question to the students as a request of the traffic section he heads. In the three previous editions, the generating question was presented by P_1 as a work that was part of the statistics subject. The students took it as an exercise and did not take the study so seriously. In this latest edition, a big change is noticeable because the final answer is no longer a simple exercise, but a response to be presented to a real request. P_1 made it clear that the importance of the external agent was in "the responsibility" transferred to the students. Of course, this also changed the final product of the inquiry $A^\heartsuit$:

P_1: Somehow we have to reach a consensus, we have to give an answer and this, for me, has opened up the world. You know they see that they must give an answer to a question that is not an exercise, that changes things.

In fact, this change did not only affect the nature of the expected answer but also the level of expectation. That is, such an intervention manifestly enforced the necessity to answer properly, meaning intelligibly and argumentatively.

CONCLUSION

This paper presents the methodology used to characterise an SRP that has been implemented during four consecutive years and acquired a certain maturity so as to be used as a pilot inquiry process for a dissemination project. The main elements used are the Herbartian schema and the dialectics of questions-answers (chronogenesis), media-milieu (mesogenesis) and individual-collective (topogenesis). The results obtained by the a priori and in vivo observation and analysis of the inquiry process point at a coherent and self-sustained instructional practice, rooted in a rather stable didactic infrastructure. Some relevant aspects to highlight affect the three dialectics. The strategy to manage the chronogenesis alternates sessions of teamwork and collective discussions while establishing a close connection between the content of the lectures and the demands produced by the SRP. What is visible in the evolution of the SRP during the three years of implementation is a radical change of the statistical content of the course, which is increasingly aligned with the needs generated by the SRP. In this respect, we can notice a rather strong transformative power of the SRP, which brings an interesting perspective on the ecology of SRPs and their potential dissemination. The didactic infrastructure created by the lecturer to support the mesogenesis includes an elaborated organisation of online resources to facilitate the students search for data and new pieces of knowledge and their sharing. The summaries of students' productions prepared by the lecturer nourish the inquiry dynamics and transform the intermediate answers in new pieces of information to be validated by the inquiry community. This strategy reinforces the topogenesis in the responsibility transferred to the students about the relevance of their productions and those of their classmates. Finally, the intervention of an external agent to present the inquiry generating question appears as a decisive factor for the students' assumption of responsibilities in the entire process and let the generating question assume a leading role during the inquiry. The fact that the lecturer is not an expert in didactics but has managed to contribute to the creation of innovative resources to sustain the SRP opens positive expectations for the ongoing dissemination project in other university settings.

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Conditions and constraints for mathematics curriculum reforms in the first year of engineering programs in Sweden

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Keywords: Curricular and institutional issues concerning the teaching of mathematics at university level, preparation and training of university mathematics teachers.

INTRODUCTION AND BACKGROUND

This study is inspired by the paper by Cuenca, et al. (2022), where the conditions and constraints of mathematics curriculum reforms in undergraduate engineering in a university in Ecuador are investigated. We build on the results of this Ecuadorian/Spanish research, to study the ongoing curriculum reforms called "Profiling engineering education" (PING) at University of Gävle (Sweden). We apply *the anthropological theory of the didactic, ATD* (Chevallard, 2019) and analyse the discrepancies between the different parties of PING regarding the perceptions on the curriculum reform. Our research question is "What are these discrepancies, and how can they be interpreted with an institutional perspective by ATD?" PING is a revision of all programs for bachelor's in engineering, which will be started in 2025 and will have common courses, including mathematics, for all first-year students. Several reform ideas have been proposed to the mathematics department. At an early stage, redistribution and integration of math contents was proposed, and this would have implied 35% reducing of the credits in the first year math courses, and providing more "motivational" lectures that would demonstrate the usefulness of mathematics in the engineering context. Those viewpoints were motivated from quite similar conditions to Cuenca, et al. (2022)'s description of the *triple discontinuity* in mathematics education for engineers (from secondary school to university, between mathematical and engineering courses, and the passage from engineering school to professional practice). The results of their study, drawn from analyses on interviews with teachers and the pre- and post-reform versions of the mathematics curricula, also show similarities with the recommendations of the revision commission in the PING project. Both reforms have mainly pedagogical concern: redistribution of the contents, a slight reduction of the teaching load, and demonstrating the usefulness of mathematics to thinking and solving problems in engineering contexts. The math teachers protested the proposal, and finally, the parties agreed to reduce credit 11% and form two math courses focusing on Linear algebra and Calculus respectively. The mathematical contents of the new syllabus proposed by the mathematics teachers was not much different from the original contents since some overlapping parts between earlier courses were omitted and some parts were removed due to better prerequisites.

INSTITUTIONAL POSITIONS

After implementing interviews with five teachers in engineering subjects (Automation engineering, Computer engineering, Energy systems engineering, Industrial engineering and management, Mechanical engineering) of their perceptions of the curriculum reform in the mathematics courses, we interviewed four math teachers to study their reflections on the issues mentioned by the engineering teachers. Three engineering teachers believed that their students do not need to learn so much of Linear algebra. Math teachers reflected that most of the math contents are connected, and it would be very difficult to cover only a few topics. Most Swedish universities provide Linear algebra in their Bachelor engineering courses. In general, university curricula are regulated by the Higher Education Ordinance, and the main authority (*noosphere* in ATD term) regarding mathematics contents for Bachelor engineering is the "Collaborative group for higher education engineering programmes". This is a network consisting of representatives of the higher education institutions. Since the first-year courses will be taken by all engineering students in Gävle, and there is an advantage to keep a program structure similar to other universities, the engineering teachers agreed with the syllabus proposal of including Linear algebra. Aside from the question of the pros and cons of Linear algebra, the guidelines from this noosphere (the collaborative group) are a constraint for the reform of the curriculum. All engineering teachers suggested providing problem solving tasks in the engineering context for motivating their students to "learn and understand mathematics". Two of the math teachers were rather against this suggestion. They considered that first-year students were "incapable" of understanding mathematics in such applied forms and the students first needed to train "pure" mathematics and understand the theories before applying them. One math teacher who had experienced teaching outside university was positive to the suggestion. These phenomena are brought about by the *inter-professional esoteric pact* (Otaki & Asami-Johansson, 2021), where math teachers act in the position of *mathematician*, committed to the idea of teaching the scholarly mathematics without pedagogical "distraction", while the engineering teachers are in the positions of *noospheric profession*, who try to transpose institutional epistemological knowledge which they consider crucial for the practice of the profession of engineering.

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STEM students' personal meanings for mathematical symbols: The case of partial and directional derivatives

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Keywords: Teaching and learning of specific topics in university mathematics, Teaching and learning of analysis and calculus, Multivariable Calculus, Derivatives.

INTRODUCTION AND LITERATURE

In the United States, multivariable and vector calculus (MVC) is typically taught as a third-semester University-level calculus course covering the differential and integral calculus of multivariable and vector-valued functions. This course spans many important ideas that are related to thermal physics (Roundy et al., 2015) and various engineering disciplines. Since MVC usually follows two semesters of single-variable calculus (SVC) where students receive a workout in taking derivatives and antiderivatives (Frank & Thompson, 2021), it is sometimes difficult for students to apply what they learned in SVC to MVC contexts (Harel, 2021). Researchers have worked to understand student thinking about partial and directional derivatives in the context of tactile and virtual surface graphs (Wangberg, & Dray, 2022), graphing two-variable functions in three-dimensional coordinate space (Martínez-Planell & Trigueros, 2021). While these research results lay the foundation for a developing body of research literature related to the teaching and learning of MVC, but there is more work to be done (Rasmussen & Wawro, 2017).

FRAMEWORK AND METHODOLOGY

From a Radical Constructivist perspective (Glasersfeld, 1995), students develop and construct their own meanings for mathematical symbols and expressions (Thompson, 2013), though our intent as mathematics instructors is to help students align their understanding of these symbols to the normative mathematics understanding as defined by mathematicians (Zandieh et al., 2017). MVC incorporates interesting mathematical notation that can be generalized in interesting ways depending on the student's intended career path and/or research interests.

In MVC, students encounter varied sets of symbols which are intended to convey partial derivatives as multivariable rate of change functions relating the values of two quantities whose values vary, with the value of the third mentally held fixed (e.g., $f_x(x, y), \frac{\partial}{\partial x} f, \frac{\partial z}{\partial x}, D_x f, \partial_x f$, etc.). Each of these different sets of symbols may or may not foreground different aspects of a student's personal meaning for the partial derivative concept. The research question for this analysis was "what are STEM students personal symbol meanings for mathematical symbols related to partial and directional derivatives of two-variable functions?"

To answer this research question and to better understand students' mathematics (Steffe & Thompson, 2000) at an experiential level, I conducted a set of exploratory interviews in the Spring 2023 academic semester with three students (Alonzo, John, and Daniel). These three students were STEM majors attending a large research university in the southwestern United States.

POSTER DESIGN

In this poster, I will demonstrate some of each students' personal symbol meanings for partial and directional derivatives. I will also elaborate on some of the more recent research results related to student understanding of the differential side of MVC. Figures and screenshots of student work will be used to highlight and illustrate some of the more pertinent research results.

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El Comportamiento Tendencial en la Comunidad de Conocimiento Matemático de Ingenieros Civiles en formación: una Resignificación de la Ecuación de Horton en la Infiltración de Agua

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Palabras clave: Comportamiento Tendencial, infiltración, enseñanza y aprendizaje de las matemáticas en otras disciplinas, modelación.

ESTUDIOS EN LA PRÁCTICA Y FORMACIÓN DE INGENIEROS CIVILES Y LA RELACIÓN ENTRE CURSOS DE MATEMÁTICAS Y DE INGENIERÍA

Estudios en la práctica y formación de Ingenieros Civiles se cuestionan las interfaces (enlaces) entre la ingeniería y el conocimiento matemático en ingenieros en ejercicio de su profesion (Kent & Noss, 2002; Gainsburg, 2006). En la relación entre cursos de Matemáticas y de Ingeniería, González-Martín et al. (2021) identifica que con frecuencia, no existe un vínculo explícito entre el contenido de los cursos de matemáticas y el contenido de cursos propios de la ingeniería por lo que es pertinente reflexionar esa problemática. La situación específica en este estudio, es el fenómeno de infiltración de agua en un suelo. Por tanto, se propone estudiar del uso del conocimiento matemático en Ingeniería Civil a través de la categoría de conocimiento de comportamiento tendencial para responder: ***¿Que elementos de la categoría de Comportamiento Tendencial emergen en la comunidad de ingenieros civiles en formación en una situación de infiltración?***

EL FENÓMENO DE INFILTRACIÓN

Para el ingeniero civil es importante el fenómeno de infiltración, ya que, recorre transversalmente problemáticas como: el abastecimiento de agua, el diseño de obras hidráulicas, contaminación de suelos, control de avenidas. La infiltración es el movimiento del agua a través de la superficie del suelo y hacia adentro del mismo, producido por la acción de las fuerzas gravitacionales y capilares (Aparicio, 2004). En los artículos originales de Horton (1939) se identifica una epistemología, que hoy se conoce como ecuación de Horton $f_p=f_c+(f_0-f_c)e^{-kt}$ donde: f_p esla capacidad de infiltración, k factor de porporcionalidad, f_c la capacidad de infiltración final, f_0 la capacidad de infiltración inicial para t=0 y t el tiempo transcurrido desde el inicio de la infiltración. De acuerdo con él, existen dos condiciones que son determinantes para identificar la infiltración como un fenómeno de decaimiento.

EL COMPORTAMIENTO TENDENCIAL EN LA INFILTRACIÓN

Una situación específica, uno de los ejes de la categoría de modelación, posibilita la transversalidad de saberes, los elementos que evidencian la emergencia de una categoría de conocimiento (Buendía & Cordero, 2005) son: significaciones, procedimientos, instrumentos y argumentos. Para dar cuenta de las ***Significaciones*** se recurre al estudio del movimiento del agua, a través de la superficie del suelo (Aparicio, 2004). Para los ***Procedimientos*** a la variación de parámetros cuando se quiere ajustar la curva teórica a la real se manipulan los parametros k, f_0 y f_c, posibilitando el análisis de dos estados. En los ***Instrumentos*** se analiza la función f_p, como una instrucción que organiza comportamientos, dando lugar a nuevas ***argumentaciones*** que ***resignifican*** el conocimiento matemático. Si bien, la noción de *Predicción* permite conocer la evolución posterior de los fenómenos de variación continua, cuantificando la relación funcional entre variables a partir de las condiciones iniciales y de las variaciones de las variables involucradas, el *Comportamiento Tendencial* determina el fenómeno a partir de la simulación, intentando reproducir un comportamiento en particular, la *graficación* es el espacio donde la variación de los parámetros de la función (f_p) se lleva a cabo y se resignifica como una instrucción que organiza comportamientos, todo esto integra la categoría de modelación.

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The transition between Van Hiele levels in graph theory through a commognitive approach

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Keywords: Commognitive approach, Teaching and learning of mathematics in other disciplines, Teaching and learning of number theory and discrete mathematics, Van Hiele levels.

This work, which is currently in its initial steps, is part of a research project aimed at characterizing the teaching and learning of graph theory at the university level. We approach this research from various theoretical perspectives. Building on the work of Gutiérrez and Jaime (1998), we have already defined the Van Hiele levels of reasoning in graph theory through the processes of reasoning (González et al., 2021). Furthermore, using the commognitive approach (Sfard, 2008), we have identified several commognitive conflicts (Sfard, 2008) that arise during the resolution of graph theory tasks (Gavilán-Izquierdo et al., 2022).

CONCEPTUAL FRAMEWORK

Now we propose to integrate these two theoretical perspectives into a conceptual framework with the goal of characterizing the transition between Van Hiele levels using the commognitive approach, with a specific focus on the concept of commognitive conflict. This characterization will yield valuable insights for enhancing the learning and teaching of graph theory.

Commognitive conflicts are relevant because they can originate learning opportunities. Moreover, the commognitive approach allows to differentiate between object-level and meta-level learning. In this line, González-Regaña et al. (2021) classified different types of commognitive conflicts in 3D geometry.

METHODS AND PRELIMINARY RESULTS

The participants in our study were 39 first-year engineering university students from a university located in a large city. The data collection instrument was the written questionnaire with open-ended questions that appears in the works of González et al. (2022) and Gavilán-Izquierdo et al. (2022).

Using the commognitive approach, we identified an object-level commognitive conflict between the discourse of graph theory and the discourse of Euclidean geometry, “since many graphs resemble geometric figures in their pictorial representation, and also possess vertices and edges” (Gavilán-Izquierdo et al., 2022, p. 194). This commognitive conflict can be situated in the transition from Van Hiele’s level 1 to level 2. We think it is possible that there will be other commognitive conflicts in this transition between levels 1 and 2.

CONCLUSIONS AND DISCUSSION

We believe that this conceptual framework can provide opportunities to delve into further aspects of research on the teaching and learning of graph theory, in fact, we intend to propose implications for the teaching of graph theory.

We also acknowledge that this study complements other studies about Van Hiele levels and the commognitive approach. For instance, Wang and Kinzel (2014) identified differences in the Van Hiele level 3 geometric discourses of pre-service primary and middle school teachers.

The poster will present the conceptual framework and the questionnaire, including some examples of students' answers and their analysis. Also, preliminary results and conclusions will be displayed.

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Integrating worlds: a priori analysis of contextualised NUMBAS questions from an interdisciplinary perspective

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Keywords: teaching and learning of mathematics in other disciplines, digital and other resources in university mathematics education, transition to, across and from university mathematics, mathematics in Natural Sciences.

In the evolving landscape of university mathematics education, interdisciplinary discourses are vital in addressing complex institutional challenges and enriching the learning of non-mathematics specialists (Trigueros et al., 2022; Viirman & Nardi, 2019; Watters et al., 2022). Adding to this growing body of literature, our contribution sets out to explore the affordances and limitations of using contextualised dynamic resources created in NUMBAS as tools to support the teaching and learning of first year Natural Sciences students.

CONTEXT

In England, students take national assessments at age 16. Between the ages of 16 and 18, students could opt in to further their studies in three or more advanced level (A-level) subjects, with mathematics among the more than 40 different subjects available. The typical university-entrance qualifications are A-levels. Universities specify entry requirements – A-level subjects, grades and equivalent qualifications (e.g., International Baccalaureate, practical-based vocational qualifications etc.) – for each degree, offering alternatives to students. It follows that there are a considerable number of students admitted to study non-specialist subjects without A-level mathematics as one of their qualifications (Hodgen et al., 2018). The study of mathematics post-16 was not found to predict success in university subjects such as biology or chemistry (Adkins & Noyes, 2018). However, the gap created due to the various entry pathways suggests a need for tailoring university curricula around the diverse needs of cohorts.

This project is a collaborative effort between colleagues at the Mathematics Resources Centre, the Department of Life Sciences and the Department of Mathematical Sciences at our university. The project aims to address the gap arising from the various entry pathways for a bachelor's degree in Natural Sciences through targeted teaching and the use of contextualised resources. We developed a set of NUMBAS questions to aid in identifying the target population and support the teaching of mathematical skills required for their scientific study and degree accreditation.

NUMBAS is a web-based system used to design open-source dynamic assessment questions for mathematical subjects. The use of NUMBAS was grounded on its accessibility and easy-to-use features. The questions were designed around contextualised examples pertinent to the discipline based on suggestions in relevant literature (e.g., Watters et al., 2022). Prior research indicates potential affordances of using NUMBAS as learning and e-assessment tool where the mathematical content is not contextualised (Hadjerrouit, 2020). However, there is limited evidence about the potential of using NUMBAS to create contextualised randomised questions.

RESEARCH QUESTION AND POSTER DESIGN

The poster aims to answer the following research question: what are the anticipated benefits and limitations of using contextualised NUMBAS questions to identify and support Natural Sciences students' mathematical needs?

The poster will present insights into the *a priori* analysis on the NUMBAS questions from three viewpoints: (a) the students' mathematical backgrounds, (b) the enrichment of the present learning experience, and (c) the introduction to discipline-specific practices. Combining expertise and learning from each other, the process of designing and analysing the potentials and limitations of NUMBAS resources enriched our professional experiences. To illustrate the results of the *a priori* analysis, the poster will focus on a specific NUMBAS question. During the poster session, attendees will have the opportunity to explore the features of the question on their mobile devices.

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The mathematical modelling in industry: the case of controller design

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Keywords: Teaching and learning of mathematics in other disciplines, Teachers' and students' practices at university level, mathematical modelling, differential equations, atd.

THE MATHEMATICS IN THE TRAINING OF FUTURE ENGINEER

University courses are evolving slowly, adhering to classical curricula and teaching methods, in which mathematical modelling has not been fully integrated and the current mathematical needs of industry have not been adequately addressed (Castela & Romo-Vázquez, 2022). On the other hand, the industry is evolving rapidly, as illustrated by several studies (e.g., Bissell & Dillon, 2000; Frejd & Bergsten, 2016; Gainsburg, 2006; Kent & Noss, 2002) that have shown the indispensability of mathematical models, computer technologies, problem-solving expertise, and the skills to acquire and process data. In fact, new industries have recently emerged, comprised of multidisciplinary teams of engineers specializing in the development of specific projects requested by large companies, an example of industrial outsourcing. This leads to questioning: How has the process of modelling evolved among engineers compared to previously documented practices? What role does mathematical modelling play in the development of specific projects? How does a multidisciplinary team propose, develop, and implement a mathematical modelling strategy? Do the mathematical models studied in the training of the future engineers appear in the process?

FRAMEWORK AND METODOLOGY

In Romo-Vázquez & Artigue (2022) the Anthropological Theory of the Didactic (ATD), provides tools for studying the tensions and relationships among institutions involved in the training and profession of engineers. These institutions, classified based on their relationship with mathematical knowledge, fall into three types: *producing*, *teaching*, and *using* institutions (industry). Likewise, Castela & Romo-Vázquez (2022) highlight that within institutions, *subjects* perform specific types of task guided by institutional idiosyncrasies, which is reflected in the particular way of generating new knowledge and, consequently, *institutional epistemologies*, which can be evidenced and analysed through *epistemic task*: *appraising*, *motivating* and *validating*. For this study, a mathematical modelling praxeology associated with control theory was identified in the development of a durability test for windshield wiper motors. This information was gathered through a series of semi-structured interviews with an engineer involved in the project, enabling the reconstruction of the modelling process.

INITIAL RESULTS

To enhance comprehension of the results from this initial analysis of the durability test, an outline has been made (see here), delineating the factors influencing the modelling process. It elucidates the role of mathematics as a language bridging the expert client and the multidisciplinary team and highlights the decisive impact of *time constraints* and *functionality* on the selection of employed strategies. A sequence of mathematical models, linked to these three different strategies, was identified. The first one began with a more theoretical approach closest to school knowledge. However, it was changed when scholarly knowledge alone proved insufficient, combining physical experiments. The last strategy, centred on parameter determination, whose validity depended on mathematical and control theory knowledge, as well as the examination of physical tests conducted with a motor, prevailed. This poster illustrates the schematic reconstruction of the modelling process undertaken by the engineer's team. It also provides a preliminary insight into the mathematics underpinning the three mathematical models employed in the development of the identified praxeology.

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Ontologies in engineering mathematics research

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Keywords: Teaching and learning of mathematics in other disciplines, Digital and other resources in university mathematics education, Anthropological Theory of the Didactics, Ontology design, Semantic Web technologies.

This contribution presents a research project that explores possible connections and intersections between mathematics education research, particularly the Anthropological Theory of the Didactic (ATD), and ontology engineering.

The possibilities of educational research in the context of data-based learning and teaching depend in particular on the potential offered by the electronic representation of knowledge. On the one hand, we need modern data structures that enable a digital representation of subject-specific knowledge going beyond e.g. content structures or taxonomies. On the other hand, we need analytical methods and approaches to reconstruct subject-specific knowledge structures from curricular documents and teaching materials. In a collaborative research project of mathematics education researchers and computer scientists, we explore the potentials of connecting ATD concepts to reconstruct institutionally specific knowledge in the form of praxeologies and knowledge graphs or, their formal representations, ontologies as electronic representation of the reconstructed praxeologies. We will first give short introductions into ontologies and the idea of the ontological representation of mathematical praxeologies, and then present initial research ideas based on our previous engineering mathematics research projects (Peters, 2023).

ONTOLOGIES & SEMANTIC WEB TECHNOLOGIES

In the context of computer science, *ontologies* are defined as “a formal, explicit specification of a shared conceptualisation” (Gruber, 1995). That is, an ontology is designed to represent the knowledge of a certain domain of interest, collecting the relevant terminology, definitions, taxonomic and semantic relationships. A domain ontology serves as a specified vocabulary, and as such enables unambiguous communication among domain experts. If formalised in a technological standard, it allows for the design of knowledge-based systems for said domain. Semantic Web technologies are a set of widely used standards for the machine-interpretable representation of ontologies. Domain knowledge is modelled in form of *entities* (contributing to the domain knowledge), *classes* (i.e. sets of entities), and *properties* (i.e. relationships which exist between two entities). They further allow the definition of logical constraints such as domain and range restrictions for properties, and rule-based statements concerning their combined meaning.

THE ONTOLOGICAL REPRESENTATION OF PRAXEOLOGIES

The basic elements of the praxeological model of ATD, i.e. *type of tasks*, *technique*, *technology*, and *theory*, are classes of the ontological model. As are ATDs *levels of codetermination*, i.e. subject, theme, sector, domain, and discipline, which are organised into a subclass-superclass-hierarchy. The class “type of tasks” contains e.g. “Draw a phasor diagram”, “Transform an algebraic expression”, or “Interpret a mathematical expression as an electromagnetical signal” as entities. The class “technique” contains descriptions of actions to solve tasks, e.g. “draw the phasors” and “indicate rotation of phasors”. Justifications and explanations of entities in the class “technique” are elements of the class “technology”. Properties are binary relations between entities. The properties “hasTechnique”, “hasTechnology”, “hasDiscipline”, etc. link entities of classes according to theoretical conceptualisations of ATD.

AN ONTOLOGY FOR ENGINEERING MATHEMATICS

For a first approach, we draw on previous results (Peters, 2023) considering mathematical practices in electrical engineering. Here connections between mathematics, as it is taught in engineering courses, and mathematics as it is taught in higher mathematics courses for engineers, play a major role. Both courses are connected to different institutions. In the ontological model, both institutions are represented as specific entities in the classes modelling the different levels of codetermination. E.g. the class “discipline” contains “Mathematics for Engineers (HM)” and “Electrical Engineering (ET)” as entities. The class “sector” has entities “Complex numbers” and “Modulation of sinusoidal signals”. In our previous analyses mathematical praxeologies emerged, that contain aspects of both institutions. To grasp this mixture of mathematical practices, properties link praxeological elements with a specific entity of “discipline”: The entity “indicate rotation of phasors” of the class “technique” can be linked via the property “hasDiscipline” to the entity “Electrical Engineering (ET)” of the class “Discipline”. The poster will present first steps in building mathematical ontologies, taking the different institutional mathematical aspects into account, and give ideas for further research especially with regard to the role of ontologies in supporting interdisciplinary collaboration.

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The definite integral in the training of civil engineers

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Keywords: calculus, definite integral, modelling, mathematics to engineering.

INTRODUCTION

The teaching and learning of integral calculus often focus primarily on procedural aspects, emphasising the calculation of antiderivatives while neglecting the significance of the definite integral (DI). In this poster we show part of a broader research whose theoretical basis is the Models and Modelling Perspective (Lesh & Doerr, 2003). We started with reviewing the bibliography for the study of the subjects of the specialisation courses in the training of civil engineers in Mexico. The aim is to identify situations that allow students to come up with ideas for solutions to problems close to their workplace. Particularly, those accumulation processes are involved and that can be characterised by the design principles proposed by the theoretical perspective employed. In this sense, a question arises: which situations from the workplace of a civil engineer addressed in the specialisation courses of his or her training can be contextualised for the study of the DI, using accumulation processes and from the approach of models and modelling?

THEORETICAL AND METHODOLOGICAL CONSIDERATIONS

To address this issue, we consider modelling as a rich context for studying the development of ideas as they are produced in response to real-world situations. One of the contextual modelling approaches is Lesh and Doerr's (2003) perspective mentioned above, characterised mainly by a focus on model eliciting activities (MEAs), model exploring activities (MXAs) and model adapting activities (MAAs). For the design of these activities, this perspective recommends following six design principles associated with: reality, model building, documentation, self-evaluation, model generalisation and simple prototyping.

To identify the types of situations required, we reviewed the recommended bibliography for the calculus curses in the curriculum of two recognised universities in Mexico City. We selected two of the most suggested textbooks, "Mechanics of Materials" by Hibbeler (2011) and Gere y Barry (2013), for the subject of mechanics or strength of materials in civil engineering. We identify some of the uses and meanings of DI reported by González-Martín and Hernandes (2018) in situations involving DI: the centroid, the interpretation of forces as an area under a curve, shear force and moment. In addition, we also identify DI as the displacement of the deflection curve in a beam and DI in continuously varying loads or cross-sections. Subsequently, we seek to address the situations encountered and pose problems involving the calculation of the centroid, shear force and/or bending moment of a beam. That is to say, to design activities that provoke students to the emergence, refinement and reuse of models focused on accumulation processes and that lead to the study of DI itself, at the same time as their uses and meanings in the workplace are seen.

RESULTS AND PROPOSAL

We consider that the identified uses and meanings of DI can be contextualised based on the six principles of designing modelling and modelling activities, enabling students to construct solution paths that explicitly use the mathematical content referred to in the DI. In other words, the context of the examples close to the workplace mentioned can be adjusted so that through generalisation processes, models emerge in the students that can be refined, interpreted and evaluated in other contexts or analogous situations. We believe that this way we can also contribute to reducing the discrepancy mentioned by Gonzales (2021) between the approaches, content and skills developed in calculus courses and engineering students' actual academic and professional needs.

It is important to note that designing the sequence of modelling activities for learning DI is underway. This consists of two MEAs, two MXAs and one MAA. The identified situations mentioned in this poster are part of one MEA, one MXA and one MAA. A couple of pilot studies have been conducted and reported in another paper. In the following study, we want to identify a different situation for the design of the other MAA.

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Mathematical functional modelling techniques for Biomedicine degrees

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Keywords: university education, health sciences, modelling, calculus, Anthropological Theory of the Didactic, study and research paths.

MOTIVATION

The development of technology in recent decades, and the need of multidisciplinary collaborations are improving the leading role of mathematics in health sciences and, therefore, the necessity of an articulated training in medicine, mathematics, and computing. In this sense, the presence of technical degrees in health sciences, such as Biomedical Engineering or Nuclear Medicine seems to respond to these needs. In the first years of these degrees it is common to find courses in Calculus, Algebra, and Statistics. However, the revised syllabuses of these courses show a standard structure, similar to those taught in any other engineering degrees, without any specific connection to the health domain. This issue reveals an "applicationist" vision of the role of mathematics in experimental sciences (Barquero et al., 2014), only connecting mathematics and the health domain once the mathematical contents are already acquired. Our research is motivated by the need to introduce new instructional devices to overcome this constraint and propose the study of mathematics as a modelling tool to address health and medical problems.

FRAMEWORK

The Anthropological Theory of the Didactic (ATD) promotes a didactic tool that have been implemented in different universities for more than a decade (Barquero et al., 2022), the so-called study and research paths (SRPs). This tool, based on a generating question Q_0 proposed to the study community formed by the teacher and the students, places mathematical modelling at the center of the study process.Through a sequence of derived questions and associated answers, the students elaborate a final answer to Q_0 under the teacher's guidance. They work in teams and throughout the inquiry process have to take on different responsibilities, combine moments of study of available information with moments of research (Chevallard, 2019).

METHODOLOGY AND RESULTS

We present the *a priori* design of two SRPs based on an epistemological reference model built around the relationship between functional modelling and elementary differential calculus. Both are designed to be implemented in technical degrees in health sciences in which realistic situations close to the students' future professional practice are presented. In both cases, the generating questions are related to drug variation in a patient's bloodstream, but each SRP evolves in different directions. One focuses on fitting functional models already built with continuous data to a set of discrete data, giving an essential role to the manipulation of parameters (Lucas, 2015). The other SRP starts from a specific model presented in a functional form. The evolution of the system over time requires different adaptations of the model combining continuous and discrete work (Serrano, 2024). The comparison of both designs is carried out by considering the sequences of questions that structure each inquiry process, the didactic tools introduced to manage them, the way to integrate the SRP in the course structure and some observed results from their empirical implementation. Special attention is paid to the type of mathematical resources about differential calculus that are requested in both cases and the specificities of the modelling processes.

The results open new questions about the *ecology* of study and research paths in different university settings. They show the students' interest and facilities to adapt to the change of contract but also the lack of mathematics resources for the teacher to design the new instructional proposals and, more importantly, to manage the inquiry process in class.

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TWG4: Teacher education and knowledge

Mathematical orientation as part of teaching competence

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Already since Felix Klein, the aim of teacher educators has been to show pre-service teachers the importance of the subject-specific part of teacher education and convey a "higher standpoint" so that the knowledge and skills acquired in university studies can be utilized in classroom situations. In this contribution we introduce the concept of "mathematical orientation" as a situational competency of mathematics teachers. We developed an analytical framework based on Felix Klein's pervasive approach to elementary mathematics from a higher standpoint and the work of the philosopher Stegmaier on orientation that can be used to describe teachers' situational reflections and actions related to mathematical content. We illustrate its different components by presenting pre-service teachers' reflections on mathematical pieces of content.

Keywords: transition to, across and from university mathematics; teaching and learning of specific topics in university mathematics; mathematical orientation; pre-service teachers; double discontinuity.

INTRODUCTION

Teachers' competencies are usually described as an interaction between their knowledge, abilities, and affective-motivational characteristics (Blömeke & Delaney, 2012). However, the research discourse was recently enhanced by Blömeke et al. (2015) with a continuum model of competence incorporating dispositions, situation-specific skills, and observable behavior, shifting the focus of research more to situational aspects of teaching competence. Most theoretical models of teachers' competence - including the one from Blömeke et al. (2015) - draw back on Shulman's (1986, 1987) terminology of professional knowledge to describe teachers' dispositions, which amongst others include mathematical content knowledge.

Assessing situation-specific skills, in the sense of Blömeke et al. (2015), makes it possible to draw conclusions about a teacher's underlying disposition. Thus, inert knowledge is determined, and cognitive knowledge and abilities are explicitly used in teaching situations. However, while cognitive abilities concerning subject didactical and pedagogical knowledge became detectable using situation-specific instruments (Kaiser et al., 2015), this has not yet been convincingly achieved concerning mathematical content knowledge and abilities. One explanation is that the influence of background knowledge is more challenging to observe within pedagogical decisions or even concrete activities due to its nature, often only being indirectly incorporated.

Complicating matters, existing mathematical abilities are often not recognized as a basis for decision-making in classroom situations and therefore are not used. This assumption is supported by numerous reports that suggest that pre-service teachers criticize their university studies in mathematics as lacking practical applicability to their future professional work (Cooney & Wiegel, 2003; Hefendehl-Hebeker, 2013).

Decades ago, Klein (1932/2016) referred to this issue as the problem of a "double discontinuity" between school mathematics and university mathematics and claimed that a teacher would need a "higher standpoint" to observe possible connections. To overcome this problem, many mathematics teacher educators and researchers aimed to clarify and describe which mathematical knowledge, or abilities are relevant for teaching and make them applicable to their later professional practice as mathematics teachers (e.g., Ball & Bass, 2000; Winsløw & Grønbæk, 2014). This approach led to a collection of good examples for teacher education practices and the development of general teaching guidelines to improve university secondary teacher education (e.g., Allmendinger, 2019; Wassermann 2018; Murray & Star, 2013).

Despite the progress in the research field on overcoming the double discontinuity, no unifying consensus exists on the "higher standpoint," how to describe the mathematical needs of pre-service teachers, and the connections between university mathematics courses and school mathematics and teaching. Multiple theoretical conceptualizations have been introduced, among them mathematical sophistication (Seaman & Szydlik, 2007), mathematical literacy (Bauer & Hefendehl-Hebeker, 2019), mathematical understanding (Kilpatrick et al., 2015), and didactical transposition (Chevellard & Bosh 2020). Among other things, the conceptualizers differ in their location of this resource in the teacher competence model by Blömeke et al. Some are built on discussions of a specific cognitive knowledge base held by secondary teachers, certain included abilities, or even affective aspects.

However, these existing conceptualizations describe the higher standpoint as part of mathematics teachers' dispositions. Thus far, no systematic description exists of how this influence how teachers act and draw on their situation-specific skills in the classroom. This problem is complex because mathematical knowledge does not necessarily come into play explicitly in every action of a teacher in the classroom but can also be part of a mathematics teacher's subconscious decision-making process, for example, when deciding on an appropriate response to a student's mathematical misconception. A conceptualization of performative aspects of mathematics competence when teaching must therefore include situation-specific aspects and capture the mathematical needs of teachers in the classroom.

We therefore developed an analytical framework in Allmendinger et al. (2023) that describes and emphasizes the connections of this orientation to concrete different teaching situations explicitly. Many conceptualizations use conceptual metaphorical language, such as "horizon knowledge" (Ball and Bass, 2020) or Felix Klein's (1932/2016) "higher standpoint." We follow this approach by adopting a network of interconnected geographical metaphors to describe the mathematical needs of pre-service teachers in specific teaching situations. In this contribution, we present this analytical framework and will illustrate it with empirical evidence.

CONCEPTUALIZATION OF MATHEMATICAL ORIENTATION

Mathematics can be conceptualized as a landscape, as articulated by Davis and Hersh (1998) or Wasserman (2016). Within this landscape, locations represent distinct pieces of mathematical content interconnected through logical relationships and axioms. School mathematics, identifiable within this terrain, comprises pieces of mathematical content primarily taught and learned in school, even though its boundaries may be indistinct (cf. Fig. 1). As students engage with mathematics in school, their exploration of these content pieces is facilitated by their teachers.

Fig. 1 Mathematical landscape

In line with the pragmatic philosopher Stegmaier, navigating the mathematical landscape involves encountering situations demanding orientation, defined as the "achievement of finding one's way in a situation and identifying promising opportunities for actions to master the situation" (Stegmaier, 2019, p. 25). Stegmaier's monograph, "What is orientation? A philosophical investigation" (2019), elucidates the conditions and structures of orientation, introducing a philosophical terminology applicable across diverse contexts. He delineates geographical sub-concepts crucial for orientation, encompassing various points of reference, perspectives, and individual standpoints: When faced with a situation, our eyes roam to gain an overview, concentrating on specific *points of reference*, as Stegmaier (2019, p. 56) notes, "what a new situation brings is not yet certain; it must first be made surveyable [...] by means of such points". In our context, these are interconnected locations within the mathematical landscape guiding the way from one point to another. To identify pertinent points of reference and make decisions in a given situation, diverse *perspectives* are essential, aiding in concentrating on relevant aspects to "differentiate the sight" (Stegmaier, 2019, p. 47). These perspectives will be taken from different *standpoints*, which can be understood as opinions on or attitude towards a specific topic, as "one does not only 'stand' in a situation, but one's mood, the psychophysical state of one's orientation, also depends on it" (Stegmaier, 2019, p. 44). This perception is known from other contexts as well, for example when considering a political or moral standpoint.

We have built on this approach and developed an analytical framework in Allmendinger et al. (2023) that additionally emphasizes affective aspects of orientation and the connections of this orientation to concrete teaching situations. This required the addition a set of "links" that express an individual's established connections between reference map, perspectives and attitudes and concrete teaching situations. Mathematical orientation encompasses thus four key aspects – a reference map, a perspective toolbox, a mathematical attitude as well as a set of links to concrete teaching situations (cf. Fig. 2), which we will illustrate below.

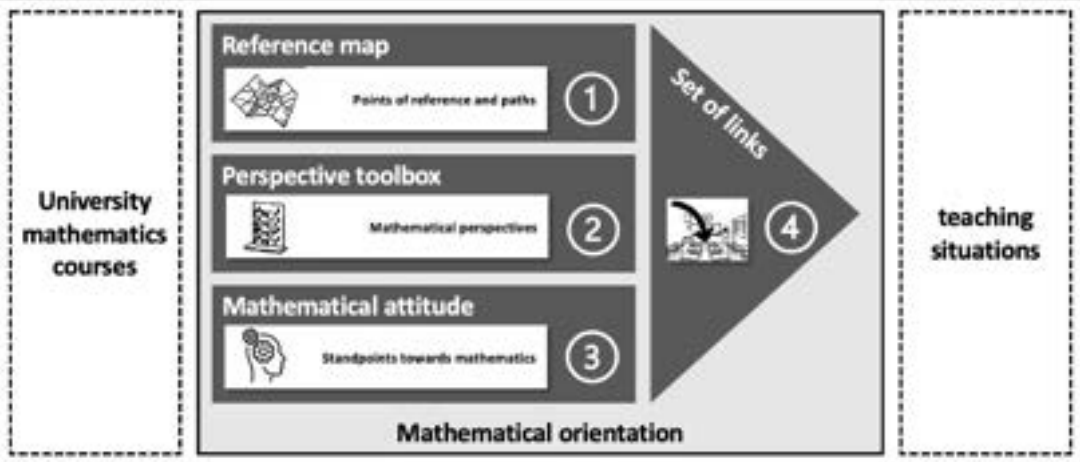

Fig. 2: Conceptualization of mathematical orientation (Allmendinger et al., 2023)

While acknowledging that decisions in teaching situations involve not only mathematical aspects but also didactical or pedagogical considerations, our conceptualization of mathematical orientation centers on the mathematical needs for teaching, aligning with prior concepts like mathematical understanding (cf. Kilpatrick et al., 2015), mathematical sophistication (cf. Seaman & Szydlik, 2007), mathematical literacy (cf. Bauer & Hefendehl-Hebeker, 2019), and didactical transposition (Chevellard & Bosch, 2020).

RESEARCH APPROACH

Obviously, mathematical orientation is a content-specific concept. In addition, as already mentioned, mathematical orientation is highly individualized. The conceptualization therefore gives rise to four research questions which we investigated in a qualitative empirical study: Which points of reference and paths are marked in PSTs' reference maps? Which mathematical perspectives are included in PSTs' perspective toolboxes? How can PSTs' mathematical attitude towards mathematical content be characterized? How do PSTs use their reference map, perspective toolbox, and mathematical attitude to link mathematical content to teaching situations?

We asked Norwegian and Swiss pre-service teachers (PSTs) which attended university mathematics courses on "school mathematics from an advanced standpoint" to reflect on the topic of decimal expansion presented in both courses and analyzed these with regard to the research questions above. We were able to reconstruct their mathematical orientation (see Allmendinger et al. 2023 for details). Their mathematical orientation could also be rediscovered in reflections on other mathematical topics, which we will use in this contribution to illustrate the characteristics of our analytical framework. We choose reflections from Swiss PSTs from Lucerne on the topic of complex numbers as examples. Our reconstruction approach showed that mathematical orientation varies across PSTs to a certain extent. In particular, different points of reference, perspectives, standpoints, and types of links between the mathematical content and teaching situations could be identified.

ANALYTICAL FRAMEWORK

Mathematical reference map

Teachers need a selection of points of reference, i.e., pieces of mathematical content from both school and university mathematics, connected by paths to guide students securely through the mathematical landscape. These points form an individual reference map mentally carried by teachers. Ideally, university mathematics courses equip PSTs with a reference map containing all school topics and connections, fostering an understanding of relationships within and beyond the school mathematical domain. When faced with a teaching situation, teachers are expected to mentally consult this map before deciding how to respond (see Allmendinger et al. 2023).

The reflections of the Swiss PSTs provided valuable insights into the placement of the given content on their reference map. For instance, they shed light on whether they position this content within or beyond the confines of school mathematics. On the other hand, different paths between mathematical pieces of content became visible. PSTs named a wide variety of *other points of reference*, that they connected to the topic of complex numbers:

> “The biggest eye-opener is probably the wider context of all the mathematical topics we have covered so far [in the course]. Different methods of representation, trigonometry, axioms, solid theory, unit circle, degrees - I found it very exciting to see all of this in a new context.”

Mathematical perspectives

Allmendinger (2019) was able to show that Felix Klein associates a “higher standpoint” with viewing mathematics from different perspectives focusing on different aspects of mathematical content. We call this collection of mathematical perspectives the individual's perspective toolbox, enclosing a variety of viewpoints enabling teachers to analyze different pieces of mathematical content or paths in teaching situations. In our study, we were able to identify three different perspectives – a *content-related perspective*, a *principle-related perspective*, and a *presentation-related perspective* (see Allmendinger et al. 2023).

While a PST taking a content-related perspective will concentrate on the content itself, e.g. identifying the piece of mathematical content as part of school curriculum, a PST taking a principle-related perspective will focus on general mathematical activities or ideas.

> "In addition, every time the number range is expanded, new rules are added and old ideas have to be adapted. For example, the complex numbers can no longer be arranged according to size. This was possible with the previous number ranges. As a result, our ideas had to be adapted again. It is precisely these thought processes that the students also go through and as a teacher you should be aware of this and support the students in precisely this area." (*principle*)

> "I think the principle of going from a specific case to a general one and then back to a specific case is cool. I would also like to use this in my lessons. It gives you the feeling that you really understand it." (*principle*)

PSTs taking a presentation-related perspective will draw conclusions for their teaching with regard to the way a topic is presented.

> "Changing the form of representation, e.g. as an adaptive learning opportunity. If pupils do not understand a new calculation rule, it always makes sense to explain the facts from a different perspective. Visualizations, e.g. with GeoGebra, offer great added value here." (*presentation*)

Mathematical attitude

Orientation demands the will and courage to make decisions based on observations from different perspectives. A teacher's willingness to do so depends on their "standpoint" toward mathematics and specific content in school or university mathematics, reflecting beliefs, values, preferences, and feelings – a teacher's mathematical attitude. This attitude influences a teacher's reactions in various situations, such as valuing the importance of applications based on their confidence in certain aspects fostered during university mathematics courses (see Allmendinger et al., 2023).

In the analyzed reflections we recognized different aspects especially concerning *confidence*, *compassion*, and *enthusiasm*.

> "I haven't had the feeling of getting to know something completely new for a long time, so I feel for the students when we look at a topic that is unfamiliar to them." (*compassion*)

> "I thought it was so beautiful to see that with the special case of the square with r = 1, you could see so beautifully why i^2 gives minus 1 and why, or that -i^2 also gives minus 1. You could see that really well on the GeoGebra diagram and it made perfect sense to me and I just thought it was really beautiful :)" (*enthusiasm*)

In particular, it became clear that this attitude is necessary in order to be willing to create links to school mathematical situations.

> "I have to be honest and say that I can't think of a single situation where I would need this subject. "

Links to teaching situations.

The three components – reference map, perspective toolbox and attitude – combined are used to connect pieces of mathematical content to teaching situations. It was possible to classify different types of links PSTs are establishing. The links could be differentiated according to whether the aspect of the mathematical content under consideration was directly incorporated into a specific teaching situation or only indirectly as background knowledge for teaching in general. It was also possible to distinguish between different teaching situations, such as the requirement to explain something in an in-class situation or the selection of suitable tasks when preparing lessons, as the following examples show.

> "So far, I have 'spit off' the students with the fact that there are no roots of negative numbers, and that the quadratic equation therefore has no solution. Now I can, for example, give an outlook on complex numbers in the form of a small digression." (*directly, explain-situation*)

> "In school different sets of numbers will be a topic. Here, for example, the background knowledge is very useful. For example, if I am asked whether there are other sets of numbers etc." (*indirectly, react to students questions*)

CONCLUSION

Mathematical orientation and Felix Klein's "higher standpoint"

Our introduction of mathematical orientation adjusts the understanding of the Kleinian term "higher standpoint". Traditionally, the term is tied to specific knowledge or abilities (e.g. Dreher et al., 2018). Our conceptualization challenges the conventional hierarchical view of mathematics, advocating for a unified awareness across the entire mathematical landscape (spanning both school and university). Within our framework, standpoints in mathematical orientation signify a personal disposition rather than specific knowledge or abilities, aligning with Klein's vision.

While adopting a higher standpoint is deemed necessary, it alone is insufficient for addressing teaching scenarios. Educators sharing a similar standpoint may differ in frames of reference, perspectives, and the way they link content to teaching situations. Due to its philosophical background (Stegmaier, 2019), mathematical orientation is highly individual, necessitating a comprehensive focus beyond the standpoint. This aligns with Klein's emphasis on the importance of orientation for secure footing, encompassing intuitive elements, vital relations, and historical development. The concept of orientation, crucial for teaching requirements, is utilized by scholars like Schoenfeld (2011) and Hannah et al. (2011). Schoenfeld defines it for "in-the-moment decision making," while Stegmaier considers it a broad category encompassing beliefs, values, preferences, tastes, and aspects of knowledge and abilities. This understanding is grounded in the original geographical meaning of orientation.

As Klein (1932/2016) emphasizes, orientation involves understanding mathematical content and its interconnections. In addressing the "double discontinuity," Klein highlights the challenge faced by teachers transitioning from university studies to teaching traditional elementary mathematics. In frameworks that highlight the differences between university and school mathematics, this discontinuity is seen as a struggle for pre-service teachers (PSTs) to connect school and university mathematical content. With this, Dreher et al. (2018) distinguish between bottom-up and top-down, while Zazkis and Mamolo (2011) focus on horizon knowledge linking school topics with university content. Both interpretations therefore concentrate on connections between different mathematical pieces. However, reading Klein's description of the "double discontinuity" literally, it is more about the inability to link university course content to concrete teaching tasks (Klein, 1932/2016, p. 1). Our approach adopts this interpretation, defining mathematical orientation as the ability to connect any mathematical content to teaching situations.

With this, the conceptualization distinguishes between two types of connections: "paths" within the mathematical landscape and "links" between mathematical content and teaching situations. Thus, mathematical orientation notably highlights the situational component of teaching and raises the question of how these links can be described and characterized with regard to the different requirements a teacher faces in practice.

Mathematical orientation and teachers' competence

If PSTs must guide their students through the mathematical landscape in their later professional work of teaching, they must have the appropriate orientation with respect to school mathematics. However, with the concept of the higher standpoint, it was difficult to identify it as a special capability of PSTs because of its conceptual fuzziness. This is now easier with the concept of orientation. It can be shown that the concept of orientation fulfills all the requirements of a competency for PSTs with regard to the understanding of competence by Weinert (2001) or Niss and Højgaard (2019). Orientation implies a specific kind of knowledge in the form of connected points of reference: "When during the process of orientation, enough (points of reference) come together and sufficiently fit with each other, then that which we call knowledge arises" (Stegmaier, 2019, p. 58). Coping with a mathematical situation in the classroom will require analyzing different paths in order to decide on how to act in this situation. In order to do that, PSTs need the "ability to view something in various ways that exclude but also complement and enrich each other" (Stegmaier, 2019, p. 47). So, taking different perspectives is a specific ability PSTs need for teaching. Additionally, orientation will include affective-motivational aspects: "Thus, the courage for an orientation under uncertainty involves the courage to decide for points of reference and to be determined to hold onto them despite persistently unsettling conditions" (Stegmaier, 2019, p. 61). So, like other competencies, orientation also requires that a person is willing to use their competency. Finally, orientation is always bound to concrete mathematical situations, making it a situation-specific competency.

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Pre-service teachers' understanding of sine and cosine functions and their inverses based on the unit circle trigonometry

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We applied Action-Process-Object-Schema (APOS) theory to study pre-service secondary mathematics teachers' understanding of the concepts of sine and cosine functions and their inverses based on the unit circle approach. We used a model (genetic decomposition) of mental constructions that students may do to understand these notions and designed research-based didactical materials and implemented them in two countries, Czech Republic and Iran. Eighteen pre-service teachers (nine from each country), who were studying a bachelor's degree of mathematics, participated in our research. The study involves three phases: initial interview, instructional intervention, and exit interview, which were separately carried out in each country. We discussed the teachers' understanding in both initial and exit interviews.

Keywords: Teacher's preparation, Trigonometric, Radian, Unit circle, APOS.

INTRODUCTION

Trigonometric functions and their inverses play a crucial role in understanding many aspects of mathematics, including calculus. Literature review shows, however, that students and teachers have problems in understanding and reasoning about trigonometric functions (Martínez-Planell & Cruz Delgado, 2016; Moore, 2013; Weber et al., 2020). This suggests that difficulties regarding trigonometric functions among future teachers may impact student understanding in the long term. The teaching and learning of trigonometric functions is not a straightforward process and can be influenced by a number of variables. It is often students' first encounter with a function that does not provide an explicit rule to compute its output. To interpret the values of trigonometric functions for real inputs, students would need to imagine the process of obtaining these values in the unit circle (Martínez-Planell & Cruz Delgado, 2016). Failing to understand this process limits students' knowledge of trigonometric functions and their inverses. To address the difficulties in learning trigonometry, some researchers have focused on angles and the relationship between radian and degree measures in the unit circle (Tallman & Frank, 2020). These studies show that students did not regard the radian as a unit of angle measure and their conception of angles was based on degree measure; therefore, students were unable to interpret the output of trigonometric functions in situations where inputs were given as real numbers. Hence, in our research we focus our attention on radian measure and helping students construct a mental image of sine, cosine and their inverse functions using the unit circle.

LITERATURE REVIEW

Some research has investigated basic constructions related to learning trigonometric functions. Brown (2005) analysed teacher's teaching and students' work regarding radian measure and the unit circle. She observed that understanding the "unit" in unit circle is a fundamental idea that causes student difficulty and is underestimated by teachers during teaching. The students of her study used the phrase 'unit circle' in such a way that was not necessarily accompanied by an appreciation that the radius was the unit of measurement (see also Moore, 2013). Bagni (1997) studied students' understanding of trigonometric equations. He reported that 80% of the 67 students in his research could provide a complete or partial solution to easy trigonometric equations (e.g., $\cos x = 1/2$ or $\sin x = -1/2$), by remembering and mentally reversing a memorized table of the values of trigonometric functions. The results also revealed that more than half of the students produced incorrect answers or no answer to questions such as find all real values x in $\sin x = 1/3$ (Bagni, 1997). In this regard, Weber (2005) conjectured that students need to be able to imagine the process of constructing trigonometric functions (from angle to circle to value) which gives rise to the unit circle definition of these functions in order to be able to understand the sine and cosine functions and go beyond the repetition of memorized procedures and facts. Weber also suggested that students should explicitly and physically construct geometric objects to help them deal with trigonometric functions and their inverses. Although inverse trigonometric functions play an important role in many secondary and university mathematics curricula, research on students' and teachers' conceptions of inverse trigonometric functions is limited. In one of them, Weber et al. (2020) investigated 14 pre-service and in-service teachers' understanding of the inverse sine function. They reported that almost all the teachers in their study were unable to explain how by restricting the domain of the sine function, an inverse function is possible.

THEORETICAL FRAMEWORK

The study used APOS theory (Arnon et al., 2014) as the theoretical framework. In APOS theory, an Action is a mathematical transformation that the student perceives as external. An Action may be the rigid application of an explicitly available algorithm or of a memorized formula or procedure. When an Action is repeated, and the student reflects on it, it might be interiorized into a Process. A Process is perceived as internal and allows the student to omit steps, anticipate results, and generate dynamical imagery of the Process. Processes may be coordinated or reversed to form new Processes. When a student is able to think of a Process as a whole and is able to do or imagine doing Actions on that whole, then one says that the Process has been encapsulated into an Object. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas having to do with a particular mathematical notion. Research in APOS typically starts by proposing a hypothetical model (a conjecture) in terms of the structures and mechanisms of the theory of how a generic student may construct a specific mathematical notion. This model is called a genetic decomposition (GD). A GD is not unique, and it does not pretend to be the best way to teach a particular notion. It has to be tested with student interviews. The following is a portion of the GD for a "unit circle approach" to sine, cosine, and their inverses designed by Martínez-Planell and Cruz Delgado (2016) and that was used in this study to design and implement activities to help students do the proposed constructions and understand the radius of a circle as the natural unit of measurement in trigonometry.

Process #1: $t \to P(t)$ process. Construction of the sine and cosine functions may start with the action of taking a given real number t and locating, as a geometric representation, the terminal point $P(t)$ of an arc along the unit circle that starts at the point $(1,0)$, has length $|t|$ radii and is traversed either counter clockwise when $t \geq 0$ or clockwise in the case that $t < 0$. As students repeat and reflect on this action they may be able to imagine taking any given real number t and assigning to it a point $P(t)$ on the unit circle without having to do so explicitly. In this case they can be said to have interiorized the action into a process, denoted $t \to P(t)$.

Process #2: Circ process. In another construction, given a point $P(t)$, represented geometrically or as an ordered pair, the student can perform the action of finding the other three corresponding points $P(-t)$, $P(t+\pi)$, and $P(\pi - t)$ on the unit circle, geometrically or as ordered pairs. These actions are interiorized into a process that enables students to locate on the same circle the geometric representation of any point of the form $P(t+2k\pi)$, $P(-t+2k\pi)$, $P(t+\pi+2k\pi)$, $P(\pi - t + 2k\pi)$ when they know a geometric representation for the point $P(t)$.

Process #3: projection process. Now a process conception of the sine and cosine functions may be constructed by coordinating the $t \to P(t)$ process with a corresponding projection process. Projecting onto the y axis [x axis] defines the sine [cosine] function. These actions of projection may be interiorized into processes of "projection". The processes of locating a corresponding point $P(t)$ and then projecting onto a corresponding axis (as described above) may be coordinated into processes which we will refer to as the definition of the sine and cosine functions.

Process #4: reversal of the projection. To reverse the projection of the sine function, start with a number k in the interval $[-1,1]$ and perform the action of representing a point on the y axis that has k as its ordinate. The next action is to locate on a physical or geometric representation of the unit circle all the points that are projected horizontally onto $(0, k)$. To reverse the projection of the cosine function, perform the analogous action, namely represent a point on the x axis having abscissa k, and then identify all points on the unit circle that project vertically onto $(k, 0)$. Repetition and reflection on these actions may be interiorized into a process of projection reversal.

Process #5: reversal of the $t \to P(t)$ process. To reverse the $t \to P(t)$ process, the student starts with one or two points $P(t)$ resulting from a projection reversal and finds a value of t determining one of the points. At this stage, finding an approximation of a real number t that determines a point $P(t)$ with a specific x or y coordinate may be done physically with a manipulative like a piece of ribbon.

Process #6: reversal of the definition. After a student reverses a projection and obtains the points that correspond on the unit circle, the student may coordinate the reversal of the $t \to P(t)$ process (to obtain one value of t) with the Circ process to obtain all values of t that determine the points he/she found on the unit circle. The reversal of the $t \rightarrow P(t)$ process followed by the coordination with Circ results in a process that allows the student to recognize that the sine and cosine functions are not one to one. The chain of actions that starts with a number, represents it as an x or y coordinate on the corresponding axis, goes on to identify the point or points on the unit circle having that number as an x, or y coordinate, and then identifies all the real numbers corresponding to the point or points on the unit circle, may be interiorized into a process that we will call reversal of the definition. This process starts with a coordinate and produces the collection of all real numbers corresponding to the points on the unit circle (one, two, or none) having that coordinate. By its nature this process does not define a function.

Process #7: Range process. To construct a process conception of Range, students could interiorize actions that explore ways of restricting the domain of the sine and cosine functions to an interval so that the resulting function is one to one and the restricted domain is as large as possible. These actions should include both, the unit circle representation and the graphs of these functions. Students that interiorize these actions into a process would recognize the need to restrict the domains of sine and cosine as well as the convenience of restricting these domains as they normally are. Students not able to argue for the need of a restriction and reasonableness of the usual restrictions of the domains of sine and cosine will be constrained to having an action conception of Range as a memorized fact.

METHOD

We conducted the study separately in two countries: Czech Republic and Iran. Nine pre-service secondary teachers in their second year of an undergraduate mathematics field from a teacher training department in Czechia, and nine pre-service secondary teachers in their second year of an undergraduate mathematics field from a teacher training department in Iran voluntarily participated in this study. We did the study in these two countries because we had the opportunity and were interested in seeing how students from different cultures and backgrounds react on the interview questions and the set of class activities. However, we did not aim to do a quantitative comparison of students' understanding between the two countries because the amount of data was not enough to make any meaningful comparison. Furthermore, in order to make such a comparison, many other factors and variables would have to be examined, which were out of the scope of this study. The students in both countries learned the trigonometric functions and their inverses in several courses in secondary school and university and applied these concepts to study other advanced mathematical notions (e.g., limit, derivative, and integral). Our study consisted of an initial interview, an instructional intervention, and an exit interview. After the initial interview, we started the intervention in both groups (one in Czechia and the other in Iran), and one week after the instruction we conducted the exit interview. For the intervention, we designed a set of activities to help the pre-service teachers interiorize the geometric processes conjectured in the GD, thus extending the study of Martínez-Planell and Cruz Delgado (2016). The intervention consisted of three class sessions (each 90 minutes) by two experienced instructors (one in Czechia and the other in Iran). In the first class, the pre-service teachers used a ribbon, to prepare a measuring tape in units of circle radii. They used the measuring tape to measure the number of radii in a half circle and found the relationship between this number and the value of π, and also the number of radii in the circumference of the circle and compare it with the well-known formula $S = 2\pi r$. Regarding the $t \to P(t)$ process, students used a ribbon to locate points $P(t)$ on the circle determined by t (radii). Then, for a point on a circle they determined three other points on the circle, reflecting through each axis and through the origin (Circ process). In the second class, they represented the projection of $P(t)$ on the y axis [x axis] in units of radii to find the value of sine and cosine of the real number t (projection process). They extended the definition of sine and cosine directly to the acute angles of a right triangle where they thought of hypotenuse as the radius of a circle. In some activities they used the sine and cosine functions on the unit circle to draw the graph of these functions in the Cartesian coordinate system. In the third class, they constructed the inverse sine and cosine functions. In this regard, they started with a number between -1 and 1 as input and produced points in a unit circle as output (from number to point, reversal of the projection). Then they started with a point on a unit circle as input and using a ribbon as a measuring tape produced an angle measure as output (point to angle, reversal of the $t \to P(t)$ process). The combination of the two previous steps, starting with a number and ending with an angle, is used to reverse sine and cosine. For this, they also did some activities to find the conditions for uniqueness when reversing sine and cosine in the unit circle and Cartesian coordinate system (Range process). The intervention activities were expected to help students interiorize Actions into Processes and help them imagine the definition of sine, cosine, and their inverse functions, as proposed in the GD. The research question is: how well did the intervention help students construct the processes in the GD?

The initial and exit interviews were audio recorded. The interviews in both groups (i.e., Czechia and Iran) were done by one of the authors of this paper who separately gathered the data in each country. The interviews were audio and video recorded, transcribed, translated to English, analysed individually, and discussed as a group. Differences in opinion were negotiated. The individual and group analysis concentrated on trying to ascertain the mental structures (Actions, Processes, Objects) evidenced by students in regards to the GD. The initial interview questions were: **1)** Show angles measuring $20°$, $\pi/2$ radians, and 4 radians in the unit circle. **2)** $\boldsymbol{a}$**.** Find the value of $\sin 30$. $\boldsymbol{b}$**.** Find the value of $\sin 30°$. **3)** Using Figure 1(a), determine a formula between r, θ, and S and justify it (r is the length of the radius, θ is in radians, and S is the length of the given arc) **4) a.** Show the answers of $\sin x = 1/3$ ($x \in [0,2\pi]$) on the unit circle. **b.** Why are $\sin(\pi/2)$ and $\cos(\pi/2)$ equal to 1 and 0, respectively? **5)** Find approximately $\cos(2.5)$ using the following circle (Figure 2(a)). **6)** Find $sin^{-1}(-1)$ and justify your answer.

The exit interview questions were: **1)** Given that the angle θ in Figure 1(b) measures 0.6 radians, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2.2 cm, 4.2 cm, and 6.2 cm. **2)** Use the unit circle (in Figure 2(a) and a piece of ribbon to approximate the value of $\sin(1.2)$ the best you can. **3)** Put in order from lowest to highest: $\cos(1)$, $\cos(1/10)$, $\cos(3)$, $\cos(-4)$, $\cos(4)$, $\cos(6)$. **4)** Find all the solutions of $\cos(t) = -3/5$. **5)** Given that $\cos(22\pi/7) = -0.90096887$. Find $cos^{-1}(-0.90096887)$. **6)** Use the unit circle to show: $sin^2x + cos^2x = 1$

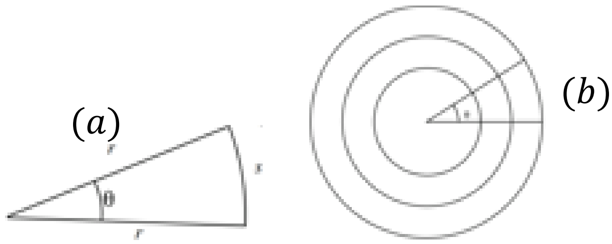

Figure 1: (a) related to Q3 of the initial interview, (b) related to Q1 of the exit interview.

RESULTS

We start by considering the number of the pre-service teachers (in both countries) constructing the different Processes of the GD in the initial and exit interviews (Table 1). The data in Table 1 shows that the teachers did not show evidence of most of the mental constructions of the GD in the initial interview. However, their answers to the exit interview improved compared to those of the initial interview.

GD / #students in:	Process #1	Process #2	Process #3	Process #4	Process #5	Process #6	Process #7
initial interview	3/18	4/18	5/18	2/18	2/18	2/18	2/18
exit interview	15/18	13/18	16/18	13/18	12/18	11/18	11/18

Table 1: Number of the teachers constructing specific parts of the GD

Initial interview results. Most of the pre-service teachers had significant challenges with the mathematical tasks involved in the initial interview. Indeed, none of them can be classified as having a process conception of sine and cosine. Only two of the 18 interviewed teachers are classified as being in transition level between the action and process conceptions, and another 16 teachers seemed to have been limited to an action conception. Here we consider some typical answers of the teachers to questions 1, 2, 3, 5, and 6. All the teachers from both countries showed the angles $20°$ and $\frac{\pi}{2}$ radians correctly. However, 15 teachers (eight from Iran and seven from Czechia) were not able to draw the angle of 4 radians. The answers of Jan and Keyvan were typical responses of these pre-service teachers:

Jan: [from Czechia] I think 4 radians is another notation for 4π, so it's on the positive side of x axis.

Keyvan: [from Iran] I don't know in which quadrant 4 radians is, actually I always have problem with radian, I don't know what it means, it's more convenient for me to use degree for angles, for example I directly and without any computation know the angle four degrees is in the first quadrant but I can't immediately realize where the angle of four radians is, but I need to do some computations, I know that π is approximately 3.14, and 4 is bigger than π so the angle 4 radians is probably in the third quadrant.

Jan's answer is consistent with the findings of Bagni (1997) where students in his study considered π as the unit for the radian measure and considered 1 radian equalled to $180°$. Also, like Keyvan, some of the teachers in both countries did not consider π as an angle measure related to the arc length of a circle. The data indicated that the pre-service teachers were more comfortable with degree measure than radian measure. Related to this issue, almost all the teachers (except one) did not realize that 30 in $\sin 30$ (question 2 part a) is expressed in radian and considered ½ as the correct answer for $\sin 30$. Although when facing with $\sin 30°$ in part b they corrected their answer and put $\sin 30° = 1/2$, they still were unable to discuss how to get the approximation of $\sin 30$ using the unit circle. The pre-service teachers still need to construct the $t \rightarrow P(t)$ process to measure angles in radians and coordinate it with the projection process to find the values of sine and cosine of real numbers. In question 3 we asked the teachers to determine a formula relating r, θ, and S and to justify it. Eleven pre-service teachers did not find the correct formula. The other 7 teachers (four from Iran and three from Czechia) found the formula using proportions and not as stemming from understanding radian measures as a multiplicative relationship between arc length and radius length. Monika's answer was typical.

Monika: [from Czechia] We know that the circumference of a circle with radius r is $2\pi r$ and the angle is 2π, umm so when the angle is θ we need to find S, So we just need to solve the proportion $\frac{2\pi}{\theta} = \frac{2\pi r}{S}$ for S. It will be $S = r\theta$.

Interviewer: Can you explain more about this formula in terms of the unit circle?

Monika: Oh, for me it's just an algebraic formula between variables r, S, and θ.

Monika's explanations showed that she did not think of the radian measure of the central angle θ as the number of radii in the corresponding arc-length. This is consistent with not constructing the $t \rightarrow P(t)$ process which helps students think of the measure of a central angle in a circle of any radius as the number of radii in the corresponding arc-length. In question 5 (approximating $\cos(2.5)$), none of the teachers (in both countries) gave correct explanations. Here we consider two of such responses.

Hamid: [from Iran] The angle 2.5 degrees is very close to 0, so $\cos 2.5$ is approximately close to $\cos 0$ which is 1.

David: [from Czechia] We learned to find the values of trigonometric functions for angles such as $\pi/6$ and $\pi/4$, umm I don't know how to find the cosine of 2.5, in such cases I use calculator.

Interviewer: Can you explain based on the unit circle approach why $\cos \pi/4$ is $\sqrt{2}/2$?

David: No, I don't, even for these convenient angles like $\pi/6$, $\pi/4$, or $\pi/3$ I just memorized the table of the values of trigonometric functions.

Most of the pre-service teachers, like David, just memorized the trigonometry table and were not able to justify the value of sine and cosine of neither convenient angles (e.g., $\pi/6$ and $\pi/4$) nor real numbers that are not integer multiples of $\pi/6$ and $\pi/4$ (e.g., 2.5 in $\cos(2.5)$). In question 6 (finding the value of $sin^{-1}(-1)$), three teachers (two from Czechia and one from Iran) found $-\pi/2$ as the correct answer using a memorized fact, showing an action conception of Range. The other pre-service teachers showed some problems with this question. We consider the answers of Samira and Kristyna as typical answers of such students.

Samira: [from Iran] I need to find an angle where its sine is -1, umm so it can be $-\pi/2$ and $3\pi/2$, or generally $-\pi/2 + 2k\pi$.

Kristyna: [from Czechia] Let's put $sin^{-1}(-1)$ equal to θ, umm by taking the sine from both sides we have $\sin(sin^{-1}(-1))$ equal to $\sin\theta$, umm I know $f(f^{-1}(x))$ is x, so $\sin(sin^{-1}(-1))$ is -1. I changed the question to the easier question $\sin\theta = -1$, we can find many angles as the answer, for example $3\pi/2$.

Both Samira and Kristyna had not constructed the Range process. The same difficulty was observed in most pre-service teachers.

Exit interview results. One week after the instruction in each group an exit interview was taken from the pre-service teachers in each country. In short, we started with 18 pre-service teachers who showed several difficulties with the construction of sine and cosine and their inverses based on the unit circle approach as evidenced by the results of the initial interview, and after an instructional intervention most of them showed an improvement in their understanding. Indeed, of the 18 interviewed teachers, we classified 10 as having reached at least the process conception of sine, cosine, and their inverses; 5 a transition level between the action and process conceptions; and 3 remained at the action conception. Here we show some typical answers of the teachers to questions 1, 3, and 5 in the exit interview. Fourteen teachers (eight from Czechia and six from Iran) determined the length of each arc cut off by the angle in question 1. We consider the case of Monika:

Monika: The angle is 0.6 radians, it means in the corresponding arc of each circle there are 0.6 radii of that circle, so for each circle 0.6 times the radius length of that circle is equal to the corresponding arc length, 0.6 time 2.2, and 0.6 times 4.2, and also 0.6 times 6.2 cm will be the length of each arc cut off by the angle.

Monika's response gives evidence of understanding the ideas behind the usual formula for arc-length in terms of radian measure and radius length (i.e., $S = r\theta$) in a way which is not dependent on the manipulation of a symbolic expression (her response in the initial interview), but rather the ideas may surface as natural relations that do not require memorization. This gives evidence of constructing the $t \to P(t)$ process. Regarding question 3, thirteen pre-service teachers (seven from Czechia and six from

Iran) ordered the values of the cosine function correctly. We consider the case of Hamid (Figure 2(a)) who was not able to find $\cos(2.5)$ in the initial interview.

Hamid: I need ribbon to find the location of these radians as points on the unit circle, then I project each point onto the x axis to find the cosine of each radian.

Like Hamid, a high proportion of the teachers under this experimental instruction gave evidence of the $t \rightarrow P(t)$, Circ, and projection processes and their coordination to estimate the values of the trigonometric functions of non-standard angles, articulate the process of finding the value of the sine and cosine of an angle, as well as to derive and explain properties of the trigonometric functions. In question 5, students had $\cos(22\pi/7) = -0.90096887$ and we asked them to find $cos^{-1}(-0.90096887)$. Consider the answer of Samira (Figure 2(b)) who had problems with the inverse of trigonometric functions in the initial interview.

Samira: We can write $22\pi/7$ as $3\pi + \pi/7$, so the angle is in the third quadrant, now I project to the x axis, the range of cos^{-1} is from 0 to π to be a one-to-one function, so the answer of $cos^{-1}(-0.90096887\ )$ can't be $22\pi/7$, I need to reflect $22\pi/7$ respect to the x-axis, the final answer will be $\pi - \pi/7$ in the second quadrant. I can see both $\cos 22\pi/7$ and $\cos(\pi - \pi/7)$ are equal to -0.90096887 but the cosine inverse of -0.90096887 is only $\pi - \pi/7$.

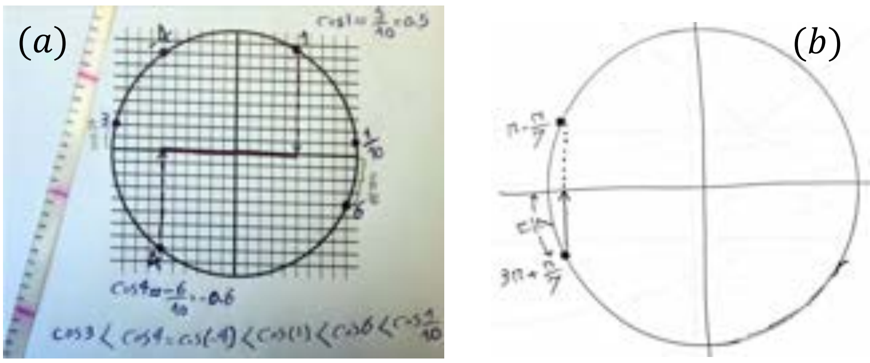

Figure 2: (a) Hamid's response to question 3, (b) Samira's response to question 5.

Overall, the pre-service teachers gave evidence consistent with the Range process when they recognized that they were looking for a value in the interval from 0 to π.

DISCUSSION AND CONCLUSION

This study was designed to examine aspects of the pre-service teachers' understanding of sine and cosine functions and their inverse based on the unit circle approach and the effect of a set of activities, designed to help construct these concepts as proposed in the GD. We found that in the initial interview, the pre-service teachers were not yet able to draw angles given in radian measure, approximate values of sine and cosine for non-integer multiples of $\pi/6$ and $\pi/4$, and apply inverse trigonometric functions. Based on the initial interview, the teachers did not express sufficient knowledge of trigonometric functions to teach the topic. However, the research-based activities, designed to help students do the constructions proposed in the GD, succeeded in having most participants overcome the difficulties observed in the initial interview and show improvement as documented in the exit interview. This is an important contribution of our research. Indeed, this study contributes to better understanding how students may construct sine, cosine, and their inverse functions based on the unit circle approach that is proposed in the GD. Our research is in line with the studies carried out by Tallman and Frank (2020) where they studied secondary teachers' knowledge of sine and cosine values and reported their difficulties. However, we add to their contribution by considering teachers' understanding of inverse trigonometric functions, emphasizing the importance of teachers' being able to imagine reverting the definition of sine and cosine (from number, to corresponding axis, to points on the circle, to considering range, to arc-length and angle). Like Weber et al. (2020) we considered the process of inverse trigonometric functions, however, we also investigated the influence of research-based activities on learning radian measure, sine and cosine functions, and the inverse of these functions. Finally, the results suggest that, in general, the participating students in Iran and Czech Republic tended to construct the same type of structures dealing with basic trigonometry, regardless of possible differences in their prior preparation.

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Klein's second discontinuity: the case of proportion theory

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This communication addresses the issue of teachers' professional development about the teaching of proportionality, taking as a perspective the so-called Klein double discontinuity. It appears that university generally proposes topics only remotely related to proportion theory: linear spaces and applications for instance. We advocate the idea that in such situations, the identification of difficulties and of praxeological needs of the teacher profession is a first step in the development of an object to be studied at the university with a view towards the training of teachers. We propose such an object in the case of proportion theory and indicate ways of studying it at the university.

Keywords: Proportion, Klein's double discontinuity, Transition to, across and from university mathematics, Curricular and institutional issues concerning the teaching of mathematics at university level, Preparation and training of university mathematics teachers.

INTRODUCTION

The study of the double discontinuity identified by Klein has been revived in (Winsløw & Grønbæk, 2014). Our aim is to determine to what extent it is possible to understand difficulties of the profession regarding the teaching of proportionality in the frame of Klein's double discontinuity. A first movement in this direction consists in developing a praxeological reference model (Chevallard, 2020; Fonseca, Gascón & Oliveira, 2014; Gascón, 2001) of proportion at the level of university mathematics in order to determine what could the *relations* to proportion be at university and in primary and secondary schools (for a theory of *relations*, see below and Chevallard, 2020). Proportion theory has been the object of many investigations. The relation between quantities and proportion is well explained in (Freudenthal, 1983). Theories of quantities are developed in (Chevallard & Bosch, 2002; Rouche, 1994; Steiner, 1969; Whitney, 1968a and 1968b); the history of teaching quantities is addressed in (Chambris, 2010; Chambris & Vinovska, 2021). Many works are focused on the issue of proportionality: in a historical perspective (Hersant, 2005; Wijayanti & Bosch, 2018; see also Comin 2002 for an instant picture of the teaching of proportionality at school); in relation to "algebrization" processes (Bolea, 2002 and Bolea, Bosch & Gascón, 2001); or as a first step towards functional relations (Garcia, 2005). Works have also been realised in the perspective of the curricular question (Burgos & Godino, 2020).

To our knowledge, no work has been dedicated to the issue of the transition "to, across and from" university mathematics in the case of proportionality. We will present the theoretical framework that will be used in the paper, then develop a possible relation to proportionality at the level of university and derive some implications on the teaching of proportionality at school. Finally, we discuss Klein's double discontinuity in the case of proportionality.

THEORETICAL FRAMEWORK

We set ourselves in the frame of the anthropological theory of the didactic (ATD, Chevallard, 2020 for instance), and more specifically we will use the theory of *relations*. We therefore denote $R(x, o)$ the relation of a person x to an object o, that is the set of ways x enters in relation to o: what x knows about o, what x can do with o, etc. In ATD, any group of persons can be considered as an *institution*: a family, a university, a school, etc. Persons are subject to many institutions and occupy *positions* in institutions: the position of father in a family, the position of student in university, the position of pupil in high school, etc. Given an institutional position (p, I), the relation of position p to an object o is denoted by $R_I(p, o)$. The so-called Klein double discontinuity can be formalized in the following way (Winsløw & Grønbæk, 2014): students at high school or lower secondary school (position (s, HS)) enter in relation $R_{HS}(s, o)$ with an object o (as they "study it"); students at university (position (s, U)) deal with an object ω (they "study it"), developing a relation denoted by $R_U(s, \omega)$; teachers in high or lower secondary school (position (t, HS)) have a relation $R_{HS}(t, o)$ to object o (as they "teach it"). The double discontinuity appears when object ω is considered to have a close relation to o (for instance: o is perceived as the didactic transpose of ω from U to HS): students transiting from HS to U have to study some object *nominally identical to o*, object ω, which however deeply differs from it, to such an extent as to not to be recognizable (first discontinuity); teachers have in turn to teach o while having been trained in knowing and using ω (second discontinuity).

One issue is to have relation $R_{HS}(t, o)$ evolve towards a new relation $*R_{HS}(t, o)$ to o: teachers should know more and better about o than what they learnt about it as students in HS. The study of objects such as ω should improve the praxeological equipment of the profession regarding o. However, Klein's double discontinuity supports the assumption that the discrepancy between $R_U(s, \omega)$ and $R_{HS}(t, o)$ is such that teachers are left with $R_{HS}(s, o)$ to develop their professional relation $R_{HS}(t, o)$:

> "since [the teacher] was scarcely able, unaided, to discern any connection between [the task of teaching o] and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching" (Klein, 1908/1932, quoted by Winsløw and Grønbæk, 2014)

An important research question is therefore the following: how, and in which institutions, can the effort to change $R_{HS}(t, o)$ be undertaken? It can be advocated that university, through teachers' training institutions, should take its part in the process, notably by developing a new relation to a possibly new object more closely related to o. For this reason, we consider in the following that both relations $R_{HS}(t, o)$ and $R_U(s,$

ω) may be simultaneously unsatisfactory, and that new relations $*R_U(s, *\omega)$ and $*R_{HS}(t, o)$ should then be developed[1].

PROPORTION THEORY: PRAXEOLOGICAL NEEDS OF THE PROFESSION

The teaching of proportion theory in high school and in lower secondary school raises a series of difficulties. The definition is often (be it implicitly) based on multiplicative viewpoints between quantities; the existence of a *constant quotient quantity*, the *coefficient of proportionality*, is the main demand for a situation to be said to be proportional. For instance, LeTourneau, Posamentier and Ford (2009, p. 416-418, our emphasis) state that “a ratio is a way of comparing two numbers or quantities by division” and that “a proportion is a number sentence stating that *two ratios are equal*”. A French textbook, Sesamath (2013), provides the following definition (our translation): “Two quantities are proportional if the values of the one are obtained by multiplying the values of the other by the same nonzero number”. The two definitions require an additional precision: how can we decide which values of both quantities must be compared? How can we say that a value of the first quantity *corresponds to* a value of the second quantity? What criterion do we have for “correspondence”?

In France, most exercises can be classified into one of the two following classes: exercises that demand whether a situation is proportional or not; exercises which ask for the determination of an unknown quantity or number (for instance when they are set in a worldly situation, where proportionality is assumed –e.g. renting eBikes without fixed fee, buying fruits at a grocer’s, etc.). It is seldom the case that an exercise asks for the realisation of both types of tasks. In fact, determining whether a situation is proportional is often made by analysing a given series of values of both quantities, and checking the equality of ratios. If one value were to be determined, it would mean that the series of values be not complete and it would be impossible to calculate at least one of the ratios. There appears a circle: if one has to check whether two quantities are proportional, they have to calculate ratios; if one tries to determine an unknown quantity, they could want to base themselves on the assumption that the situation is proportional, but they couldn’t check it since one value is missing. How could we define proportionality, so that it could be checked without appealing to the calculation of ratios? As the case of worldly situation suggests it, determining whether a situation is proportional could be considered a *modelling issue*. However, in classrooms in France, the model is usually accepted without further study of the modelled system. The question is raised: how should we decide if a system can be modelled by a proportional model?

[1] The symbol “*” indicates that the object or relation it precedes is new in comparison to another state of the same object or relation. Object *ω thus replaces object ω.

These difficulties point towards praxeological needs of the profession: some types of tasks should be mastered by the profession, which are not even acknowledged as problematic.

PROPORTIONALITY: THEORIZATION AT UNIVERSITY LEVEL AND CONSEQUENCES

The "revisited" Klein double discontinuity, as proposed by Winsløw and Grønbæk (2014), presupposes that an object $*\omega$ corresponds at least grossly to the object *o* to be studied in high school or secondary school. In the case of proportion theory, we consider that this object $*\omega$ is not easily found: it would possibly be "linearity", that is, in *U*, linear spaces and linear applications among linear spaces. However, "proportion", as such, does not appear in these courses, and they do not provide tools for the modelling of worldly situations: our claim is that a thorough theory of quantities is needed for this, in order to anchor proportionality in the ground of concrete systems under study.

As a consequence, solving the discontinuity requires the production of an object of study at university, say $*\omega$, and of two relations $*R_U(s, *\omega)$ and $*R_{HS}(t, o)$. In the sequel, we set ourselves the task of describing an investigation into this problem (see Planchon, 2022 for an example of a similar study in the case of integral calculus). More precisely, we indicate some difficulties of the profession of mathematics teacher in *HS*, and the way they can translate into mathematical needs at the university level. We also try to provide some elements of university mathematics that could fulfil these needs.

An object-based theory of proportionality ($*R_U(s, *\omega)$)

In the sequel, we assume that the reader is familiar with a theory of quantities (Whitney, 1968a and 1968b; Steiner, 1969; Rouche, 1994; Freudenthal, 1983; Chevallard & Bosch, 2002). Let X be a set of objects, equipped with a pre-order $\prec$, an internal law (called "combination of objects") $\oplus$, and an equivalence relation $\sim$ which is compatible with the pre-order and the combination. Roughly speaking, a quantity is the quotient space $Q = X/\sim$ equipped with the projection of the combination and the pre-order on the quotient space (they give respectively addition and order on quantities). Combination, pre-order and equivalence are subject to a series of axioms that we leave it to the reader to discover in, for instance, (Chevallard & Bosch, 2002). We will call $(X, \oplus, \prec, \sim)$ a *structured family* if it satisfies these axioms. The same notation Q will be used to denote both the quotient space $X/\sim$ and the canonical projection from X onto Q.

This object-based theory of quantities *could be* part of $*\omega$. Another part would be the conceptualisation of proportionality based on it. We take advantage of the previously introduced theory of quantities to propose a possible definition of *situations of proportionality*. We believe that proportion, ratios and rates are better understood in the context of *situations*[2] relative to objects and their social uses; a consequence is that, rather than coefficients of proportionality, rates or equal ratios, we will consider *situations* as the grounding material of the theory of proportions we wish to develop.

Vocabulary: A *double situation* $(\Xi; X_1, \oplus_1, \prec_1, \sim_1; X_2, \oplus_2, \prec_2, \sim_2)$ is given by a family of objects Ξ on which are defined two quantities Q_1 $(=X_1/\sim_1)$ and Q_2 $(=X_2/\sim_2)$. In such a double situation, we will say that a quantity q is *associated to an object* ξ in Ξ for quantity Q_1 if

- *q is the value of a quantity of type Q_1,*
- *and $q = Q_1(\xi)$.*

Two quantities q_1 and q_2 *correspond to each other* in a given (double) situation if q_1 and q_2 are associated to one and the same object ξ in Ξ.

This solves a difficulty of the profession: "corresponding quantities" are but quantities *associated to the same object*. Let us now introduce a definition of proportionality, which is entirely formulated at the level of of objects and their social uses, that is, of *structured families*, without reference to quantities.

Definition: Let $(\Xi; X_1, \oplus_1, \prec_1, \sim_1; X_2, \oplus_2, \prec_2, \sim_2)$ be a double situation. It defines a *situation of proportionality* if:

For all ξ and ξ' in Ξ and for $i = 1$ and $i = 2$,

$\xi \sim_1 \xi'$ if and only if $\xi \sim_2 \xi'$,

$\xi \prec_1 \xi'$ if and only if $\xi \prec_2 \xi'$,

$\xi \oplus_1 \xi' \sim_i \xi \oplus_2 \xi'$.

This definition says that, for a situation to be proportional, the objects should be ordered and classified in the same way for both quantities: if any metal rod from a family of metal rods is shorter than another if and only if it is also lighter, if a metal rod weighs as much as another one if and only if they are of the same length, and if the combination (say, gluing both rods together) for length and the combination (say, again, gluing both rods) for weigh, yield equivalent objects for both length and weigh, then, the situation provided by this family of metal rods is said to be proportional.

The following theorem is here to state that this viewpoint is *equivalent* to traditional views on proportion.

Theorem: Let $(\Xi; X_1, \oplus_1, \prec_1, \sim_1; X_2, \oplus_2, \prec_2, \sim_2)$ be a double situation. Let Q_1 and Q_2 the quantities naturally induced by $(X_1, \oplus_1, \prec_1, \sim_1)$ and $(X_2, \oplus_2, \prec_2, \sim_2)$ respectively, by passing to the quotient spaces.

Then the following three statements are equivalent:

[2] Let us emphasize here that, in this article, the word "situation" will *not* be ascribed the meaning given this word in Brousseau's *theory of didactic situations* (TDS; see Brousseau, 1997).

1. For any object ξ in Ξ, and any number a, quantities $a\,Q_1(\xi)$ and $a\,Q_2(\xi)$ correspond to each other.
2. There exist a quotient quantity λ such that, for any object ξ in Ξ, $Q_2(\xi) = \lambda\,Q_1(\xi)$ (or, for any two objects ξ and ξ' in Ξ, $Q_1(\xi)Q_2(\xi') = Q_2(\xi)Q_1(\xi')$).
3. The double situation is a situation of proportionality in the sense of the previous definition.

This theorem is proved in (Bourgade & Durringer, submitted; see also Bourgade & Durringer, 2023). Items 1 and 2 correspond to classical definitions of proportionality. Item 3 states that the two structured families (X_1, $\oplus_1$, $\prec_1$, $\sim_1$) and (X_2, $\oplus_2$, $\prec_2$, $\sim_2$) yield the same classification and ordering of elements of Ξ, in compatibility with both combinations. To check if a situation is proportional, it is required to investigate *at the level of the modelled system*: what do people do, in practice, with objects? How do they classify them, how do they order them, how do they combine them? In particular, this provides a setting for modelling issues: instead of assuming proportionality, students could investigate to determine whether the axioms in the above definition are satisfied.

Consequences for teaching proportionality (*$R_{HS}(t, o)$)

More specifically, it is important to note that determining whether a "situation" can be modelled as a situation of proportionality, or not, requires investigations: one has to know about paint buckets, about postal rates, about the growth of human beings, etc. Significantly enough, exercises usually leave aside the issue of *determining whether a situation is proportional or not*. It can be asked to determine whether a table is proportional, but the very issue of identifying *situations* of proportionality remains seldom studied – or left to the intuition of the student.

Let us consider an exercise, given to grade 7 students in France: "15 kg of wheat yield 12 kg of flour. *We assume that* the quantity of wheat and the quantity of flour are proportional. a. Calculate the coefficient of proportionality. What is its signification in this situation? b. Calculate quantities 2 and 3 [missing in a table provided in the exercise]. Interpret these results for the situation" (our emphasis). One could ask for more: could it be that we *knew* that proportional modelling is fit for this situation without having to *assume* it? According to the formal setting that we have introduced in this paper, it would be sufficient to investigate whether 1. a lesser quantity of wheat would lead to a lesser quantity of flour and conversely; 2. an equivalent quantity of wheat would lead to an equivalent quantity of flour and conversely; 3. milling two sacks of wheat separately or together would produce the same quantity of flour. The situation can correctly be modelled by a situation of proportionality if the answer to the three questions is, up to a desirable level of precision, 'yes'. Let us underline that it is most frequently a *choice* to model a system by a relation of proportionality: in many cases, the fit is only approximate and it is however decided to model the system by a proportion relation in order to yield interesting features of the system.

Rather than leaving it to the intuition of the student, such questions could be raised in classrooms and investigated thoroughly in order to design a set of types of situations that can be "assumed" to be of proportionality. Without such investigations being performed by the students, 1. it is left to their intuition to decide whether the proportionality model is adapted or not, 2. part of the mathematical activity –the modelling activity– is left aside. In our opinion, it is important that students' relations to proportionality be not limited to the managing of tables of numbers: actual situations in daily world require an investigation into quantities and their measurement, which entails a direct contact with objects.

Another important consequence for the teaching is the improvement of techniques to calculate unknown quantities: additive techniques derive from the definition; multiplicative techniques derive from the Theorem above. Also, to check whether a situation is *not* proportional, it is sufficient to exhibit two quantities that do not evolve monotonously one with respect to the other.

KLEIN'S DOUBLE DISCONTINUITY AND PRAXEOLOGICAL NEEDS

As we mentioned it in the previous section, it was quite difficult to identify *a priori* an object $*\omega$ close enough to o = "proportionality", living in university mathematics. Developing a consistent theory of proportion starting from an object-based theory of quantities made it possible to identify university mathematics that are more closely related to proportionality than it appeared at first glance. Specifically, general algebra (internal laws, relations, quotient spaces, etc.) is at the core of our mathematization of proportion. Our investigation into proportionality theory was led in a procognitive attitude—i.e. assuming that the necessary mathematical tools are probably not yet available; in contrast, we could have assumed that what we knew about proportion (its relation to linearity in particular) was all there was to know about it—retrocognitive attitude. More generally, the identification of a relevant piece of university mathematics in relation with an object o studied at school must be the result of a genuine investigation: one must consider that things may be learnt in the process about o and its relation to university mathematics. We do not pretend to have derived the one and only solution to the Klein double discontinuity in the case of proportion theory—even though we advocate that any attempt in this direction should take it seriously to avoid reduction of proportion theory to linearity: linearity appears to us as a consequence of proportionality rather than as a cause for it.

Our proposal for aiming at a possible solution of the second part of Klein's double discontinuity here takes the following form:

$$R_{HS}(s, \text{proportionality}) \rightarrow R_U(s, \text{general algebra})$$
$$\rightarrow *R_U(s, \text{object-based proportionality theory}) \rightarrow *R_{HS}(t, \text{proportionality})$$

The first hypothetical relation $*R_U(s$, object-based proportionality theory) makes it necessary to imagine due training in object-based theories of quantities and proportion at university (solving the second discontinuity); the second hypothetical relation

*R_{HS}(t, proportionality) is that of a teacher trained in these theories and capable of adapting his or her teaching of proportion theory at School level—in particular to be able to engage in modelling processes and asking such questions as the following ones: given an exercise exhibiting a double situation, which families of objects are considered? Which combinations of objects are used? What are the quantities under consideration? However, teaching proportionality requires that teachers be confronted, during their initial training, to the question of the mathematization of proportionality: it could take the form of an investigation into the actual difficulties of the profession related to teaching proportionality, giving rise to the need of mathematical complements. At this point several paths could emerge, hopefully including the one we propose in this communication.

CONCLUSION

Dealing with Klein's double discontinuity should be coupled to two supplementary considerations. From an institutional point of view, it should be stressed that identifying university mathematics relevant to the training in the teaching of a given object is a difficult task. In the case of proportionality, we ended up considering abstract algebra as a good setting for theorizing modelling issues. Teachers' professional development should take into consideration such issues and their mathematical treatment. We advocate that the mathematical setting we propose may be part of an answer to these issues.

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Frequency calculation of trigonometric polynomials when modeling sound waves

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In the teaching of trigonometric functions, the periodicity property is one of the most important, since traditionally, from it, periodic functions are formalized. In the present research we have proposed to characterize the mathematical work of preservice teachers when solving a modeling task of periodic functions. We present the results of the third lesson of a sequence whose objective was to determine the conceptualization of periodic functions based on sound phenomena. The situation was implemented with 13 mathematics teachers in initial training at a Chilean university. The data analysis was carried out using the Mathematical Working Space. The results show that the preservice teachers construct the trigonometric polynomial and calculate the frequency of this, coordinating semiotic and instrumental aspects both in Geogebra and Audacity.

Keywords: teaching analysis, periodical functions, trigonometric polynomial, modelling, digital technologies.

INTRODUCTION

There is a close relationship between periodic functions and trigonometric functions, which can be evidenced in two aspects. On the one hand, periodicity emerges as a constitutive property of trigonometric functions that historically arise from the mathematization of the motion of objects. On the other hand, periodic functions are related to physical phenomena such as the pendulum or spring-mass system. In this sense, research reports that there is a predominance of algebraic work on these functions (Kepceoğlu & Yavuz, 2016; Tun, 2017), making it difficult to visualize periodic functions in the graphic register, which implies that the property of periodicity remains invisible in front of others, appearing as a transfer of the sine function that emerges as a solution of differential equations that model certain phenomena (Buendía, 2006).

It is evidenced in the current curricular organization, that the teaching of trigonometric functions is carried out from an extension of geometry, generating an obstacle in the learning of these, as there is a difficulty in the relationship between angles and radians (Moore, 2013; Demir & Heck, 2013; Tuna, 2013). Therefore, favoring the learning of trigonometric functions from other variables, different from those traditionally addressed, can enrich the conceptualization of trigonometric functions, which, in turn, can favor the process of conceptualization of periodic functions.

We consider that designing a sequence for the teaching of periodic functions from sound phenomena, whose variables involved in their modeling are associated to time and wave pressure, will contribute, on the one hand, to enrich the understanding of periodic functions and, on the other hand, to contribute to the conceptualization of trigonometric functions from phenomena in which the variables are not associated to angles, avoiding the problems associated to trigonometric ratios and the confusion between ratio and function.

As a result of the above, we have proposed to characterize the mathematical work of preservice teachers in solving a modeling task of periodic functions.

FRAMEWORK

Based on the proposed objective and the design of the task, we used the Theory Mathematical Working Space [ThMWS] (Kuzniak et al., 2022)

To characterize mathematical work, the theory considers epistemological and cognitive aspects that are organized in two planes: the epistemological plane and the cognitive plane. In each of these planes, three components are considered, organized in three dimensions, also called genesis, which are considered essential in mathematical work (figure 1).

- Semiotic genesis that considers visualization as a cognitive process where an individual gives meaning to the mathematical signs (considered, for example, in semiotic representation registers) of the representamen component.

- Instrumental genesis where an individual transforms an artifact (which may be material, digital or symbolic) into an instrument for use in a cognitive process of construction.

- Discursive genesis in which an individual uses the mathematical knowledge (definitions, theorems, properties, …) of the theoretical referential in the cognitive process of verification (in a broad sense).

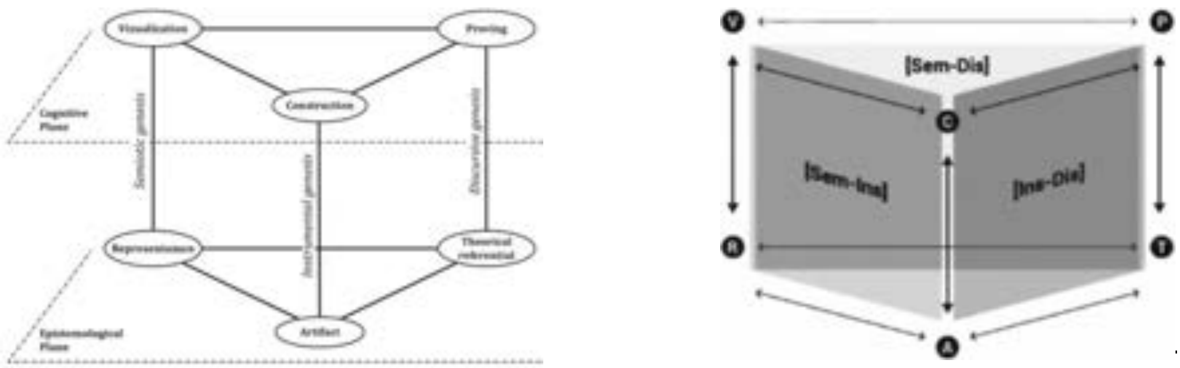

Figure 1. Genesis, horizontal and vertical planes of MWS (Kuzniak et al., 2022)

These geneses are related since knowledge is needed to be able to visualize mathematical signs or to use an artifact effectively; demonstration uses semiotic representations of mathematical objects; construction creates new mathematical signs, etc. ThMWS seeks to understand the role of each of these dimensions and components in an integrated system that gives rise to mathematical work. The articulation and interrelationships between the different genesis are considered through vertical planes. Each of them highlights the preponderant role of two genesis: the semiotic-instrumental plane [Sem-Ins], the instrumental-discursive plane [Ins-Dis] and the semiotic-discursive plane [Sem-Dis].

CONTEXT AND METOD

The methodology that guides this research is Didactic Engineering (Artigue, 1995). Implementation is made up of 13 preservice teacher who are in the fifth semester of the nine-semester program at a Chilean university. They formed six working pairs, which were identified as G1 to G6; only the G6 group consisted of three participants.

The implementation was developed in the context of the subject of didactics of functions. This is a compulsory subject in the training of future teachers. The objective of this course is to deepen the study of different functions present in the Chilean curriculum based on research carried out in the area of didactics of mathematics. The sound phenomenon was used to give an interdisciplinary sense to the mathematical work with the preservice teachers in which they had to model sound waves for the learning of periodic functions.

The design of the proposal aimed to generate a technology-mediated modeling process for learning periodic functions from sound phenomena. For this purpose, the incorporation of technological tools such as Geogebra and Audacity was considered. This sequence is composed of four didactic situation whose objectives promote the achievement of the proposed research objective. The four situations were implemented in 6 sessions of 90 minutes each.

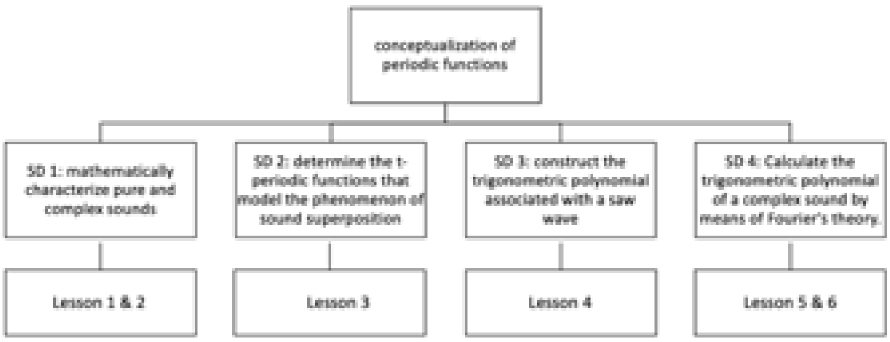

Figure 2. Objective of each didactic situations and lesson implemented.

In the following section we will present the results obtained in lesson 3, in which preservice teachers are expected to construct a trigonometric polynomial that models the superposition of waves. To accomplish this objective, in the second lesson the teacher institutionalized the concept of pure sound as one that could be modeled by a sinusoidal function and complex sounds as those that could not. Preservice teachers modeled pure sounds from the instrumental mathematical knowledge of the Geogebra software and their knowledge of music and physics (Cabrera et al., 2022).

The data were extracted from the written productions of the preservice teacher, from the recording of the screen of the work computer of each of the groups when solving the proposed tasks. The mathematical work will be analyzed in the sense of the ThMWS, in each of the proposed tasks, analyzing the activation of the genesis, the planes [Sem-Ins], [Ins-Dis], [Sem -Dis] and the circulations between them, carried out by the different groups of teachers in training that compose the case.

In addition to the above, in the procedures centered on sound phenomena and, in which - a priori - mathematical knowledge is not mobilized, we will analyze the semiotic and instrumental dimensions, which we will put in correspondence with the ThMWS.

In relation to the a priori analysis, students were expected to construct and mix sounds based on the instrumentalization of the Audacity software. From the mathematical work developed in session 2, students exported the data in numerical format to plot the waves in Geogebra. The frequency calculation is expected to be determined from the Audacity instrumentation, as in session 1, or from the Geogebra instrumentation. To determine the frequency of the trigonometric polynomial, students are expected to identify through an exploratory process, as harmonics are added, that it corresponds to the mcd of the mixed frequencies.

RESULTS

In the following, the results obtained by the 6 groups when solving the second didactic situation will be presented. The objective of this situation was: *determine the function that models the phenomenon of superposition of pure sounds.* As mentioned above, this task was implemented in the third class of the process.

The first task assigned to the preservice was: *Construct in Audacity the pure sounds of 220 Hz with an amplitude of 1 and 330 Hz with an amplitude of 0.8, mix them and determine the mathematical function that models it.* On the one hand, preservice build each of the sounds correctly as performed in lesson 2. Once the preservice teacher constructed both sounds, from the instrumental knowledge of the Audacity software, they mixed both sounds obtaining a graphical representation of this new wave.

Preservice teachers mathematically determined the function that modeled each of the pure sounds, thanks to the work done in the previous lesson. Subsequently, preservice teacher began to identify relationships in the Audacity interface that allowed them to understand what mathematical operation could be associated with the sound mix and the result obtained.

Within the strategies used, G3 recognizes, from their knowledge of what mixing sounds represents, that the mix created is the sum of both sounds. This is because they compare certain values of each of the sounds and observe the amplitude of the mixed sound in the Audacity software, generating an instrumental semiotic process in Audacity.

Subsequently, they determine the function that models each pure sound from their knowledge of trigonometric functions and graph them in Geogebra to construct the sum function of these. This function is compared graphically with the one visualized in Audacity, establishing a comparison process through an instrumental semiotic process in Geogebra and Audacity simultaneously. G3 recognizes from the graph of each pure sound in Geogebra, the behavior of the sum function by analyzing specific instants of each sound and observing the behavior of the sum function. In figure 3, it can observe the graphical view in Audacity and Geogebra of the same sound.

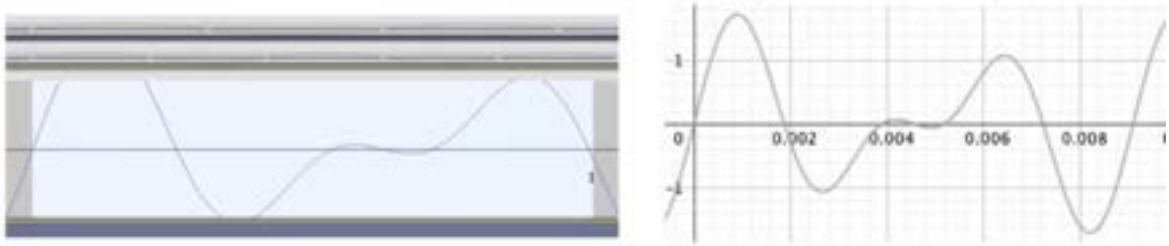

Figure 3. Graphical comparison between Audacity and geogebra

In this process, we can observe that G3 develops a coordination process by activating the [Sem-Dis] plane and then the [Sem-Ins] plane of his personal ThMWS, since from the theoretical knowledge about functions and the concept of sum associating it to the mix, they determine graphically some values that allow them to confirm this hypothesis. Finally, they plot the sum function in Geogebra to compare it visually with the Audacity waveform. Figure 4 shows the transition between the planes describing the mathematical work done by the preservice teachers.

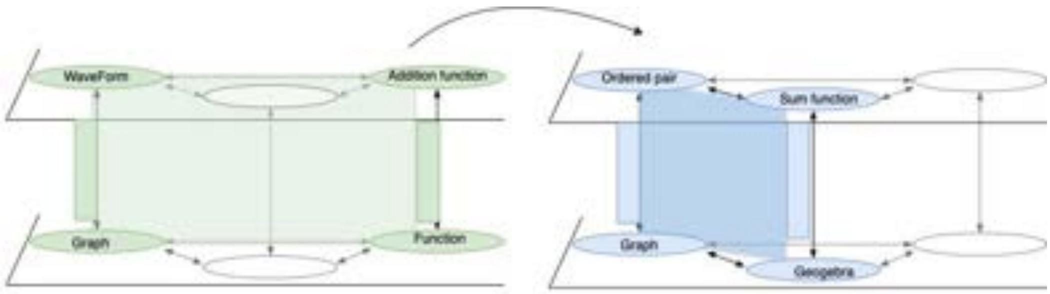

Figure 4. Transition between [Sem-Dis] and [Sem-Ins] plane of G3.

In general terms, each of the groups determined that the function modeling the mixture of sounds was: $f(x) = \sin(2\pi * 220x) + 0{,}8\sin(2\pi * 330x)$. The preservice teacher verified that this function was the one that modeled the sounds from the confrontation between the Geogebra graph and the Audacity graph, determining its behavior locally in some specific intervals.

From the work done by the preservice teacher, we recognize a semiotic process that guides the work, as they make comparisons between the graphs to validate their constructed models. This semiotic process is supported by the instrumental work that allows to obtain approximations, both in Audacity and Geogebra in a coordinated way, therefore, we recognize that preservice teachers, in the process of validation of the obtained model, coordinate the Geogebra interface with Audacity. In this instance, a process is developed where they circulate between plane [Sem - Ins] from Geogebra with the instrumental semiotic dialectic in Audacity (figure 5).

On the other hand, the semiotic process of the preservice teachers is directed by the knowledge that they possess about the mix understood as a sum of functions, this knowledge is typical of the first years of schooling in which the mixing of elements from different sets is associated with the sum of elements that form a new larger set. Due to the above, the circulation [Sem-Ins] is influenced by the discursive genesis that allows understanding the mix as sum.

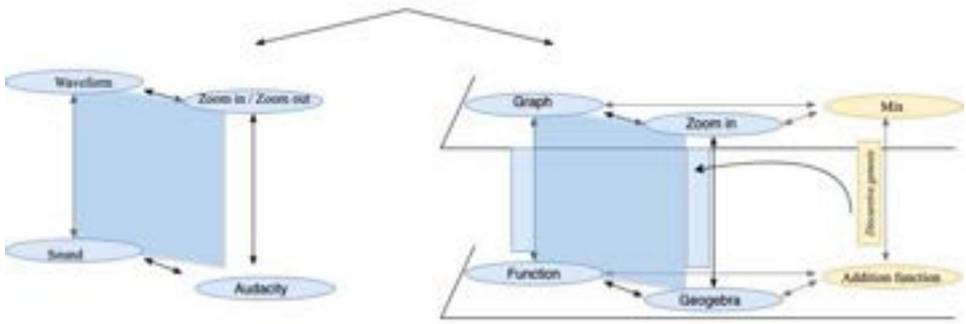

Figure 5. Coordination between plane [Sem-Ins] and instrumental semiotic dialectics in Audacity.

To determine the frequency of the sound created from the sound mix, the groups used three different strategies to determine it. These strategies focused on graphically recording the sound wave in either Audacity or Geogebra to determine the interval at which the wave meets a cycle.

Groups G1, G4 and G6 plotted the function $f(x) = \sin(2\pi * 220x) + 0.8\sin(2\pi * 330x)$, located two points that are in the same phase and subsequently, determine the distance between these values. Subsequently, they calculated the multiplicative inverse of this value, which corresponds to the frequency. In this strategy, preservice teachers use the instrumental knowledge of Geogebra and the concept of period to locate two points that are in the same phase and that are one period apart. The concept of frequency is based on physical knowledge as they calculate it from the inverse of the period.

On the other hand, G5 calculates the frequency of the sound, as in the previous groups, as the inverse of the period. The period is calculated from instrumental knowledge of Audacity, in which they select approximately one interval where the wave cycles to determine the period of the sound. Unlike the work done in Geogebra, in Audacity the preservice teachers determine approximately the interval where the wave makes one cycle. Finally, G2 and G3 use the graphical representation of the wave in Audacity to approximately determine the value of the period from the data table exported in Excel. In particular, G2 determines when the wave performs five cycles approximately and then, they place in the table the time it takes for the wave to perform five cycles, finally, they divide this time obtained by five to find the period of a wave.

From the three processes described above, we can recognize three different processes: in the first one, the calculation of the frequency is based on the circulation [Sem-Dis], which is supported by the instrumental work done by the group based on the precision with which they locate the selected points on the graphed function. This process is similar to the third strategy, however, in the latter, the semiotic process is based on the search for numerical regularities, unlike the calculation of the frequency through the Geogebra software. In this third strategy, there is no instrumental process to support the circulation [Sem-Dis]. In the second strategy, the frequency calculation is based on the Audacity software and, therefore, is performed from extra-mathematical knowledge. In figure 6, it can observe the different process preforming for the groups to calculate the frequency.

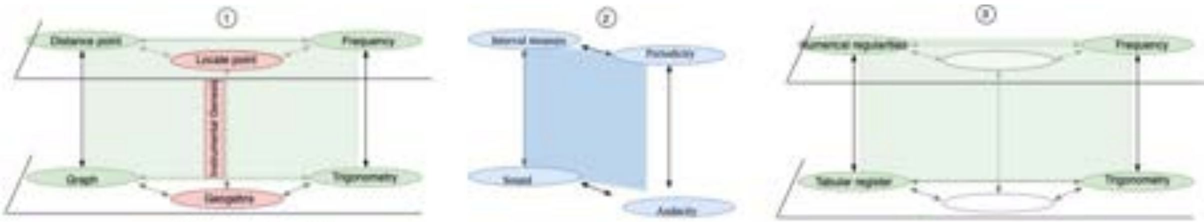

Figure 6. Three different processes to calculate the sound frequency.

After performing the first task, the teacher presented a second task to the preservice teachers: *modify the 330 Hz sound by a 440 Hz sound and determine the function that models this sound and then calculate the frequency.* In this task the groups repeat the strategy used previously, i.e., they recognize that the function that models the new sound is the sum of the functions that model both pure sounds, therefore, they determine that this is: $f(x) = \sin(2\pi * 220x) + 0.8\sin(2\pi * 440x)$. As before, the groups repeat the strategy used to calculate the frequency of the sounds. Unlike the previous case, some groups do not construct the sound mix in Audacity and focus purely on the function graph in Geogebra.

In this process, we can observe how the mathematical knowledge of the preservice teachers allows to determine the function that models this sound without the need to construct the sound, prevailing an instrumental-discursive work from the instrumentalization of Geogebra to graph the function that models this new mix.

From this situation, G6 began to deduce that the frequency was related to the frequencies of each pure sound, in this case, the group considers that the frequency corresponds to be the additive difference between the values of the frequencies, this hypothesis was proposed by the groups G2 and G3 in different instances of the session.

This strategy of searching for additive regularities to determine the value of the frequency may be due to strategies developed in the schooling process of the preservice teachers, since Chilean curricula give scarce emphasis to the search for non-linear patterns. Therefore, this groups strategy is based on the knowledge belonging to the theoretical referential of the preservice teachers. We recognize the activation of the [Ins-Dis] plane in groups that directly plot the function in Geogebra, without constructing the mix, and establish an additive relationship to determine the frequency based on the graph of the function (figure 7).

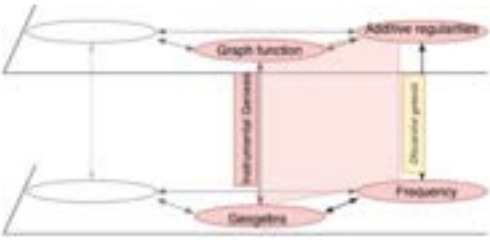

Figure 7. Activation of [Ins – Dis] plane to solve the second task.

Based on this hypothesis, the teacher presented the third task of the session: *add a third sound of 660 Hz with an amplitude of 0.6 and then a fourth sound of 880 Hz with an amplitude of 0.4 to the mix, and to determine the function that models this sound and calculate the frequency.* On the one hand, preservice teachers were expected to recognize that these third and fourth sounds were in addition to the previous ones and, therefore, now the function is a trinomial and then a quadrinomial. In relation to frequency, adding three sounds can no longer be considered as the additive difference between the mixed values, therefore, groups should look for other strategies to argue the value of frequency.

The 6 groups correctly determine the polynomial $f(x) = \sin(2\pi * 220x) + 0.8\sin(2\pi * 440x) + 0.6\sin(2\pi * 660x) + 0.4\sin(2\pi * 880x)$ that models the sum of the three mixed sounds and then four mixed sounds.

We can conclude that four groups use the strategies validated in question 1 to construct the polynomials that model the sounds, recognizing that the sound mix is the sum of these. This validation was made from the graphical confrontation of Geogebra and Audacity to analyze the graphical form of the sound mix and the function that modeled the superposition of sounds.

In relation to the calculation of frequency, the 6 groups determine the frequency of the trigonometric polynomial from instrumental knowledge that allows them to replicate previous strategies, determining the period in Geogebra from the intersection of the curve with the X-axis. Then, they determine the frequency as the inverse of the period by integrating knowledge of physics.

Regarding the arguments used by the groups to justify the value of the frequency, G1 indicates that it is due to the value of the difference between consecutive terms of pure sounds 880 - 660 = 660 - 440 = 440 - 220 = 220 Hz.

Groups G2, G5 and G6 determine that the frequency of the trigonometric polynomial is the greatest common divisor of the sounds, groups G5 and G6 also complement the analysis of the frequencies of the sounds with the relationship between the amplitude of the sounds. G2 is the only group that recognizes the phenomenon of wave interference as the one that represents the superposition of sounds, integrating the knowledge of physics to the construction of the mathematical model.

Groups G3 and G4 calculate the frequency correctly, however, they do not justify the reason for this value, focusing their attention only on the numerical calculation of the frequency and not on its relation between the components of the sound.

The procedures described above, both to validate the model and to calculate the frequency, are based on instrumental semiotic work by the preservice teachers. From the second task onwards, the preservice teachers focus their attention on the algebraic model, which they plot in Geogebra, without building the sound mixes in Audacity. In relation to periodic functions, preservice teachers recognize the periodicity of sounds from the software interface, which allows them to calculate the periodicity of polynomials through Geogebra instrumentation.

In this session the teacher formalized the concepts of addition of function which allowed him to formalize the trigonometric polynomials where the frequency of this polynomial corresponds to the maximum common divisor of the frequencies of each of the terms that compose the polynomial. In relation to sound phenomena, fundamental and harmonic concepts were formalized, as well as the construction of complex sounds thanks to the superposition of pure sounds that have a relationship between their frequencies.

CONCLUSION

We can conclude that the six groups build the model of the sound mix in Audacity from the instrumental knowledge of the software. In the same way, the six groups obtain the mathematical model expected for the sound mix. The validation of the models took place through the graphical confrontation between the sound wave represented in Audacity and the curve in Geogebra from local values of the function. In this process, preservice teachers mobilize the mathematical knowledge they have about periodic functions, sinusoidal functions and sum of functions, while the concept of frequency is recognized from their knowledge of physics.

In relation to the sound phenomenon, preservice teachers construct the first sound mix using Audacity software, however, in the following tasks of the situation preservice teachers do not use Audacity software and only vary the coefficients in the trigonometric polynomial, decontextualizing the task of the sound phenomenon and only staying in the mathematical context. We consider that this is due to the fact that the preservice teachers focus their attention on the algebraic expression of functions rather than on the sound phenomenon. This may be due to the context in which the research was developed, since the subject was focused on the study of functions and their didactics.

In respect of the proposed research objective, we can conclude that the preservice teachers construct the trigonometric polynomial of each proposed sound mix and correctly calculate its frequency, coordinating semiotic and instrumental aspects, with a predominance of Geogebra software over Audacity due to what was pointed out in the previous paragraph. Although the activation of discursive genesis is observed at the moment of establishing and basing the numerical relations to determine the frequency, this genesis complements the semiotic-instrumental circulation developed by the preservice teachers.

We can conclude from the described circulations, in relation to what is proposed by the ThMWS, that the designed situations are conducive to the learning of periodic and trigonometric functions, overcoming the difficulties associated with angles and radians.

The Chilean curriculum requires future teachers to design interdisciplinary and mathematical modeling tasks. Therefore, we consider it fundamental for initial teacher training to confront preservice teachers with modeling tasks that require knowledge of other disciplines in order to favor the conceptualization of different mathematical objects. This can help preservice teachers to design teaching proposals that favor an adequate conceptualization of mathematical objects, overcoming the didactic and epistemological obstacles reported in the literature, integrating digital tools.

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Didactic paradigms in the study of real numbers in the Degree in Mathematics

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We present an analysis of the current didactic paradigm of real numbers in the Degree of Mathematics. For this, we make a description that takes into account the strong relationship between the Degree of Mathematics and the community of researchers in mathematics. With this description at hand, we present an explanation for a certain well-known didactic phenomenon whose avoidance would require a completely new approach to the study of real numbers.

Keywords: curricular and institutional issues concerning the teaching of mathematics at university level, teaching and learning of analysis and calculus, didactic paradigm, epistemological model, praxeology.

INTRODUCTION

Certain features of real numbers (for instance, student's conception of completeness) have attracted the interest of many researchers in Mathematics Education, and have been studied in previous editions of the INDRUM (Berge, 2010, 2016; Durand-Guerrier, 2022; Hochmuth, 2018; Kidron, 2016; Tanguay & Durand-Guerrier, 2016; Vivier & Durand-Guerrier, 2016). Our work adopts an institutional approach, so let us start with the very idea of 'institution'. A *social institution* consists in a set of constitutive rules, stated by convention, which fix the following: A series of institutional positions; allowed, compulsory and forbidden actions for each position, and rewards and punishments for certain actions (Searle, 2010). Social sciences study the working of social institutions. *Didactic institutions* are those social institutions stablished with the aim of studying something. Here we will consider the following two examples of didactic institutions: the Degree of Mathematics, denoted by DegMath, and the community of researchers in mathematics, denoted by ResMath. We know that certain features of DegMath can be different at different countries, and also that certain characteristics of ResMath can be different at different research groups or branches in mathematics. Nevertheless, we think those are unified enough institution with respect to the features we focus on.

In this work we would like to analyse how the study of real numbers is carried out in the didactic institution DegMath. A complete analysis would entail the following questions:

- *descriptive analysis*: How does the study of real numbers in DegMath work? What is studied? How is it studied? What are the didactic ends pursued? Is what is studied coherent with the didactic ends?
- *backwards analysis*: Why does the current study of real numbers in DegMath work as it does? Where does this kind of study come from?
- *forwards analysis*: What should be done to change the current study of real numbers in DegMath in a certain direction?

DIDACTIC PARADIGMS

The notion of didactic paradigm

To deal with these questions we will use the notion of *didactic paradigm* introduced by Gascón and Nicolás (2018, 2019, 2022a). On one hand, it is a way of describing how the study of a certain field works at a certain institution. On the other hand, the explicit use of didactic paradigms in the analysis of the current modality of study of a didactic institution can help to emancipate ourselves from the self-vision of that didactic institution.

Given a didactic institution, I, and a certain field of study, F, in I, we say that a *didactic paradigm for F in I* is a 4-tuple, $DP_I(F) = [EM_I(F), DE_I(F), DM_I(F), D\varphi_I(F)]$, where:

- The *epistemological model*, written $EM_I(F)$, describes what is specifically studied (what types of questions, definitions, techniques, theorems, proofs, kind of proofs, etc.) in order to study F in I.
- The *didactic ends*, written $DE_I(F)$, explain the purpose of studying F in I.
- The *didactic means*, written $DM_I(F)$, state what is done in order to study F in I.
- The *didactic phenomena*, written $D\varphi_I(F)$, state what is to be avoided in the study of F in I.

Of course, there are strong links between these four components. For instance, $DE_I(F)$ and $D\varphi_I(F)$ determine to a large extent $EM_I(F)$ and $DM_I(F)$. On the other hand, $EM_I(F)$ and $DM_I(F)$ also condition each other. Below, we will see examples of didactic paradigms and of the links between the corresponding components.

Concerning notation, when we use the idea of didactic paradigm to describe a certain current study, we speak of a *current didactic paradigm for F in I*, and we write $CDP_I(F) = [CEM_I(F), CDE_I(F), CDM_I(F), CD\varphi_I(F)]$. When the idea of didactic paradigm is used to describe a possible way of studying, we speak of *reference didactic paradigm for F in I*, and we write $RDP_I(F) = [REM_I(F), RDE_I(F), RDM_I(F), RD\varphi_I(F)]$. When there is no need to specify F (for instance, because we are considering every possible field of study in I), we just write $DP_I = [EM_I, DE_I, DM_I, D\varphi_I]$.

Description of an epistemological model

In order to specify the $EM_I(F)$, we will use the idea of 'praxeology' (Chevallard, 1999), which is useful to describe what someone knows or is supposed to know about F in I. A *praxeology* is a 4-tuple $\Pi = (T, \tau, \theta, \Theta)$, were:

- T is a family of *types of tasks*,
- τ is a family of *techniques* (typically described in terms of actions), devoted to carry out the types of tasks of T,
- θ is the *technology*, devoted to explain why the techniques of τ work, and to assess the scope, the economy and the reliability of those techniques, and
- Θ is the *theory*, devoted to specify everything else concerning the knowledge of F in I.

In order to describe properly certain didactic phenomena, and even to describe properly a process of study, we need to specify certain ingredients in Θ. For this, we will use the general description of the theory of a praxeology proposed in (Gascón & Nicolás, 2022b), according to which a theory would have, at least:

- An *ontological* component, denoted by $\mathcal{O}$, which provides the language L used to speak of the field of study F, the interpretation *Int* of the non-logical terms of L (that is to say, terms other than $\neg, \wedge, \vee, \rightarrow, \forall, \exists$), and a list of *axioms*, which are elementary statements expressed in terms of L that we assume to hold (under *Int*) without any supporting argument.
- A *nomological* component, denoted by $\mathcal{N}$, made of *theorems*. Each theorem is a statement expressed in terms of L, and, contrary to what happens with axioms, theorems are not a starting point in the study of F, but the conclusions of a *valid arguments* with premises regarded as true under *Int* (either because they are, in turn, theorems, or because they are axioms).
- An *epistemological* component, denoted by $\mathcal{E}$, which states what kinds of arguments are valid to support theorems.

Therefore, in this work we will consider a praxeology to be $\Pi = (T, \tau, \theta, \Theta)$, with $\Theta = (\mathcal{O}, \mathcal{N}, \mathcal{E})$ and with $\mathcal{O} = (L, Int, Axioms)$.

If we need to specify the institution, I, and the area of study, F, we will use the notation $\Pi_I(F)$, $\mathcal{O}_I(F)$, $\mathcal{N}_I(F)$, $\mathcal{E}_I(F)$, etc. If these are elements of the current DP, then we will write $C\Pi_I(F)$, $C\mathcal{O}_I(F)$, $C\mathcal{N}_I(F)$, $C\mathcal{E}_I(F)$, etc. If the area of study in I is clear, or it is everything studied in I, then we will just omit the letter "F".

We can use a single (possibly big) praxeology to provide a *synchronic description* of $EM_I(F)$, that is to say, a description in one go of all that one has learned along the process of study. In a synchronic description we have just to specify the elements of the big praxeology obtained at the end of the study process. However, if we want to provide a *diachronic description* of $EM_I(F)$, taking into account the order in which the steps have been achieved along the study process, then we might better use a directed graph, were the nodes are praxeologies, and the arrows indicate that the praxeology in the tip is a single step evolution[1] of the praxeology in the tail. Such a diachronic description of $EM_I(F)$ is said to be a *diachronic epistemological model* of the study of F in I. In a diachronic epistemological model, we have what we call *initial praxeologies*, which are those praxeologies which are not in the tip of any arrow, and *final praxeologies*, which are those praxeologies which are not in the tail of any arrow. Among final praxeologies, we can distinguish between *failed praxeologies*, which are discarded attempts to advance in the study of F in I, and *successful praxeologies*, which provide the answers we were looking for at the beginning of the study process. Those praxeologies which are neither initial nor final are called *transition praxeologies*. Progressing through a diachronic EM involves finding praxeologies that offer increasingly better solutions to the issues raised by the initial praxeologies. Failed praxeologies represent dead ends in the study process, while successful praxeologies offer conclusive solutions. Of course, not every collection of praxeologies placed in the nodes of a directed graph is capable of describing the praxeological development of a possible study process. On the contrary, there must be some kind of epistemological plausibility in the passage from one praxeology to another.

Some relationships between didactic paradigms

Notice that in a didactic institution there might be two fields of study, F and G, such that $F \subseteq G$. For instance, in I = DegMath we can consider F = {real numbers} and G = {Calculus}, or G = {Differential Geometry}, or even G = {Mathematics}. In those cases, we have that any (not only the current one) $DP_I(F)$ is going to be strongly conditioned by the corresponding $DP_I(G)$. For instance, one of the didactic ends for the study of F will be to contribute effectively to the study of G. Also, $EM_I(F)$ will be somehow 'contained' in $EM_I(G)$. To express the relationship stablished between $DP_I(F)$ and $DP_I(G)$ when $F \subseteq G$ we will write $DP_I(F) \trianglelefteq DP_I(G)$.

Notice that there are didactic institutions where the main goal is not to produce new knowledge, but rather to teach something that is already known, at least by some experts in the subject. In these institutions there are two main positions: one devoted to study, denoted by X, and one devoted to help X to study, denoted by Y, and one individual is always either X or Y. This is the case, for instance, of DegMath. These particular kinds of didactic institutions will be called *educational institutions*. On the other hand, there are didactic institutions whose main goal is to produce new

[1] Of course, it is worth to analyse further the notion of 'single step evolution', the notion of contiguity of ideas, but not in this work.

knowledge, new even for the experts in the subject. In these institutions individuals play, at different moments, either the position of Y or the position of X. This is the case of ResMath, where one individual can play at different moments the role of X, for instance when attending a conference or reading a paper, and the role of Y, for instance when giving a lecture or writing a paper. These particular kinds of didactic institutions will be called *research institutions*.

Notice that, as said in every official programme, one of the didactic ends of DegMath is to educate members X to become suitable members of ResMath. We will write DegMath $\uparrow$ ResMath. This relationship entails that any $DP_{DegMath}$ is strongly conditioned by the corresponding $DP_{ResMath}$ (we write $DP_{DegMath} \uparrow DP_{ResMath}$), because $EM_{DegMath}$ has to be compatible with $EM_{ResMath}$.

THE STUDY OF REAL NUMBERS IN THE DEGREE OF MATHEMATICS

In this section we will sketch a descriptive analysis of the current didactic paradigm in DegMath for the study of $\mathbb{R}$.

In order to understand the current study of $\mathbb{R}$ in DegMath, we must take into account the relationships $DP_{DegMath}(\mathbb{R}) \trianglelefteq DP_{DegMath} \uparrow DP_{ResMath}$. With respect to the epistemological model, this implies that $CEM_{DegMath}(\mathbb{R})$ is made of elements of $CEM_{ResMath}$, or rather, that the final praxeologies of a graph providing a diachronic description of $CEM_{DegMath}(\mathbb{R})$ are contained in $CEM_{ResMath}(\mathbb{R})$. In few words, we could say that the mathematics learned in DegMath are part of the mathematics used in ResMath. In particular, we have that $\mathcal{O}_{DegMath}(\mathbb{R}) \subseteq \mathcal{O}_{ResMath}$, $\mathcal{N}_{DegMath}(\mathbb{R}) \subseteq \mathcal{N}_{ResMath}$ and $\mathcal{E}_{DegMath}(\mathbb{R}) \subseteq \mathcal{E}_{ResMath}$. Therefore, in order to understand the current study of $\mathbb{R}$ in DegMath, we must understand how $\mathbb{R}$ is conceived in ResMath, and how mathematics in general are conceived in ResMath.

Let us point out some general features of $\mathcal{O}_{ResMath}$ and $\mathcal{E}_{ResMath}$, which can be found, for instance, in (Hintikka, 1996):

- Concerning $C\mathcal{O}_{ResMath}$: The language does not include, in an essential way, terms referring to magnitudes. We can certainly find some references to magnitudes in motivating textbooks introductions, or in examples of applications. Anyway, definitions and theorems are not about magnitudes, but about certain abstract objects (numbers, geometric figures, maps, equations, sets, etc.). Even if, as many texts accept, the ultimate motivation for many objects and properties of basic mathematics can be find in human work on magnitudes, current mathematical language does not include magnitudes. More precisely, every mathematical statement can be reduced to a first order language where we have the following terms: terms for variables, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, $\in$. The interpretations of the *logical terms*, namely $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$, are the standard ones, and can be provided by using *semantic games* in the sense of (Hintikka, 1996). Variables are interpreted as sets, and the statement $x \in y$ (which is made with the only non-logical term, $\in$, of the set theory first order language) is interpreted as the set x being an element of the set y. Axioms can be, for instance, those of the Zermelo-Fraenkel theory. As we can see, even if magnitudes are acknowledged to be at the origin of mathematics, they are not mentioned in the official language of current research in mathematics.

- Concerning $C\mathscr{E}_{\text{ResMath}}$: the only allowed arguments are the so-called *deductive* ones. An argument is a pair (Γ, φ), where Γ is a family of statements, called *premises*, and φ is a statement called *conclusion*, together with a way of showing how the truth of premises would support the truth of the conclusion. In many daily life arguments, and even in natural sciences, the support provided by the premises is falible. Nevertheless, there also exist arguments where the support is such a strong one, that any interpretation of the non-logical terms of the language making true the premises makes true the conclusion. These are *deductive arguments*. A typical example of a deductive is given by Γ = {(1) All humans are mortal, and (2) Socrates is a human.} and φ = {Socrates is mortal}. This argument is valid regardless the interpretation of the terms "Socrates", "human" and "mortal", and so this argument is deductive. Since deductive arguments are valid regardless the interpretation (namely, the semantic) of the non-logical terms, then the validity is due to syntax. In deductive arguments validity cannot rely in the use of a particular interpretation of a non-logical term to reach a certain conclusion.

In order to sketch a brief description of $\text{CEM}_{\text{DegMath}}(\mathbb{R})$ we have examined some textbooks which are quite standard and internationally shared. The analysis of textbooks is quite a frequent method in didactics in order to check how the current teaching is done. Notice that the alternative would be to attend to actual lessons in many countries, but this is hardly doable, and, actually, this is something that the standard textbooks already do for us, since they are present in many course. After having examined many standard textbooks (for instance, Apostol, 1991; Fernández Viña, 1994; Ortega, 1993; Rudin, 1976; Spivak, 2008), and in coherence with $C\mathcal{O}_{\text{ResMath}}$ and $C\mathscr{E}_{\text{ResMath}}$, the sketch for a $\text{CEM}_{\text{DegMath}}(\mathbb{R})$ could be as follows:

1) One starts with praxeologies taking care of the construction of the structures $(\mathbb{N}, +, \cdot, \leq)$, $(\mathbb{Z}, +, \cdot, \leq)$ and $(\mathbb{Q}, +, \cdot, \leq)$ in terms of sets. This is actually the main type of tasks: to construct well-known numbers in terms of sets. Then one proves the existence of injective maps $(\mathbb{N}, +, \cdot, \leq) \rightarrow (\mathbb{Z}, +, \cdot, \leq) \rightarrow (\mathbb{Q}, +, \cdot, \leq)$ compatible with the addition, the multiplication and the order.
2) The following theorem is proved: "Any two Archimedian totally ordered fields for which every Cauchy sequence is convergent are isomorphic via an isomorphism preserving addition, multiplication and order".
3) The following theorem is proved: "Given an Archimedian totally ordered field R for which every Cauchy sequence is convergent, for any element of R there exists a unique corresponding decimal representation, and, conversely, any decimal representation defines a unique element of R".

4) One can prove that $(\mathbb{Q}, +, \cdot, \leq)$ is an Archimedian totally ordered field for which some Cauchy sequences are not convergent. This is the case, for instance, for the sequence $(p_n/10^n)_{n\in\mathbb{N}}$, where $p_n = \max\{a \in \mathbb{N} \mid a^2 < 2 \cdot 10^{2n}\}$, or for the sequence $(1 + 1/1! + 1/2! + \ldots + 1/n!)_{n\in\mathbb{N}}$.

5) Next, one proves the existence of an Archimedian totally ordered field where every Cauchy sequence is convergent. This can be done, for instance, by considering the set formed by certain equivalence classes of Cauchy sequences of $(\mathbb{Q}, +, \cdot, \leq)$, or by considering the set of Dedekind cuts of $(\mathbb{Q}, +, \cdot, \leq)$. Such a field, which we already know that is unique up to an isomorphism of ordered fields, is said to be the field of real numbers, $(\mathbb{R}, +, \cdot, \leq)$.

In all this construction of real numbers, there is no structural need to mention any magnitude at all. It is everything about sets. On the other hand, proofs are always deductive arguments, not relying on any interpretation of the non-logical terms. All that is used as starting premises are the axioms of set theory. Of course, those axioms and many definitions (for instance, that of the addition of natural numbers) rely on the interpretation of the variables of the language as sets. However, proofs no longer use this interpretation.

At this point we identify a didactic phenomenon, $\mathrm{RD}\varphi_{\mathrm{ResMath}}(\mathbb{R})$, whose avoidance will guide our forwards analysis, namely, the loss of the raison d'être of the presentation of $(\mathbb{R}, +, \cdot, \leq)$ via seemingly artificial constructions. Why cannot we presents real numbers just as (possibly non-repeating) decimal numbers?

THE NEED OF A REFERENCE DIDACTIC PARADIGM

Given the descriptive analysis, in an ambitious backwards analysis we should explain, at least, the origin of the following features concerning $\mathrm{CEM}_{\mathrm{ResMath}}$:

1. Disappearance of magnitudes of the current official language of mathematics.

2. Obligatory nature of the use of a (language reducible to a) first order language.

3. Exclusive use of deductive arguments.

4. Presentation of $(\mathbb{R}, +, \cdot, \leq)$ via seemingly artificial constructions.

In our backwards analysis we should provide an *evolutionary explanation*, showing these features as the result of a series of tries to avoid a certain didactic phenomenon, $\mathrm{CD}\varphi_{\mathrm{ResMath}}$. If we are able to present a diachronic $\mathrm{REM}_{\mathrm{ResMath}}$ so that the corresponding successful praxeologies form $\mathrm{CEM}_{\mathrm{ResMath}}$, then this $\mathrm{REM}_{\mathrm{ResMath}}$ would serve us to learn the usefulness of $\mathrm{CEM}_{\mathrm{ResMath}}$, insofar as it would show $\mathrm{CEM}_{\mathrm{ResMath}}$ as the best solution to a problematic situation. That situation could be hypothetical rather than historical, being an oversimplification of what historically occurred. But even in that case, this diachronic $\mathrm{REM}_{\mathrm{ResMath}}$ could be seen as an enlightening counterfactual history [1] of the origin of $\mathrm{CEM}_{\mathrm{ResMath}}$. That is, a story that asks us to consider an imaginary problematic situation in the light of which we might evaluate the importance of $\mathrm{CEM}_{\mathrm{ResMath}}$, enlightening us about its nature, its *raison d'être*, and allowing us to imagine what kinds of problems we would encounter in the absence of such successful praxeologies, and what kinds of solutions they provide. Notice that, with this kind of evolutive explanation of $\mathrm{CEM}_{\mathrm{ResMath}}$ via a diachronic $\mathrm{REM}_{\mathrm{ResMath}}$ we not only present a backwards analysis, but also a forwards analysis, because a study process based on this diachronic $\mathrm{REM}_{\mathrm{ResMath}}$ would avoid $\mathrm{RD}\varphi_{\mathrm{ResMath}}(\mathbb{R})$, namely, the loss of the *raison d'être* of the features already explained by the backwards analysis.

In (Gascón & Nicolás, 2022a), we suggested that the $\mathrm{CD}\varphi_{\mathrm{ResMath}}$ to be avoided with the $\mathrm{CEM}_{\mathrm{ResMath}}$ is the occurrence of paradoxical results, which led to the crisis in the foundations of mathematics in the late 19th and early 20th century. The solution provided by $\mathrm{CEM}_{\mathrm{ResMath}}$, the one suggested by Hilbert and called *formalism*, was to formulate every mathematical notion in a first order language via axiomatic definitions, and to prove theorems using only deductive arguments (Kline, 1990; Hintikka, 1996).

A diachronic $\mathrm{REM}_{\mathrm{ResMath}}$, showing the process of formalisation of all the mathematics is nearly an immeasurable endeavor. However, it seems affordable the construction of a diachronic $\mathrm{REM}_{\mathrm{ResMath}}(\mathbb{R})$ providing the *raison d'être* of the seemingly artificial constructions of $(\mathbb{R}, +, \cdot, \leq)$, showing them as nice solutions to problems that arise in the effort to give a formal description of $\mathbb{R}$ based on sets. This diachronic $\mathrm{REM}_{\mathrm{ResMath}}(\mathbb{R})$, which is already under construction, will be the next step in our analysis of the current study of $\mathbb{R}$ in DegMath.

CONCLUSION

In our descriptive analysis, we show that the current didactic paradigm for the study of real numbers in DegMath, $\mathrm{CEM}_{\mathrm{DegMath}}(\mathbb{R})$, is due to certain features of the current didactic paradigm for the study of mathematics in ResMath, $\mathrm{CDP}_{\mathrm{ResMath}}$. This, in turn, responds to a didactic phenomenon, $\mathrm{CD}\varphi_{\mathrm{ResMath}}$, that ResMath wants to avoid. This explains a well-known didactic phenomenon $\mathrm{RD}\varphi_{\mathrm{DegMath}}(\mathbb{R})$ in the current epistemological model $\mathrm{CEM}_{\mathrm{DegMath}}(\mathbb{R})$, namely, the loss of the *raison d'être* of the construction of $\mathbb{R}$. Our point here is not the discovery of this phenomenon, but rather the outline of an explanation in terms of didactic paradigms.

A study process based on a diachronic $\mathrm{REM}_{\mathrm{DegMath}}(\mathbb{R})$ that flows into $\mathrm{CEM}_{\mathrm{DegMath}}(\mathbb{R})$ would avoid $\mathrm{RD}\varphi_{\mathrm{DegMath}}(\mathbb{R})$.

NOTES

1. The use of counterfactual histories by historians (Maar, 2014), and in other social science disciplines (Morgan and Winship, 2007; Elster, 1994), is not surprising, as the notion of counterfactual fact has proven useful in the analysis of the notion of causality (Menzies and Beebee, 2020) .

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Didactics in the university teaching of mathematics

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The goal of this work is to underline the relevance of including in the university teaching of mathematics a didactic training. That would allow future mathematicians, not only future teachers, to have a broad view of mathematics and a reflective perspective on the kind of mathematics they are studying in the Degree of Mathematics.

Keywords: teaching and learning of specific topics in university mathematics, curricular and institutional issues concerning the teaching of mathematics at university level, epistemological model, didactic paradigm.

INTRODUCTION

Should didactics form a part of the university mathematics curriculum for all students majoring in mathematics?

In this work, we are going to present a positive answer. For this, first we will introduce some theoretical tools needed both to support our positive answer and to sketch the proposal of a course in didactics for the Degree in Mathematics. Then we will briefly outline the possible contents of this course. Finally, we will present in more detail some examples that could be a part of that course in didactics.

THEORETICAL FRAMEWORK

Intuitively, the notion of 'didactic paradigm' refers to a modality of study, an organized way of studying something. Although intensively used in the Anthropological Theory of the Didactic, it has been only recently analyzed. In (Gascón & Nicolás, 2018, 2019, 2022; Gascón, in press) we proposed a description of this notion in terms of four components.

But, before presenting these components, we first need to introduce the idea of 'didactic institution'. Social sciences study the working of social institutions. A *social institution* consists in a set of constitutive rules, stated by convention, which fix: a series of institutional positions; allowed, compulsory and forbidden actions for each position, and rewards and punishments for certain actions (Searle, 2010). Examples of social institution are: chess, language, marriage, paternity, authorship, law, jobs, religions, etc. *Didactic institutions* are social institutions having among their goals that of studying something, and this purpose explains (at least partially) what the constitutive rules are in this case.

Given a didactic institution, I, and a certain field of study, F, in I, we say that a *didactic paradigm for F in I*, written $DP_I(F)$, consists of the following components:

- The *didactic ends*, written $DE_I(F)$, explain the purpose of studying F in I.
- The *didactic phenomena*, written $D\varphi_I(F)$, state what is to be avoided in the study of F in I.
- The *epistemological model*, written $EM_I(F)$, describes what is specifically studied (what types of questions, definitions, techniques, theorems, proofs, kind of proofs, etc.) in order to study F in I.
- The *didactic means*, written $DM_I(F)$, state what is done in order to study F in I.

We will see two examples of didactic paradigm below, but first we should make some remarks.

Notice that there are strong links between the four components of a didactic paradigm. For instance, it seems clear that the didactic ends (we want to get with the study of F in I) and the didactic phenomena (we want to avoid with the study of F in I) determine to a large extent the epistemological model (how F specified) and the didactic means (how the study of F is made). Also, it seems clear that the didactic means we want to use determine to a large extent the epistemological model. Thus, for instance, if we want to follow a student-centered way of teaching (perhaps because among the didactic ends we find that of promoting certain researcher abilities), then the epistemological model of F cannot be just made of a series of axiomatic definitions and deductive proofs, but rather to include also some questions whose exploration by students (guided by the teacher) should be able to lead to some enlightening conclusions. In few words, the didactic means and the epistemological model are mutually conditioned, because certain didactic means and certain epistemological models might or might not be compatible.

Note also that the field of study, F, can be quite broad, for instance mathematics in general, or more specific, for instance systems of linear equations. Actually, in a didactic institution where mathematics are studied, there is a general didactic paradigm for the study of mathematics (that is, there is a general conception of how to study mathematics), and several specific didactic paradigms for the study of the different parts of mathematics studied therein.

When we use the idea of didactic paradigm to describe the current mode of study of a certain educational institution I, we speak of a *current didactic paradigm (CDP)* in I. When we use the idea of didactic paradigm to describe a possible mode of study of a certain educational institution I, we speak of *reference didactic paradigm (RDP)* for I. A reference didactic paradigm can be regarded as the hypothesis according to which the didactic ends would be achieved and the didactic phenomenon would be avoided if the didactic means were used based on the epistemological model. Typically, what has been done in the framework of the Anthropological Theory of the Didactic concerning didactic paradigms is the following. In a given $CDP_I(F)$

$=[CDE_I(F), CD\varphi_I(F), CEM_I(F), CDM_I(F),]$ a certain didactic phenomenon, $RD\varphi_I(F)$, is detected. Then one wonders how $CDP_I(F)$ should be changed in order to avoid $RD\varphi_I(F)$, and, perhaps, even in order to achieve certain educational ends $RDE_I(F)$ not necessarily identical to $CDE_I(F)$. Then, one reconstructs F by building an alternative epistemological model, $REM_I(F)$, and perhaps also proposing alternative didactic means, $RDM_I(F)$. In this way, one constructs a whole new didactic paradigm, $RDP_I(F)=[REM_I(F), RDE_I(F), RDM_I(F), RD\varphi_I(F)]$. The construction of the $RDP_I(F)$ allows a better understanding of the $CDP_I(F)$. This is what has been done, for example, in the reconstructions of proportionality in secondary school (García, 2005), number systems in teacher training (Sierra, 2006), elementary algebra in compulsory secondary education (Bolea et al., 2001; Ruiz-Munzón et al., 2015), real numbers (Licera et al., 2019) and limits of functions (Barbé et al., 2005) in high school, or elementary differential calculus in the transition from secondary school to university (Lucas, 2015).

Example 1: We can consider the current didactic paradigm for the Degree of Mathematics not relative to a specific field, but concerning mathematics in general. We put:

- The didactic institution is denoted by I = DegMath.
- The field of study is denoted by F = Math.
- The current didactic ends, $CDE_{DegMath}(Math)$, respond to the idea that society should continue to have, in future generations, experts in mathematics. Then, mathematics graduates should be able to teach mathematics at various levels (secondary and tertiary education), and should also be in a position to embark on a research career. Thus, the study of mathematics in DegMath should convey a basic body of knowledge that gives access to the most important branches of mathematics but also, and this is crucial, should transmit the epistemological model currently used in mathematics research.
- One of the current didactic phenomena, $CD\varphi_{DegMath}(Math)$, that the institution DegMath seems to be trying to avoid is the use of non-deductive arguments which, at the time (by the end of the 18th century and in the 19th century), gave rise to paradoxes and contradictions.
- Consequently, and following in part Hilbert's ideas, the current epistemological model, $CEM_{DegMath}(Math)$, presents mathematics as a part of Set Theory, which is a formal theory expressed in a first order language, where definitions are axiomatic and where the only possible arguments are the deductive ones (that is, those not supported by possible meanings of non-logical terms). Of course, students only receive a light flavor of Set Theory, and in lessons mathematics are written in a mixture of formal and natural language. But, still, students are trained in the idea that definitions must be axiomatic (and not, for instance, a recreation of a mental image) and proofs must be reducible to formal language, only made of a few permitted logical steps. This will be called *axiomatic-deductivistic set-theoretical*

epistemological model. In this model, presentations of the topics begin with the enunciation of a list of axiomatic definitions and lemmas, presented in an artificial and authoritative manner, and then they are followed by the theorems, carefully stated and loaded with suitable conditions, and proved without hesitating.

- Although one of the didactic ends is to allow students to embark on a research career, the Degree of Mathematics is not specially interested in training in research activity (this is confined in postgraduate degrees), but rather in transmitting a sufficient amount of mathematical knowledge. Thus, the current didactic means, $CDM_{DegMath}$(Math), are teacher-centered. Students are supposed to understand the definitions shown, the proofs explained, and the techniques used by the teacher, and students are also supposed to be able to solve exercises by applying the techniques and/or using the kind of deductive reasoning explained during lessons.

Example 2: We can also consider the current didactic paradigm for the DegMath relative to the study of real numbers. As before, I = DegMath, but now F = $\mathbb{R}$.

- The current didactic ends, $CDE_{DegMath}(\mathbb{R})$, is to introduce the field of real numbers by showing their difference from rational numbers and underlining the completeness, so that it could be used as a basis for many future deductive proofs (notably in analysis and differential geometry).
- The current didactic phenomenon, $CD\varphi_{DegMath}(\mathbb{R})$, to avoid could be very well the same as $CD\varphi_{DegMath}$(Math). In practice, and concerning specifically to the study of real numbers, another current didactic phenomenon is perhaps to avoid too long a presentation of $\mathbb{R}$, which would leave no time for other basic topics in a first analysis course (sequences, functions, limits, derivatives, series, integral calculus, etc.). This would explain that, frequently, the construction of $\mathbb{R}$ (as equivalence classes of Cauchy sequences over $\mathbb{Q}$ or as Dedekind cuts of $\mathbb{Q}$, for instance), is not taught, opting instead for an axiomatic definition.
- According to many standard textbooks (Apostol, 1991; Rudin, 1976; Spivak, 2008), the current didactic paradigm concerning the study of real numbers, $CEM_{DegMath}(\mathbb{R})$, assumes the construction of natural, integer and rational numbers in Set Theory, and establishes by definition that the ordered field of real numbers is a non-empty set, $\mathbb{R}$, together with two inner binary operations, + (called addition) and $\cdot$ (called multiplication), and with a binary relation, $\leq$, satisfying certain axioms which say that $(\mathbb{R}, +, \cdot, \leq)$ is a *Cauchy complete Archimedean field.* At some moment, the students might be proved that a Cauchy complete Archimedean field is unique up to an isomorphism of ordered field. Also, regardless it is actually studied or not in the Degree of Mathematics, textbooks also include a proof of the fact that there exists a Cauchy complete Archimedian field. Typically, this is constructed either by considering the set given by the Dedekind cuts of $\mathbb{Q}$, or by considering the set of certain equivalence classes of Cauchy sequences on $\mathbb{Q}$. After having proved the existence and the unicity of Cauchy complete Archimedean

field, textbooks feel the need of proving that such a field corresponds, after all, with the set of all decimal numbers, which is the initial naive image of real numbers we might have.

- In DegMath didactic means do not change depending of the field of study. Thus, the kind of teaching remains similar when studying algebra, analysis, topology or anything else. Hence, $CDM_{DegMath}(\mathbb{R}) = CDM_{DegMath}(Math)$.

As we said before, one can note in specific examples the interrelationship between the components of didactic paradigms.

DIDACTICS IN THE UNIVERSITY TEACHING OF MATHEMATICS

Possible contents for a course in didactics

The didactic institution of Bachelor's Degree in Mathematics is governed by a didactic paradigm. If the didactic ends of this paradigm consisted solely of training experts in mathematics by transmitting the axiomatic-deductivistic epistemological model, then neither the current epistemological model nor the didactic means (see the corresponding descriptions above) should be replaced.

However, it is reasonable to assume that the educational ends of DegMath include also the purpose of giving a more global view of mathematics, and not just the one transmitted by the *axiomatic-deductivistic set-theoretical* epistemological model. Actually, it seems clear that those students that will end up being teachers in Secondary Education will eventually need an alternative epistemological model (Gascón & Nicolás, 2022). But, on the other hand, it is impossible not to teach this epistemological model in the Degree in Mathematics, that being the epistemological model currently admitted in research. How to solve this problem?

Our proposal is to offer a course in didactics in order to explain the following:

1) The idea of didactic paradigm. The study of this idea is aimed at providing students with a vision of themselves as individuals at a certain position in a didactic institution, where there is a certain pervasive paradigm of study of mathematics. This would allow them to get a reflective perspective on the vision of mathematics they have been receiving so far.

2) The relativity of mathematical knowledge, that is to say, the fact that a field of study does not exist independently of a didactic institution and a didactic paradigm therein. For example, there is no such thing as 'the study of natural numbers', in absolute terms, but rather their study according to different didactic paradigms holding in different institutions, from Early Childhood Education to the Tertiary Education, and even beyond, in the research activity in mathematics. None of these visions of natural numbers offers the 'authentic' description of what natural numbers are. In particular, the 'authentic' description of natural numbers would not be the one offered today by Set Theory. This description has its *raison d'être* as part of the axiomatic-deductivistic set-theoretical epistemological model (see $CEM_{DegMath}(Math)$ above), which responds to didactic ends (see

$CDE_{DegMath}$(Math) above) and tries to avoid certain didactic phenomena (see $CD\varphi_{DegMath}$(Math) above). But there is no way of seeing that description as more 'authentic' than the one that professional mathematicians themselves had in another era, or more 'authentic' than the one that could be studied in Early Childhood Education (where natural numbers, far from being certain recursively defined sets, are written symbols used to refer to the size of certain sets). Each of those descriptions were actually part of a certain didactic paradigm in a didactic institution, and were strongly connected to certain didactic ends aimed to be reached and certain didactic phenomena aimed to be avoided. Consequently, if F is a field of study in mathematics, the $CEM_{DegMath}$(F) is not 'the' description of 'the true F', but only one of the possible ways of presenting this part of mathematical knowledge.

3) Different didactic paradigms that have appeared along history. In this part of the course, we would review several didactic paradigms of mathematics holding for the community of mathematicians regarded as a didactic institution. It would not be just a survey of history, because there will be an emphasis on the strong relationship between the epistemological model (the organization of mathematics) and the other components of the didactic paradigm. Among the reformulations of different areas of mathematics that have historically given rise to new didactic paradigms, we can mention: Euclid's Elements, the creation of analytic geometry, the axiomatization of the natural numbers, the axiomatization of probability, the classification of geometries, the axiomatization of Euclidean geometry, the constructions of the real numbers, the axiomatization of set theory, and the arithmetization of analysis. We can consider each of these reformulations of mathematical knowledge as a process aimed at facilitating either its diffusion (and, in particular, its teaching-learning), its development, its use and/or access to new mathematical problems. In short, the reconstruction of each of the aforementioned fields was intended to circumvent a didactic phenomenon (in the broad sense of "phenomenon related to the study") through a redefinition of the field in question, and with the consequent construction of a new didactic paradigm that sets new goals for the study of this field, and proposed new means of study to achieve these goals. Thus, for example, the "arithmetization of analysis", which took place in the second half of the 19th century, materialized in a new epistemological model that redefined and made it possible to construct new foundations of mathematical analysis. This reorganization of analysis was aimed at bringing to light and avoiding an undesirable didactic phenomenon from the perspective of the mathematical community of the time: the insufficiency of the old geometrical foundations to support analysis in a certain way. This reorganization of analysis, in turn, allowed further progress with access to new types of problems, new techniques and new justifying and interpretative discourses of the new mathematical practice.

4) Possible alternative ways of organizing the study of some mathematical field of study. Below we will sketch possible reference didactic paradigms in DegMath, one for F = $\mathbb{R}$ and another one for F = Math in general.

Of course, we are not claiming that this is the only possible content of a course in didactic for DegMath. Rather, we say that a course with this content would solve the tension existing between the aim of providing students with the view of mathematics presented by the axiomatic-deductivistic set-theoretical epistemological model, on one hand, and the aim of providing them with a broad vision of mathematics, on the other hand.

Possible changes of didactic paradigm in the study of real numbers

As we have seen above, in mainstream handbooks real numbers are introduced in a seemingly artificial way: either as Dedekind cuts or as equivalence classes of Cauchy sequences of rational numbers. Moreover, it might be surprising the existence of a theorem that, after all, identifies real numbers with decimal expressions. It seems that we have made a huge circumlocution to end up admitting that the real numbers are all decimal numbers. Also, completeness is the crucial difference between $\mathbb{R}$ and $\mathbb{Q}$, but why is it important? Is it a desirable intuitive property or rather a technical property needed in order to prove certain theorems? Which ones? Are there different ways of proving those theorems?

In short, could real numbers have been introduced in a different way?

Imagine we define real numbers as pairs $(a_0, (a_n)_{n\geq 1})$ where a_0 is an integer number, and $(a_n)_{n\geq 1}$ is a sequence of natural numbers such that $0 \leq a_n \leq 9$ and not all the elements are 9 from a certain position on. This would be a familiar definition of real number, perfectly coherent with the definition of natural and integer number, and with the decimal expression of rational numbers. Which would be the problem? Is it possibly, with this definition, to prove completeness? Is it even possible to define addition and multiplication? What would be the limitations of this presentation of real numbers compared to those in terms of Dedekind cuts or Cauchy sequences? What problematic question does each of these constructions solve?

The first irrational numbers human beings encountered were related to measurement. For instance, after the Pythagorean theorem we know that $\sqrt{2}$ is the length of the diagonal of a square of square of side 1. However, in the current epistemological model, magnitudes and measurements are not mentioned. Why is that? If we accept to talk about magnitudes and measurements, can we do a more intuitive treatment of real numbers?

One of the didactic ends for the study of real numbers is to lay the foundations for the study of differential and integral calculus. In the standard analysis, this entails a cumbersome interplay between ε and δ in definitions and proofs. However, one could decide to do the so-called *non-standard analysis* (Robinson, 1966; Nelson, 1977). There, one can distinguish, among the real numbers, the so-called *infinitesimal* *numbers*. Then we say that two numbers are *infinitely close* if their difference is 0 or infinitesimal. By using this notion, we can outline a definition of limit of functions without using ε and δ.

Non-standard analysis has proved to be perfectly rigorous, and, after a careful work in considering what we mean with 'infinitesimal numbers', proofs of classical results in analysis become much more natural. So, there does not seem to be sound mathematical reasons to avoid teaching non-standard analysis. Rather, the reasons seem to be of didactic nature. Which are them? Why is non-standard analysis generally not taught?

All the questions above can be regarded as embodiments of the following general question: Why are real numbers taught in DegMath as they do? Could real numbers be taught in a different way? Of course, corresponding suitable answers would lead to didactic-mathematical considerations which, in turn, would contribute to a more deep understanding of real numbers and, more generally, of how mathematics develops.

Lakatos' proposal regarded as a change of didactic paradigm

Now we will briefly present a hypothetical reorganization of mathematics globally considered. It was proposed by Imre Lakatos (Lakatos, 1976) and we could call it the passage from an *axiomatic-deductivistic paradigm* to an *heuristic paradigm* (although Lakatos does not speak of paradigms but of "styles" or "approaches" in the way of presenting mathematics). This reorganization of mathematics obeys didactic reasons, because it seeks to modify the study of mathematics to the point that it would require rewriting textbooks. But also because it seeks to profoundly transform the diffusion of mathematical knowledge within the mathematical community. Lakatos stresses that, if the heuristic paradigm were to be imposed, the irrelevance of many of the results published would be clearly revealed, which would simplify and reduce the volume of publications while at the same time increasing access to new mathematical problems that remain hidden in the axiomatic-deductivist style.

In the axiomatic-deductivist style, all definitions are suitable and fruitful, propositions are true, theorems are fully developed, original conjectures, refutations and criticism of the proofs are suppressed. Consequently, the axiomatic-deductivist epistemological model hides the struggle and conceals the adventure. In contrast, Lakatos proposes a heuristic paradigm, characterized by placing at the origin and heart of the mathematical activity the problematic situation, the initial conjectures, the tentative proofs, the criticism of the proof, the counterexamples, the refutations and the definitions generated by the proof.

This heuristic paradigm would be a completely new modality of study of mathematics, with new didactic ends and new didactic means based on the interplay between conjectures, proofs and refutations, the true logic of mathematical discovery (from Lakatos' perspective) that replaces deductivism. In addition, this new paradigm brings to light and aims to circumvent a didactic phenomenon, undesirable from

Lakatos' perspective, which is manifested in the artificial and authoritarian character of deductivism.

CONCLUSION

What we propose in this work is the convenience of a didactic education in the study of mathematics at university, rather than just a mathematical education. Indeed, mathematical education can be enriched with an epistemological (that is, concerning the development of ideas) or a historical approach. Didactics is the science that studies how current (or possible) didactic paradigms function in all kinds of social institutions. Consequently, future researchers in mathematics can only fully understand the development and the social and scientific goals of the mathematical community, as well as the role they play in that community, with adequate didactic education. In turn, consideration of the institutional relativity of didactic paradigms would avoid the paradigmatic delusion that the only real mathematics is that which is currently done by the research community and taught in the Degree of Mathematics. A delusion that makes the transition from secondary to university education, and vice versa, so difficult (Gascón & Nicolás, 2022). But, beyond future researchers in mathematics and future secondary and university teachers, didactic education is also convenient for graduates who will carry out professional work in other social institutions, be they financial, biomedical, related to engineering, artificial intelligence, computer science or data analysis. In all of them, to use and apply mathematics appropriately, it will be necessary to reformulate and reconstruct mathematics structured in an axiomatic-deductivist paradigm. In all "mathematician" professions, it will be very useful to have had didactic education that has relativized this didactic paradigm and provided experience, even if limited, in tackling open problem situations, in formulating conjectures and counterexamples and in constructing tentative models along the lines advocated by the heuristic paradigm proposed by Lakatos.

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Analogical transitions between teachers' mathematical knowledge: the category theory as a frame

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Studying analogies among the teachers' mathematical knowledge between high school and university remains a challenging research issue. This paper reports on a research that deploys Vergnaud' theory of human activity along with the category theory to collect and analyze data related to high school and university teachers' activity of producing mathematical statements in classroom. The major result involves the systematic organization of potential analogical transitions involving teachers' knowledge and their meaning across the two levels.

Keywords: analogy, category theory, human activity, knowledge, transition.

INTRODUCTION

Research on teachers' knowledge between secondary and tertiary education predominantly concentrates on the examination of similarities and differences. Suggestions for a more all-encompassing exploration of analogical aspects are rare. Indeed, while comparison facilitates the attainment of correspondence between two distinct settings in terms of similarities and differences, analogy enables the generation of novel inquiries and the identification of potential connections (Brown et al., 2006). In the context of teachers' knowledge, analogies may be used to propose shared foundations that facilitate communication between calculus teachers in the transition. As underscored in Thomas and Klymchuk (2012), there is a discernible imperative to enhance this communication. For example, the calculus knowledge employed by university and high school teachers in similar types of calculus problems can be structurally aligned not only to unveil commonalities but also and mainly to deduce inferences from one to the other. Analogies can also be used to merge the ostensibly distinct bodies of knowledge possessed by high school and university teachers into a more comprehensive body of knowledge (Arzi-Gonczarowski, 1999), thereby permitting the deduction of implications for a more general setting encompassing high school and university knowledge. To illustrate, Corriveau and Bednarz (2017) have highlighted disparities in the mathematical knowledge of secondary and tertiary education teachers with regard to symbolism and its use; these variations are reflective of instructional differences and are likely to impact the transition. Analogies may be employed to formulate hypotheses about a more inclusive setting that bridges both interpretations of symbolism.

It is well recognized that teachers' knowledge of the mathematical content being taught influences classroom instruction. Vergnaud's concept of schema along with symbolic representations systematizes such knowledge (Vergnaud, 1998) and affords the opportunity to analyze it for a specific class of mathematical issues. Our focus centers on the teacher's specific activity of producing mathematical statements

(Vergnaud, 1998) in the instructional context rather than the entirety of the teaching activity. In the following, 'teachers' calculus knowledge that manifests within the specific activity of teachers producing mathematical statements in the classroom' is abbreviated as 'teachers' calculus knowledge. The aim of this paper is to study analogical transferences between two settings of teachers' calculus knowledge, each related to one side of the calculus transition. The body of research related to analogy within cognitive domains is extensive; ongoing debate among researchers continues to shape this field, and we do not take a specific stance in this debate. Nevertheless, in order to facilitate a meaningful comparison between these two settings, it is imperative to establish a precise formulation for each setting, avoiding excessive verbosity and redundancy. Equally important is the development of a frame for formalizing analogical transitions. In recent decades, the prominent focus of category theory on the analysis of mappings between settings fosters the application of categorical tools in order to formalize analogies (Brown et al., 2006; Arzi-Gonczarowski, 1999). Accordingly, the research questions addressed in this paper are: 1) What may constitute an effective approach to establishing a standardized representation of teacher calculus knowledge, making it both comparable and transferable across the two educational levels? 2) What is an effective way in which to generate tools for analogical transitions of this knowledge, spanning both sides of the calculus transition? In this paper, these two questions will be explored at a theoretical and methodological level. This investigation will be illustrated using the preliminary results of an empirical study conducted in the case of class of mathematical problems related to the use of the formal definition of sequence convergence, both as a procedural means to calculate limits and as an argument to substantiate these calculations.

TEACHERS MATHEMATICAL KNOWLEDGE: CONNECTING SCHEMA AND SYMBOLISM WITHIN BOOLEAN PREDICATE

Building upon the Piagetian perspective, Vergnaud's concept of schema (1998) provides the possibility for examining teacher's activity occurring within a particular class of situations that are structured around the same goal of the activity. In the context of teaching mathematics, the teacher has several activities such as managing classroom communication, designing mathematical lesson, organizing students' work, using technological tools, learning about technology in education, producing mathematical statements, selecting resources, etc. Each activity takes place in a particular class of situations depending on the goal assigned to the activity. For instance, the situations related to the issue of enhancing students' engagement in interactions with both their peers and the teacher may align with the overarching goal of the activity of managing classroom communication. Thus, the schema involved is one related to the managing of classroom communication.

The general goal of the teacher's activity of producing mathematical statements in the classroom can be broadly resumed to the resolution of mathematical issues such as the resolution of mathematical problems, the proof of mathematical assertions, the explanation of definitions and proprieties, etc. In accordance with Vergnaud's theory, this teacher’s activity is rooted in categories of implicit knowledge and inferences that enable the selection of valuable knowledge and the generation of rules of action (if-then statements) and anticipations including goal and subgoals (Vergnaud, 1998). Deductive inferences operate on observed regularities from which generalizations can be drawn. Knowledge become explicit through the utilization of speech, gestures, and various linguistic and non-linguistic forms employed to convey symbolic representations namely words and symbols used in conventional systems (Vergnaud, 2009). The role of these forms goes beyond mere symbolic representations of knowledge; it also serves to enhance the operationality of schemas: the designation and identification of implicit knowledge and the execution of inferences. They play a pivotal role in the linkage between the categories of mathematical thoughts, within the schemas and the categories of mathematical objects, within symbolic representations (see figure 1). The Vergnaud model facilitates an examination of the meaning assigned by the teacher to their knowledge, where meaning is defined as the schemas evoked by the mathematical issues or symbolic representations. The imperative of investigating the meaning of knowledge represents one of the criticisms within the domain of research focused on mathematics teachers’ knowledge (Thompson, 2016). However, this teacher’s activity is not disconnected from other teacher’s activities that arise in the classroom such as teacher’s activities of supervising students’ mathematical work, and answering students’ mathematical inquiries. But these activities do not alter the components of the schema involved in the activity of producing of mathematical statements. On the contrary, these activities serve to illuminate other components of the schema by effectively freezing the teacher's behaviour in mathematical production, behaviour assumed to involve a significant degree of automatism. The resulting awareness fosters the activation of additional implicit knowledge, hence leading to new subgoals and rules of action within the same schema.

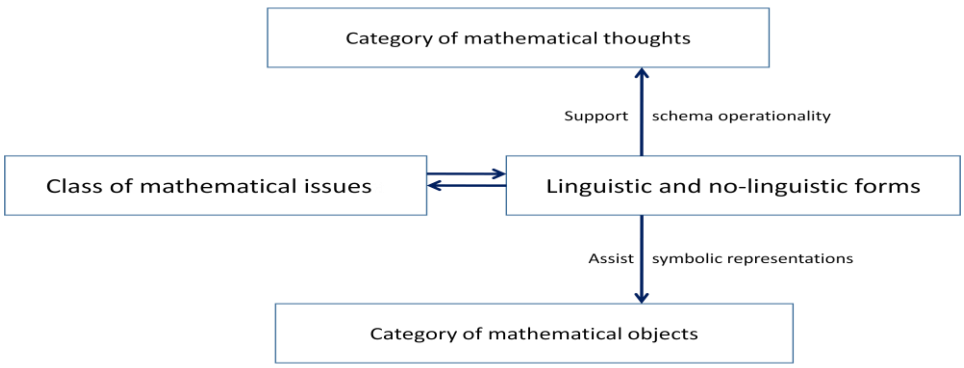


Figure 1: Schema and symbolism in mathematical activity

In this paper, the pair of implicit knowledge contained in the schema and explicit knowledge emerging through symbolic representations allows a primary standardization of the setting of teacher’s calculus knowledge for a specific class of mathematical issues in both sides of the calculus transition. The benefits of mathematical formalizations encompass attributes such as generalizability, empirical verifiability, and capacity to represent complex cognitive phenomena beyond the scope of verbal explication. In this study, the utilization of category theory tools are informed by a parallel drawn from the formalization of analogical transitions in artificial intelligence cognition that is established in Arzi-Gonczarowski (1999). The abstract idea of the setting of teacher's calculus knowledge for a particular class of mathematical issues is postulated as a mathematical construct which relates between mathematical objects, a set of symbolic representations of knowledge called o-elements and mathematical thoughts, a set of internal representations of knowledge called p-elements. Following Vergnaud, the p-elements are implicit knowledge precisely theorems-in-action considered as true propositions and concepts-in-action considered as efficient concepts that serve as basis for the formulation of propositions. The o-elements are explicit knowledge that are expressed through symbolic representations. These two settings are disjoints since their elements don't have the same cognitive status (Vergnaud, 2009). To exemplify, let us consider a scenario in which a teacher is working through mathematical problems that involves studying the convergence of a sequence using the formal definition of the limit. Furthermore, assume that the teacher's discourse, whether linguistic or non-linguistic, has been divided into several segments based on the subgoals of the teacher's activity. In this particular segment, the subgoal pertains to 'elucidating the relationship between ε and N'. The discourse is accompanied by symbolic representations, including o_i–'graphical instantiation of the formal definition specific to the sequence under study', o_j–'the algebraic inequality involving ε', and o_k–'ε is arbitrary'. Drawing upon the aid-function of the discourse, it becomes possible to formulate implicit knowledge, which includes the theorem-in-action p_l–'as the band becomes more and more smaller, the position, from which all terms of the sequence fall within this band, becomes more and more larger.'; and the concepts-in-action p_m–'the band equals ε.' and p_n–'the position is dependent on the band.' From the analysis of the teacher's activity, it is possible to verify that in this segment, o_i mobilizes p_l, o_i mobilizes p_m, o_j does not mobilize p_m, o_j may either mobilize or not p_l, etc. More specifically, within the examined segment, the set of "concepts-in-action" and "theorems-in-action" comprises three elements. The teacher's discourse analysis serves to: 1) substantiate the mobilisation of an element in the case of a specific symbolic representation (o_i, p_l), 2) substantiate the non-mobilization of this element in this specific symbolic representation (o_j, p_m), 3) avoid substantiating either its mobilization or non-mobilization when it is ambiguous (o_j, p_l).

Accordingly, the setting of teacher's mathematical Knowledge for a specific Class of mathematical issues is a triplet CK = (O, P,) where O is the set of o-elements, P is the set of p-elements, O and P are finite and disjoint sets, and is a 3-valued predicate. The three-valued predicate is the CK-predicate which relates o-elements and p-elements as following: (o, p) = t (true) if o mobilizes p, (o, p) = f (false) if o does not mobilize p, (o, p) = u (undefined) if o may either mobilize or not p. This undefined value might eventually become defined. For the sake of clarity and simplification of the employed categorical tools, we rely upon rather trivial examples as presented in table 2. It is evident that the utility and role of these tools in capturing significantly more intricate examples are not aptly demonstrated by this type of illustration.

ABOUT THE EMPIRICAL RESULTS USED FOR EXEMPLIFICATION

Methodological considerations

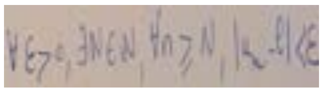
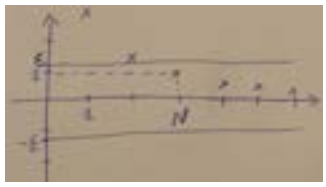
The main theoretical constructs derived from Vergnaud's theory of human activity-knowledge-in-action, symbolic representations, etc. may be subject to varying interpretations according to individual knowledge background. However, unlike other types of activity, mathematical activity is organized to communicate mathematical notions and meaning. Thus, there should be reasonable agreement among mathematicians and mathematics education researchers about what the main ideas, goal and the subgoals of the teacher's activity should be and, what the link between knowledge-in-action, and the symbolic representations might be. In this study, two of the authors are mathematics education researchers and one is both mathematician and mathematics education researcher. The three authors are committed in research on calculus and transition. In this paper, the general goal of teacher's activity relates to producing mathematical statements involved in the use of the formal definition of sequence convergence. Identifying the activity's subgoals firstly requires examining chronologically the entire activity. This examination focuses on changes in the teacher's mathematical concerns-whether these changes are stimulated by students' interventions or not-rather than on the implicit and explicit knowledge entailing them. Each new concern implies a specific subgoal of the activity. More precisely, the analyzed schema comprises rules of action that enable the generation of a sequence of teacher actions with the aim of achieving a specific subgoal. These rules of action are challenging to discern, primarily because they rely on inferences established by the individual based on their knowledge of similar class of mathematical issues. To alleviate interpretational biases concerning these rules, the identification of teacher's activity subgoals hinges on the sequences of their explicit actions and moments of transitioning from one precise mathematical concern to another. It is important to note that there are not chronological boundaries regarding the teacher's activity within each subgoal nor an initiation and a conclusion of a mathematical concern. In fact, the second step consists in the researchers' organisation of the entire activity around how teachers' calculus knowledge supports each other within these subgoals. To illustrate, suppose that 'elucidating the relationships between and N', and 'computing algebraically the value of N ' are two subgoals of the teacher's activity: the o-element ' is arbitrary' of the former may provide focus on the p-element 'the N value must be expressed with epsilon' of the later, in such a way that the link between the two elements could be systematized. Thus, while the first step permits to identify the subgoals of teacher's activity, the second step deploy the analysis of teacher's calculus knowledge to connect between them. Furthermore, the identification of knowledge-in-action begins with the tentative identification of the ways the symbolic representations are described or produced. Accordingly, these symbolic representations are explained in terms of knowledge-in-action. The potential knowledge-in-action undergoes iterative refinement to correspond to the process of symbolic representations from which they emanate. Throughout this bipartite analysis, implicit and explicit knowledge are identified and interlinked.

Some method's aspects

This study is part of a broader project initiated two years ago, which aims to investigate the knowledge and instructional practices of secondary school and university teachers in the field of calculus. The initial phase of this project involves the recording of instructional lessons in calculus on both sides of the transition. This paper concentrates on two lessons concerning sequence convergence, one in the high school and the other in the university. There were about twenty students present at the high school and at the university. Both high school and university teachers' activities of producing mathematical statements relates to the use of the formal definition of sequence convergence, both as a procedural means to calculate limits and as an argument to substantiate these calculations. Each lesson was video-recorded. In this paper, we selected and transcribed the 23-minutes excerpt from high school teacher related to the study of (1/n) and (4/n), and 20-minutes excerpt from university teacher related to the study of (3/n) and $((-1)^n/n)$.

Preliminary results

To illustrate the analysis of the excerpts, we present a university teacher's extract lasting 8 minutes in the table 1. This 8-minutes relates to the study of the sequence (3/n). The teacher's speech and writing are translated in verbatim from French. However, it is essential to know that the researchers have delineated two distinct subgoals for the whole excerpt: (Sub1) elucidating the relationship between and N; and (Sub2) computing algebraically the value of N.

	Speech	Gesture and writing	Symbolic representations
1	If you want to prove something quantified universally, as is the case here, for any epsilon, you must begin by giving yourself a positive epsilon	Points to the epsilon of the formal definition while writing the whole definition in the blackboard	
2	At that point, you must exhibit a large N ... such that for any n greater than the value N, 3 under n minus zero is less than epsilon	Points to the 'algebraic inequality using epsilon' while writing it in the blackboard	
3	So, the value of N must depend on epsilon, which represents the width of the band	Points to the band in the graphic of the sequence represented in the blackboard drawing more terms in the band and placing N	

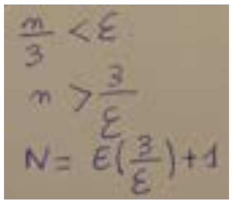
4	How will I choose a value for N? I will take a value for N that is... the integer part of 3 under epsilon plus one, for example, since it needs to be an integer, OK?	Points to 'integer part' while writing it in the blackboard	
5	The 'for any epsilon' is interpreted as me choosing an arbitrary epsilon now	Points to 'epsilon is arbitrary' while writing it in the blackboard	is arbitrary
6	As it is arbitrary, in the end it will tell me that for any epsilon, whatever I want to happen occurs		

Table 1: Extract from the university teacher's activity

In this extract, we describe the chronological appearance of activity subgoals as follows: Sub1 (segment 1-2-3), Sub2 (line 4), and Sub1 (segment 5-6). Several symbolic representations related to Sub1 appears in this extract: o_{1u}–'formal definition of sequence convergence ', o_{2u}–'graphical instantiation of the formal definition specific to the sequence under study', o_{3u}–'the algebraic inequality involving ε', and o_{4u}–'ε is arbitrary'. However, to illustrate with this extract, we focus on o_{4u} as the teacher's speech, gesture and writing mainly refer to it. Based on the bipartite analysis mentioned in the methodological considerations section, we identify and connect knowledge-in action to o_{4u}. More precisely, we identify two theorems-in action: p_{1u}–fixing epsilon allows for determining the corresponding value of N', and p_{2u}–''finding a specific value for N given a generic epsilon enables the determination of all corresponding values of N'; and four concepts-in-action: p_{3u}–'the band equals ε', p_{4u}– 'the position is dependent on the band', p_{5u}–'ε is universally quantified, it can be set', and p_{6u}–'setting epsilon permits to restore its generality'. The fifth line of table 2 allows understanding the connections of knowledge established within the 3-value predicate $_{U}$ for the university until p_{6u}. The analysis of the entire excerpt of the university also led to the following additional knowledge: two o-elements o_{5u}–'the integer part of a number' and o_{6u}–'formulation of the formal definition using the computed value of N'; one theorem-in-action p_{7u}–'as the band becomes more and more smaller, the position, from which all terms of the sequence fall within this band, becomes more and more larger.', and two concepts-in-action p_{8u}–'the N value must be expressed with epsilon', p_{9u}–'taking the integer part is sufficient to calculate N'. The analysis of the entire excerpt of the high school led to identifying and connecting the following p-elements with the o-elements: p_{1s}– 'exemplifying epsilon allows for determining the corresponding value of N', and p_{2s}–'finding a specific value for N given a generic epsilon enables the determination of all corresponding values of N', p_{3s}–'The band equals the interval', p_{4s}–'The band and the position are interdependent', p_{5s}–'ε can be set', and p_{6s}–'setting epsilon does not altered its generality', p_{7s}–'within the band, we can find a position, more and more larger, from which all the terms of the sequence fall in it', p_{8s}–'the N value must be expressed with epsilon', p_{9s}–'taking the integer part is not necessary to calculate N'. The o-elements are homologous to those of the university in terms of specifications.

EXAMINING ANALOGICAL TRANSITIONS

Category theory tools facilitate the elucidation of one setting in terms of another, the discernment of structural relations, and the proposition of a more comprehensive setting that encompasses both settings. Due to space constraints, the prerequisites category tools used in this study will not be fully detailed. The search for a set mapping is a primary categorical tool that can be deployed for analogical transition. More precisely, let CKS = (O_S, $P_{S,}$ $_S$) and CKU = (O_U, $P_{U,}$ $_U$) be the settings of the high school and university respectively. h is a CK-morphism if h defines the following set mappings: h: O_S O_U , h: P_S P_U in such a way that for all o-elements in O_S and all p-elements in P_S, if $_S$(o, p) u, then $_U$(h(o), h(p)) = $_S$(o, p) (h satisfies the no-blur condition). The definition does not presuppose that both settings exist prior to the transition, with the CK-morphism following thereafter. There are scenarios in which the transition can be regarded as resourceful, in the sense that either one of the two settings can generate the morphism, effectively giving rise to the other setting, though this study does not focus on this aspect. A rigid CK-morphism retains the structure in a stringent manner, where h is considered a rigid CK-morphism if the definition is applicable for all three values of $_S$(o, p). In this mathematical framework, CKU, CKS, and CK-morphism collectively form a mathematical category, with CKU and CKS serving as categorical objects, and CK-morphism, in conjunction with the identity morphisms, representing the arrows within the category. Each object can be conveniently described by a CK matrix, where lines represent o-elements, columns represent p-elements, and entries consist of the p-predicate value for the corresponding coordinates. In the examples of table 2, CKS and CKU are identified from the entire excerpt of each level (p_{iu} and o_{iu} denoted the elements of CKU and p_{is} and o_{is} for CKS). The CK-predicate values are not discussed in this paper.

$_S$/ $_U$	**p_{1s}/p_{1u}**	**p_{2s}/p_{2u}**	**p_{3s}/p_{3u}**	**p_{4s}/p_{4u}**	**p_{5s}/p_{5u}**	**p_{6s}/p_{6u}**	**p_{7s}/p_{7u}**	**p_{8s}/p_{8u}**	**p_{9s}/p_{9u}**
o_{1s}/ o_{1u}	t/t	t/t	t/t	t/t	t/t	t/t	u/t	t/t	u/u
o_{2s}/ o_{2u}	t/t	u/u	t /t	t/t	t/t	t/t	t/t	t/t	f/f
o_{3s}/ o_{3u}	t/t	u/t	t/t	f/f	t/t	t/t	u/u	t/t	t/t
o_{4s}/ o_{4u}	u/t	t/t	t/t	t/t	u/t	t/t	u/t	t/t	u/t
o_{5s}/ o_{5u}	t/t	u/t	u/u	u/t	u/t	u/t	f/u	u/t	u/t
o_{6s}/ o_{6u}	t/t	t/t	t/t	t/t	t/t	t/t	u/t	t/t	u/u

Table 2: High school and university class of calculus knowledge

The CK-morphism h: O_S O_U and h: P_S P_U are based on the following mapping: $h(o_{is})= o_{iu}$ and $h(p_{is})= p_{ju}$, i=1, ...,6 and j=1, ..., 9. It is easy to see that h is no-blur as required by its definition, yet it is not rigid (grey cells). Two examples of comments could be derived from the analogy between CKS and CKU: 1) the h-mapping may explicate CKU in terms of CKS. For example, the graphical instantiation of the definition of sequence convergence at the university (o_{2u}) may focus on the association between the band and the position of the required term (p_{3u}, p_{4u}, and p_{7u}), because it is the case in the high school (o_{2s}); 2) the h-mapping may be used to suggest principles for a transitional setting of CK. This issue will be elaborated in the following to systematize arguments such as: In both CK there is graphical instantiations of sequence convergence, which may be performed in different ways along with the other o-elements. The process consists of generalizing the two settings into a third generalized CKT setting (CK of Transition). Such a setting consists of 'abstracted' p-elements that should be activated in graphical instantiations of the definition while being mobilized within the other o-elements (o_i, for all i 2). The notions of CK-product and CK-pullback systematize such CKT. The first step is to start by juxtaposing all possible pairs in the cartesian products O_S O_U and P_S P_U; the product CKS CKU = (O_S O_U, P_S P_U, $_S$ $_U$) is defined through $_S$ $_U$ as follows: $\rho_S \times \rho_U((o_S, o_U), (p_S, p_U)) = t\ (\text{resp. } f)$ if, $\rho_i(o_i, p_i) = t\ (\text{resp. } f)$ for all i=S, U, otherwise it is undefined. In the general case, most of the CK-predicate values in a product are going to be undefined, since most pairs should consist of two essentially different coordinates. For example, a juxtaposition of (p_{7S}, p_{7U}) is meaningless for more than half o-elements of the product containing o_{2s} or o_{2u} which is captured formally by the undefined CK-predicate value for all such pairs of o-elements (computation from table 2). Accordingly, at both end of the transition, a common potential meaning assigned to the graphical instantiation of the formal definition should not be grasped through the link between the width of the band (epsilon strip) and the value of the position from which the terms of the sequence fall in it. This can be explained by the observation that this link is thought as generic to all sequences at the university level and rather specific to monotonic sequences in high school. The second step will be to select a subset of O_S O_U-elements and a subset of P_S P_U-elements in the CK-product that results in a CK-pullback (CKT, T for Transition) offering analogs. Formally interpreted, a CKT CK-product consists of pairs of O_S O_U-elements and pairs of P_S P_U-elements chosen to follow two conditions: (1) For all pairs (o_{jS}, o_{jU}) and all pairs (p_{ks}, p_{ku}), either for all i=S, U, $_i$ (o_{ji}, p_{ki}) t, or for all i=S, U, $_i$ (o_{ji}, p_{ki}) f. For instance, the pairs (o_{5s}, o_{2u}) and (p_{7S}, p_{7U}) cannot be included in the same CKT because $_S(o_{5s}, p_{7s})$= f and $_U(o_{2u}, p_{7u})$= t, (2) given a CKT, for each o-element, there exist a p-element such that (o, p) u. Likewise, for each p-element, there exist a o-element such that (o, p) u. Accordingly, a CKT that can be generated from table 2 may consist of 900 pairs from the extracted matrix (o_1 , o_2 , o_3 , o_4 , o_6) (p_1, p_2, p_3, p_5 p_6, p_8). As a result, at both side of the transition, an operational schema invoked in graphical instantiations of the definition of sequence convergence while being mobilized in the formal interpretation and the arbitrariness of could be expressed as follows: the band's width can be defined, facilitating the determination of the corresponding position. This, in turn, enables the restoration of the generality of the band width, thereby yielding a general position contingent upon it. Nevertheless, the university teacher emphasizes the role of universal quantification of epsilon for validating the feasibility of setting epsilon, whereas this justification is completely naturalized for the high school teacher. Furthermore, the extracted matrix (o_3 , o_5) (p_1, p_3, p_5 p_6, p_8, p_9) shows a CKT of 144 pairs that permits to explore teachers' knowledge pertaining the utilization of the integer part in expressing the variable N within the parameter ε. Accordingly, if algebraic computation on the epsilon inequality follows the same pattern on both sides of the transition, employing a consistent scheme to determine the position from the band width (i.e., starting by fixing this width until restoring the generality of the found position), the use of the integer part is considered essential by the university teacher but concealed by the high school teacher through the utilization of numerical examples of epsilon that immediately yield integer values for N. However, in both levels, the algebraic meaning of the dependence between epsilon and N is not evoked while expressing the inequalities involving epsilon (see the cell f/f of table 2 for o_{3i} and p_{4i}). These rather trivial examples can be completed by more complex ones using the same tools.

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Sensitising teacher students for institutional obstacles in analysing content for lesson planning: Preliminary analysis of an intervention project

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Lesson planning includes an analysis of the mathematical subject matter to be taught and is based on pedagogical, methodical, and didactic considerations. These elements then need to be integrated into a coherent lesson design. Apart from difficulties with the subject matter, students find it particularly difficult to represent the mathematical content in such a way that methodical and didactic decisions could be built on it. This paper reports on an intervention that aims to support students in lesson planning through the introduction and subsequent use of ATD tools. We present a preliminary a priori analysis, that particularly takes into account institutional obstacles like a naturalisation of mathematical content and a subordination of lesson planning to pedagogical considerations.

Keywords: Lesson planning by student teachers, Teachers' and students' practices at university level, Novel approaches to teaching, Paradigm of questioning the world, Anthropological Theory of the Didactic.

INTRODUCTION

Lesson planning is a task that student teachers in Germany face both in phases of their university studies, where they attend schools to observe teaching and to gain first experience in teaching, and subsequently during their teacher traineeship (Referendariat). On these occasions, students typically have to prepare written lesson plans, especially before lessons that are observed and assessed by lecturers or experienced teachers. The lesson plans have to include in particular a content analysis of the topic to be taught, as well as methodical, didactic and pedagogical considerations. On the one hand, coherence of the overall planning is a central criterion for a good lesson design (Heckmann & Padberg, 2012). On the other hand, understanding the relevance of and recognising opportunities for realising coherence in the mutual relationship between the analysis of the mathematical content and the methodical, didactic, and pedagogical design is a particular hurdle for student teachers. We have also observed this hurdle in other university teaching projects in particular, in which learning units are created for platforms such as Moodle (e.g. Hochmuth & Peters, 2023) or in Bachelor's theses in the context of subject-matter didactics, where the mathematical content is first elaborated taking into account scholarly literature and then teaching proposals are developed separately from this, some of which also include lesson plans (Hochmuth & Peters, in press).

This contribution reports on an intervention that aims to support student teachers in developing more coherent lesson plans by sensitising them to institutional obstacles to content analysis (e.g. a naturalisation of mathematical content), and an accompanying research project. The intervention builds on the Study and Research Path for Teachers' Education (SRP-TE) concept developed by Barquero et al. (2018) and intends in particular to counteract tendencies to naturalise mathematical content and its subordination to pedagogical considerations in lesson planning. Accordingly, the research question we are focussing on is: How might specific activities in university courses be designed to sensitise student teachers to naturalisations of mathematical content in lesson planning that hinder the explication and development of connections between subject matter and methodical, didactic and pedagogical considerations?

In order to achieve the objectives addressed, we propose to introduce concepts and tools from the Anthropological Theory of the Didactic (ATD) (Chevallard, 2019), in particular teaching-learning paradigms, didactic moments, question-answer maps, and media-milieu dialectics. In a research project accompanying the intervention, we want to identify the limits and the potential for a successful implementation.

These aims provides a background for the presentation of the intervention design and our preliminary a priori analysis. The particular novelties of the contribution lie in the adaptation of the SRP-TE concept and in the connection of ATD tools with standard tools for lesson planning in German teacher education.

The contribution is structured as follows: we first address elements of German teacher education, that are relevant for the intervention design. Next, we present the theoretical framework on which the intervention design and the accompanying research project are based. To answer the focused research question, we then describe the intervention, its content and its organisation, together with a preliminary a priori analysis reflecting also the institutional context. Finally, we pose open questions concerning the a priori analysis and the accompanying research project that need to be addressed in the future.

ELEMENTS OF GERMAN TEACHER EDUCATION

An undisputed goal of German teacher education at university is to foster student teachers' ability to act with a view to later professional needs. In Germany, the teacher training is divided into two consecutive parts: firstly, future teachers go through an university degree programme. This programme provides content from, roughly speaking, three different sectors: the most extensive sector, also in terms of credit points, covers scientific knowledge (usually) in two disciplines, such as mathematics and sport. The respective content is somewhat reduced compared to that of a Bachelor's degree. The second most extensive sector comprises courses in educational science, pedagogy, psychology and sociology. The smallest part, also in terms of credit points, concerns the respective subject didactics, which in mathematics includes a small selection of courses about the didactics of fractions, geometry, stochastics, or analysis in addition to supervised discipline-specific practical trainings, of which two must be successfully attended during the course of the degree programme. After successfully completing the university degree programme, the subsequent traineeship (Referendariat) usually lasts one and a half years, during which the future teachers are supervised by experienced teachers in schools. Classroom visits by mentors after submission of written lesson plans and bi-weekly organised seminars are the main elements of this phase. The prevailing vision is that the second phase serves to apply the knowledge previously acquired at university, both in the planning of lessons and in the reflection and consolidation of practical experiences. In the second phase routinisation should also begin to emerge, which is considered helpful in order to be prepared to teach around 26 hour of lessons per week as regular teachers after the traineeship.

The described organisation of teacher training may appear to be largely unproblematic. However, a less superficial examination reveals that the subsequent organisation is based on the prevailing vision that scientific knowledge is in principle directly applicable in practical contexts. Conversely, this idea leads to the expectation that taught scientific knowledge should be more or less directly applicable. Therefore, scientific knowledge that resists immediate applicability is in need of justification. To what extent and especially in what respect scientific knowledge about mathematics, its didactics or from the educational sciences, psychology, and sociology is seen to be directly applicable in turn depends on the conceptualisation of professional practice problems. How the intended curriculum works also reflects the actual requirements (e.g. teaching load, the combination of chosen disciplines) in the second phase and how these are perceived by the student teachers. The same applies to student teachers in the first phase: their evaluation of the courses offered depends heavily on their expectation on professional practice demands and what role the acquired knowledge can and should play in coping with them. A reduction to the immediate mastery of action problems, which are regarded as essentially individual and which arise with regard to personal competences, then makes it appear at least questionable to what extent scientific knowledge can be helpful. The problem situation outlined above is reflected, among other things, in the widespread view that university teacher education is ultimately uselessly theory-heavy. Regardless of whether students learn the content of the sectors mentioned at all, which would of course be a prerequisite for their later ability to use it, political institutions have long been calling for reforms to teacher training programmes in Germany, above all to ensure greater practical relevance. Currently proposed reform models address earlier and more extensive practical phases in schools (Wissenschaftsrat, 2023). This reinforces a concept of applicability, in which, for example, individual usefulness and an essentially personalised conceptualisation of competences are conceived. With respect to this vision of applicability, one could speak of a subversive function of practice including a tendency to a softening or even destruction of the academic demands of university teaching and learning (Wenzl et al., 2018).

The aforementioned interrelations are particularly evident in the following phenomenon regarding students' lesson planning: the content analysis is dominated by an academically oriented presentation of mathematical knowledge. In the methodical and didactic design of the planned lessons, the content is then only addressed superficially and not connected to possible content-related learning processes. Both parts of the lesson plan are discussed more or less isolated from each other. This is reinforced by the focus on competences, which primarily address general subject-related skills. These can also be observed in study projects and Bachelor's theses in which lesson plans are presented and justified (Hochmuth & Peters, 2023; in press).

The intervention reported here is intended to counteract these tendencies. It takes place in a university course, which is located in the field of mathematics education. The students are usually at the end of their Bachelor's degree programme, have already successfully attended at least three one- or two-hour courses in mathematics didactics, and are approved for writing a Bachelor's thesis in mathematics didactics. At the end of the course, students should have prepared an exposé for their thesis. This includes the aims of the thesis (general question, definition of the mathematical topic), the theoretical framework, the specific scientific research questions, the methodological approach, the expected results, a discussion, and, if applicable, an outlook.

The focus of the Bachelor's thesis should be on analysing, modifying and supplementing given lesson plans (Heckmann & Padberg, 2012) with an emphasis on content analysis and its connection to methodical and didactic decisions on lesson design. In particular, the following questions should be addressed: Which subject-related questions and answers (e.g. operations, actions, and reasoning) can and should be stimulated? Which concepts should be learnt beyond aspects of calculation? How can the concepts be described and characterised in terms of actions (practice and justification-discourse)? What does "Pupils have understood a concept" mean? How are these aspects embedded in knowledge about other topics and concepts (in particular prerequisites, continuations) and taken up in a selection of tasks?

THEORETICAL BACKGROUND

Connecting and integrating subject-specific analyses into methodical and didactic considerations and the design of teaching sequences was a central problem identified in previous projects (Hochmuth, Peters, 2023; in press). There, we had already formulated some hypotheses regarding hurdles or restrictive ideas inherent in the actions of student teachers: applicationism (Barquero et al., 2011) with regard to the application orientation of the teaching of school mathematics; subject-related deficits; small-step learning as an expression of the desired strong control over teaching by the teacher (uneasiness of unexpected learning paths); focus on competences that replace subject related considerations and formulations of objectives. A main hurdle identified in our previous studies was insufficient subject-specific knowledge, which could have been acquired but was not. Reasons were: available time, skills and positionings in relation to institutional conditions, and students' perceptions of related tasks (such as the establishment of defensive forms of learning, see (Holzkamp, 1995)).

With the intervention reported here, we take up these observations, also with the aims of differentiating them with regard to students' preparation of subject-specific content and developing possibilities to counteract the hurdles. In designing the intervention, we are guided by the SRP-TE concept (Barquero et al., 2018): the sequence of the course is organised along several modules conceptualised in the SRP-TE. Here, the content is designed to support students in taking subject-specific aspects and related content analyses into account in depth when planning lessons, but not to design lessons in the sense of an SRP. Our methodological approach, the research question of the paper and our preliminary answer could also be considered as part of didactic engineering (Artigue, 2020). In the following, we give a short sketch of our adaptation of the SRP-TE and its Modules 0 to 4. Further details about the intervention presented in the next section are essentially focused on Module 2.

First of all, we agree that it is important to create a common empirical milieu shared by student teachers and educators-researchers in order to approach the institutional constraints that are involved in intended and actually realised subject-related activities in lesson planning. Accordingly, we start (Module 0) with the following professional teaching questions: how can we analyse, adapt, develop and integrate subject-specific considerations in lesson planning and their connection to methodical and didactic considerations? How can we support understanding-orientated learning processes based on subject-specific analyses? What limitations are there and how could they be overcome?

The adapted Module 1 consists of jointly reflecting on selected published lesson plans (Heckmann & Padberg, 2012, pp. 129-364) on the basis of prior knowledge, e.g. from previous didactics courses. In particular, this includes key questions and basic elements of lesson planning (Barzel et al., 2012; Heckmann & Padberg, 2012, pp. 33-120) and content-related knowledge about the official curriculum as well as basic ideas, objectives and forms of tasks, with a focus on their contribution to content analysis in the lesson planning. This should also contribute to the creation of a common empirical milieu. Analogously to Module 2 in (Barquero et al., 2018), deficits in the content analyses and their coherent integration into the overall concept of the lesson plans are then discussed on the basis of concepts and questions used up to that point. This is followed by an introduction to some concepts and approaches of ATD. In particular, a reference is made to the ideal-typical distinction between two types of teaching-learning processes, namely a) teaching-learning processes in the sense of the Paradigm of Visiting Works (PVW) and b) teaching-learning processes in the sense of discovery-based learning, the Paradigm of Questioning the World (PQW) (Chevallard, 2015).

After the ATD concepts have also been illustrated with examples, the students carry out their own analyses of lesson plans in pairs (Module 3). These are initially based on the standard procedure against the background of which they were created. Then, on the basis of further literature (including textbooks), exemplary ATD analyses are carried out with a view to PVW and PQW. The results are presented and discussed by the whole group. Special emphasis is put on the modifications of the given designs stimulated by the ATD analyses and their reflection with regard to possibilities of overcoming obstacles to a coherent integration of subject-specific analyses in lesson plans. The final step (Module 4) then consists of continuing to work on the already modified lesson plans against the background of the obstacles and opportunities discussed in the group and preparing an exposé for writing a Bachelor’s thesis. This includes: A mathematical theme chosen by the student but agreed with the supervisors, an existing lesson plan of a teaching unit that is to be analysed and modified according to the outline above using additional academic literature on the theme. The main focus is on analysing the content and connecting it coherently with the other elements of the lesson plan. The modifications have to be justified in a dedicated section and the prior deficits discussed, particularly with regard to institutional obstacles.

With a view to the goals of our research project addressed in the introduction, we will conduct an a posteriori qualitative analysis of the intervention based on empirical data (e.g. students’ presentations, reworked lesson plans and Bachelor’s theses by the students, interview data). Regarding institutional obstacles and constraints, we will refer to the ATD concept of the levels of codetermination. Here we expect that all higher levels are relevant. In the following preliminary a priori analysis, as in the introductory remarks of this section, we refer to insights from previous studies.

THE TEACHING PROJECT: A PRELIMINARY A PRIORI ANALYSIS

The starting point of the intervention is a joint discussion of lesson plans presented in standard didactic literature in Germany (Heckmann & Padberg, 2012). The majority of those plans are very good. However, they contain no systematic proposal with regard to the following questions: how can the content be modelled in order to increase the coherence of decisions on objectives, content and methods? And more concretely: how can a subject-specific analysis be structured and designed in such a way that it might be used to reflect on subject-related objectives and, with this in mind, methodical and didactic specifications? These questions guide our introduction of ATD-notions and tools including teaching-learning paradigms and related praxeological dimensions of analysis, in particular praxeologies, didactical moments and question-answer maps as well as the notions of chronogenesis, mesogenesis and topogenesis.

Accordingly, we present the notion of “praxeology” as something referring to four fundamental questions and corresponding answers: a) What do people do in a given context? They perform tasks of some type (type of tasks). b) How do they do this? By using a certain method or technique (technique). c) Why do they do it this way? Because the technique seems appropriate and can be justified, but also described (technology). The justification is usually connected to a supporting and embedding discourse at a higher level (theory). And related to c: Why do people do what they do in the context? (raison d'être).

On this basis, we then analytically distinguish between two types of teaching-learning processes and situations, PVW and PQW (see previous section), which can interlock and/or follow on from each other in a teaching unit. Regarding PVW, we explain the following: The notion of works (W) usually covers several praxeologies (P). Related questions with regard to lesson plans and possibilities for analyses include the following two interrelated steps: (1) To answer the questions: What does W(P) consist of? How is W(P) used? What is W(P) for? The aim here is to generate a simple reference epistemological model (REM) (Lucas et al., 2019) with regard to the mathematical themes in the considered lesson plan. The REM should consider types of tasks, techniques and aspects of technology and theory. On this basis, the questions (2) How is the visiting of this knowledge organised? and Should it be modified? can then be addressed. The model of didactic moments can then be addressed against the background of the REM: moment of first encounter with the type of tasks and of the identification thereof; moment of the exploration of the type of tasks and of the emergence of the technique; moment of the construction of the technological-theoretical block; moment of institutionalisation; moment of praxeological work; moment of evaluation. Potential didactic phenomena are then considered by the following questions: Do the respective moments occur? How are they related to each other? How are they organised, e.g. timing, milieu, responsibilities? What is the intention in each case? What is to be expected? Of course, this should not be seen as a linear sequence of separate steps. In terms of content, they are mutually dependent, so that it is obvious that one can and should refer back to the previous steps if their modification and expansion appears appropriate. Explicit praxeological modelling and consideration of didactic moments are intended to counteract the naturalisation of the content by distancing. By going back to the aforementioned questions of lesson planning, the praxeological considerations should be experienced as a structured content-based modelling that can be used to answer them and might guide the analyses and finally the modification of a lesson plan. PVW is then illustrated using an example from a school textbook: a praxeology is presented and, against this background, exemplary passages in the textbook are identified in which didactic moments are addressed in relation to it. In the next step, students are required to do this themselves.

We take a similar approach with regard to PQW. It focuses more strongly on the raison d'être of the knowledge addressed in the lesson plan: Where does the question come from that is answered by this knowledge? What other questions are related to it? What other answers could arise? We introduce question-answer maps as a form of representation and point to the issue that the initial question does not pre-exist, in particular, that there is no clearly specified initial question. For this reason, we show an example that presents various options and discuss the issues that can be connected to them.

It is equally important that students understand that dimensions and questions of the standard analysis can be addressed using the presented ATD tools, e.g.: The introduction of the sequence of the teaching unit and its development could in particular be addressed in the context of didactical moments. Teaching-learning methods in particular relate to the topogenesis (the dialectic of individual and collective), the lesson progression relates also to the chronogenesis (the dialectic of questions and answers) as well as the mesogenesis (the dialectic of medias and milieus). We expect that coherence issues regarding the content and methodical and didactic decisions arise in particular by working on the following two questions: Do the didactic moments occur? What questions and answers are generated?

The infrastructure thus provided should then be used by the teacher students (see also the description of the Modules 3 and 4 in the previous section): first in individual work, then cooperatively and finally, in the seminar discussion, in working with ready-made good lesson plans. At the end, students choose a lesson plan on a topic and prepare an exposé. Once agreement has been reached with the responsible supervisors, the Bachelor's theses are written.

Regarding the professional goals addressed in the research question, we expect the following from the students with respect to the analysis and modification of lesson plans: a) an expansion of the otherwise primarily local reflections and b) an increased integration of the various parts with a view to the raisons d'être of the content. In particular, we expect that objectives in terms of content and process-related competences can now be addressed in a more substantive way. In view of the above-mentioned students' expectations on professional practice demands, we consider links to known standard procedures to be helpful in order to prevent motivation problems in the sense of "What we are learning here is offside and not very helpful for our further path, at least beyond writing a Bachelor's thesis.".

With regard to PVW, we expect that the students will work on the topics in a more differentiated way: reflection on the didactic moments focuses on specific aspects of content and activities, such as the formulation of tasks, the use and justification of certain calculations, the presentation of argumentation steps and the explicit questioning of the methodical and didactic design. Since these steps still largely follow the prevailing PVW, we essentially expect hurdles in terms of the mathematical content with regard to its broad and deep elaboration.

The implementation of PQW requires further distancing from the standard curricular structure. In previous studies (Hochmuth & Peters, in press), we observed that the questions formulated in question-answer maps essentially reflected the curricular structure, but now in the form of questions instead of learning goals. This can be useful and effective in individual cases. As a rule, however, these questions are not motivated by a previous question or answer and the answers to these questions do not raise any new questions of their own. In other words, the question-answer relationship is not dialectical and does not generate a process. By analysing the lesson plans in advance according to the standard guiding questions and the deficits identified in the process, we expect greater autonomy and distancing from the dominant processes in this respect.

From an ecological point of view, however, moments of applicationism are to be expected: in analyses and justifications of modifications, tendencies could emerge to continue to organise the teaching process essentially on the basis of pedagogical and methodical decisions, with the content remaining subordinated and its structure and logic largely not reflected upon. Content-wise, questions of correctness may dominate and the development of technologies and theory and the raisons d'être will not be decisive for lesson planning. This could also apply to the reflection and validation of learning outcomes. Moreover, we still expect to see a tendency to follow PVW also in students work with question-answer maps. Overall, we expect a cognitive and therefore individualistic focus in their considerations, which limits the development towards a more institutional point of view, the adoption of which would be helpful for using ATD tools. We expect that we need to support students in particular to become aware of those constraints.

DISCUSSION AND OUTLOOK

We propose an intervention in which ATD tools are introduced with regard to professional teaching-related issues to counteract known weaknesses in the preparation and analysis of lesson plans by student teachers. It will be necessary to examine the extent to which the stimulated activities and reflections appear to be a step towards overcoming the deficits and actually sensitise student teachers to institutional obstacles when analysing content for lesson planning. In addition, it will be important to analyse which institutional obstacles and limitations are visible in our implementation of the intervention and to identify opportunities to address them with regard to its objectives. The task will then be to establish a cycle of increasingly thorough a priori and a posteriori analyses and associated implementations.

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How Do Advanced Pre-Service Teachers Develop Congruence Theorems for Quadrilaterals?

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In our paper, we present a study in which we investigate which strategies pre-service teachers (PSTs) use to find and, if necessary, reject possible candidates for congruence theorems for quadrilaterals. This study was conducted before the PTSs attended a university geometry course. In this way, statements about learning prerequisites can be made. For the study, we analyzed group discussions of PSTs to identify typical approaches and evaluate them from a mathematical perspective. The results can be considered for the further development of courses for PSTs and generate hypotheses for further research.

Keywords: Teachers' and students' practices at university level; Transition to, across and from university mathematics; Teaching and learning of specific topics in university mathematics; Congruence; Quadrilaterals

INTRODUCTION

Addressing the transition problem in the university education of mathematics teachers, which Klein (2016) refers to as the *double discontinuity*, has been a relevant field of research in university didactics of mathematics for many years. A distinction must be made between the *first discontinuity* (transition from school to university) and the *second discontinuity* (transition from university to the teaching profession). In a design research project at Paderborn University, we have been working for several years on developing, implementing, and researching innovations to overcome the *second discontinuity* using the example of a 6$^{\text{th}}$-semester geometry course for pre-service teachers (PSTs) (e.g., Hoffmann, 2022; Hoffmann & Biehler, 2023a). In doing so, we have found that on a theoretical level, the distinction between a *subjective* and an *objective facet* of the *second discontinuity* is fruitful. The objective facet describes that it is possible that PSTs objectively lack mathematical knowledge and skills relevant to working as a teacher. The subjective facet refers to possible subjective attitudes of PSTs that their mathematical knowledge is not relevant or helpful for coping with future teaching jobs. In particular, the different facets also require different countermeasures, which are described in Hoffmann and Biehler (2022a) under the design principle of *profession orientation*. This involves implementing innovations in mathematics courses (e.g., *interface tasks*) through which the PSTs establish links between the mathematical content of the course and typical mathematics-related *jobs* (Ball & Bass, 2022; Prediger, 2013) and then reflect on these links.

Two typical *jobs* that are the starting point for our study are that, firstly, teachers must be able to master requirements for students at different levels themselves and, secondly, they must be able to analyze and evaluate approaches in teaching materials (e.g., textbooks) (Prediger, 2013). With these jobs in mind, in our course "Geometry for PSTs," we used the task shown in Fig. 1 from a mathematics textbook for lower secondary school as a situation for constructing an *interface task* (Hoffmann, 2022).

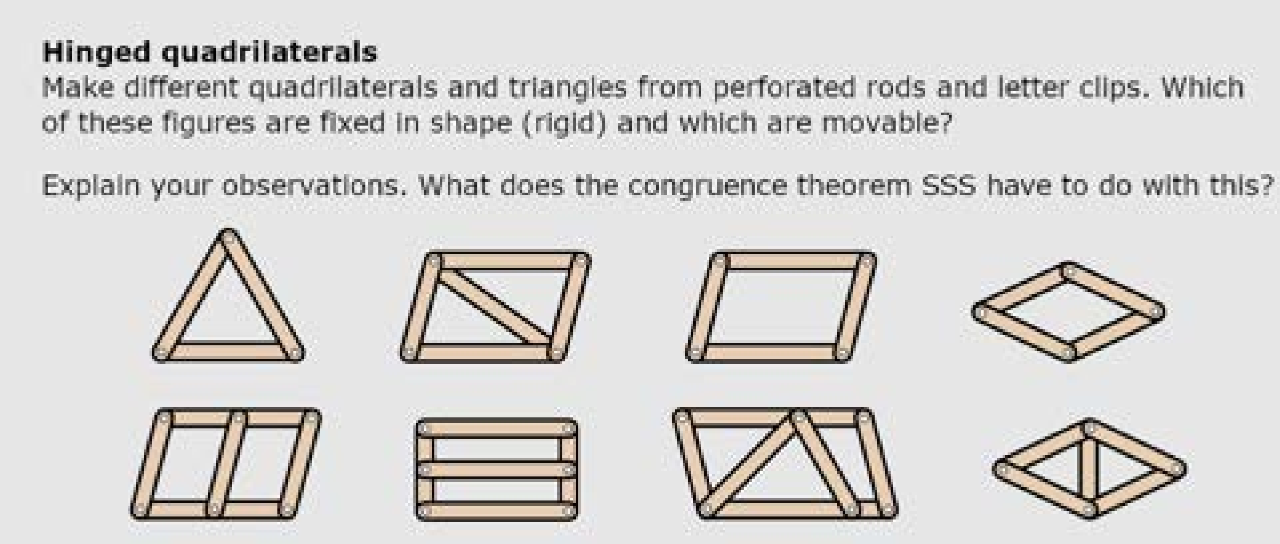

Fig. 1: Task from the textbook „Neue Wege 7“ (Körner et al., 2014, p. 195, translated).

From a mathematical perspective, this task deals with the congruence of quadrilaterals against the background of known congruence theorems for triangles. This provides a context that is very close to geometry lessons at lower secondary level (keyword: congruence of triangles) on the one hand and is significantly more complex from a rigorous mathematical perspective on the other. This means that the potential of the task can be exploited particularly well in the classroom if the teachers can fall back on a solid mathematical foundation.

In initial exploratory observations of PSTs’ activities in the context of this textbook task, we have found that many variants of task processing occurred at different mathematical levels that are worth exploring further. In terms of our methodological approach, design research (Prediger et al., 2015), our interest in empirical insights into the congruence-related learning prerequisites of the PSTs can be assigned to the research step *specifying and structuring the learning object*. It provides an essential basis for the design of suitable learning opportunities.

In this sense, we carried out a study at the beginning of the summer semester of 2023 with the aim of assessing the PSTs’ prior knowledge of the congruence of quadrilaterals. The PSTs first dealt with definitions of mathematical concepts, especially special and general quadrilaterals, and then made assumptions about valid congruence theorems for quadrilaterals. Especially the identification of false conjectures and their refuting is addressed. Both validating and refuting are important mathematical practices at all levels of education, especially in the field of geometry (e.g., Patikin, 2021).

The following task guides our activity:

From your math lessons, you know the congruence theorems for triangles, e.g.:

(SSS) Two triangles are congruent if the lengths of all three sides are pairwise equal.

(SAS) Two triangles are congruent if the lengths of two sides and the size of the angle enclosed by the sides are pairwise equal.

(ASA) Two triangles are congruent if the lengths of one side and the sizes of the angles adjacent to this side are pairwise equal.

Discuss which congruence theorems could apply to quadrilaterals and justify their validity. If you reject an idea for a congruence theorem, document how you reached this conclusion.

For our study, we videotaped four groups of PSTs in a laboratory setting. In this paper, we focus on how the PSTs proceeded in developing and rejecting ideas for possible congruence theorems for quadrilaterals and how this procedure can be analyzed from a mathematical perspective. Based on this, we discuss possible consequences for treating the topic of congruence in our course.

CONGRUENCE OF QUADRILATERALS

A common definition of the congruence of a figure $F \subset \mathbb{R}^2$ to a figure $G \subset \mathbb{R}^2$ is the existence of a (bijective) isometry $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ with $\varphi(F) = G$. Congruence is an equivalence relation on the power set of $\mathbb{R}^2$. More detailed background, also on alternative definitions, can be found in Hoffmann and Biehler (2022b). Congruence theorems are usually referred to as statements in which a few common geometric properties of two figures can be used to infer their congruence. In other words, a congruence class is already determined by these properties. Typical examples relevant to school mathematics are the well-known congruence theorems for triangles (SSS, SAS, ASA, SsA). Details on the relevance of these congruence theorems for mathematics teaching and teacher training can be found in Hoffmann (2022).

Congruence theorems can also be formulated for quadrilaterals. While three different statements are sufficient for the formulation of a congruence theorem for all common definitions of triangles, the minimum number of quantities that must match to be able to reliably conclude the congruence of two quadrilaterals depends on the precise definition of the quadrilateral used. However, four-size specifications (which, in a way, is an analogy for the three-size specifications of the triangle) are not sufficient. This can be seen from the counterexamples shown in Fig. 2, from which the other combinations can also be derived.

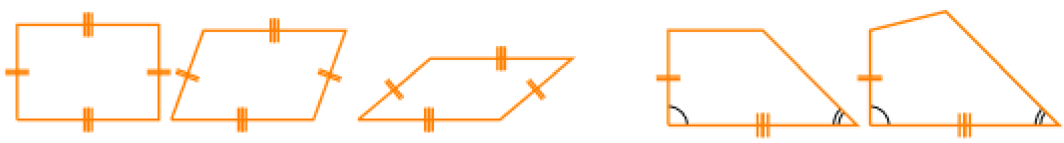

Fig. 2: Counterexamples for the validity of theorems SSSS and SASA for quadrilaterals.

If five congruent quantities are allowed instead, the congruence theorem SSSSD (where "D" stands for diagonal) is valid as long as the order of the sides remains fixed and the quadrilaterals are convex. The proof follows directly by applying SSS twice. Similarly, the validity of the congruence theorems SASAS and ASASA can also be shown under these conditions. However, if one of the two requirements is omitted, SSSSD does not provide a correct congruence theorem (Fig. 3).

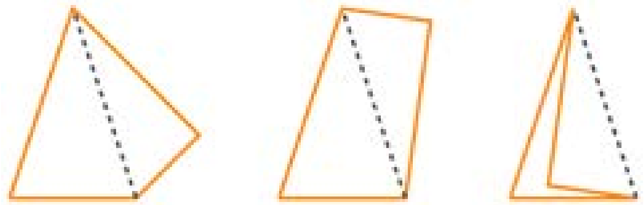

Fig. 3: If one neglects a fixed order of the sides (center) and/or the requirement of convexity (right), congruence in the sense of SSSSD does not yet follow from congruence.

If, by definition, one allows non-convex quadrilaterals, e.g., SSSSDD is a valid congruence theorem that is based on six sizes. If one also allows the sides of a quadrilateral to intersect, the situation becomes even more complicated. Further mathematical details on the congruence of quadrilaterals can be found in Laudano and Vincenzi (2017).

LEARNING PREREQUISITES OF PSTS FOR THE CONCEPT OF CONGRUENCE

To complete the task on congruence theorems of quadrilaterals presented in the introduction, the PSTs usually have to fall back on the knowledge of the concept of congruence that they acquired in their own school years, as they have not normally attended any other course at Paderborn University in which the concept of congruence is systematically dealt with. In addition, they have access to the general academic mathematical skills that they have acquired in other mathematics courses during their first two years of study. The empirical literature on PSTs' prior knowledge of the concept of congruence is very sparse. However, comprehensive studies on the elementary mathematical knowledge of PSTs (e.g., Buchholtz et al., 2013) suggest that little systematic prior knowledge can be expected in relation to elementary geometry.

Hoffmann and Biehler (2023b) found in a qualitative study that most PSTs associate congruence primarily with congruence theorems for triangles and geometric constructions. In addition, problems were found with congruence definitions in general and the adept differentiation from congruence theorems in particular. This was also evident in the part of the study where the PSTs were asked to complete a congruence definition for quadrilaterals. Many students gave candidates for congruence theorems as definitions. Although the evaluation was based on the assumption that the PSTs only think of convex quadrilaterals, most answers were incorrect. Among other things, inadmissible generalizations of congruence theorems for triangles often appeared. However, the correct congruence theorems for convex quadrilaterals mentioned in the previous section were also occasionally found.

These results already indicate that possible difficulties in dealing with quadrilateral congruence may also stem from general difficulties in dealing with definitions as described, for example, in Miller (2018) and Salinas et al. (2014).

RESEARCH QUESTION AND STUDY DESIGN

With this study, we follow up on the results described in the last section with the aim of gaining deeper insights into PSTs' prior knowledge of the concept of congruence. The following two research questions guide our study:

(RQ1) What strategies do PSTs with no systematic prior academic knowledge of the concept of congruence use when finding and rejecting congruence theorems for quadrilaterals?

(RQ2) How can the strategies found be evaluated from a mathematical perspective?

To answer the research questions, we had four groups of PSTs (group G1: 3 PSTs; G2: 2 PSTs; G3: 3PSTs; G4: 3PSTs) work on the quadrilateral congruence task ("Find congruence theorems for quadrilaterals") presented in the introduction in a laboratory setting. The task was used for all PSTs in the geometry course mentioned in the first week of the semester as part of the exercise groups. The groups in the study were PSTs who voluntarily agreed to be filmed in a separate room while working on the task. We chose the group setting to create authentic and low-threshold discussion situations in which the PSTs could demonstrate their prior knowledge of the concept of congruence.

We videotaped and transcribed these work processes and then analyzed them according to our research interests using qualitative content analysis (QCA) methods. We followed an explorative approach in which we focused on the variance of the strategies used by the PSTs and not on quantitative statements about typical or frequent procedures.

In our analysis, we first identified the passages in the transcripts in which a specific candidate for a congruence theorem is discussed. For each of these passages, we distinguished and analyzed in detail the justification and rejection processes that occur (RQ1). In addition, we examined the extent to which the identified strategies could be developed into viable arguments from a mathematical perspective.

SELECTED RESULTS

In this section, we summarize the key findings of our analysis. First, we provide an overview of the potential congruence theorems discussed by the individual groups. Then, we present the three main strategies identified in all groups when analyzing the PSTs' discussions. Finally, we reflect on expected strategies we could not code in our data.

Overview of the possible congruence theorems found

When working on the task, the four groups dealt with the candidates for congruence theorems for quadrilaterals presented in Tab. 1.

Possible congruence theorem	G1	G2	G3	G4
SSSS (*)	x		x	x
SASAS	x	x	x	x
ASASA	x	x	x	x
SSSSA		x	x	
other theorems with four sizes (*)			x	x

Tab. 1: Candidates for congruence theorems that the groups have dealt with. "(*)" means that the congruence theorem is false even for convex quadrilaterals.

It turns out that, essentially, the statements we identified above in our theoretical background were used. As expected, only congruence theorems for convex quadrilaterals were discussed.

Strategy A: Generalization of congruence theorems for triangles

> *But SSSSA definitely works. But with the triangle, it was only three. Who should expect that we can now suddenly use five?* (Alex, G3)

This strategy was found in all groups and is also triggered to a certain extent by explicitly mentioning the congruence theorems for triangles in the task. The PSTs develop a possible congruence theorem for quadrilaterals by adding another size to a congruence theorem for triangles. This is based on the understandable but incorrect assumption that if three sizes are required for triangles, exactly one more size is needed for quadrilaterals (see Alex's quote). All groups find a counterexample for SSSS relatively quickly (c.f., Strategy B). While G1 and G2 directly switch to congruence theorems with five quantities, G3 and G4 discuss other combinations of four quantities in detail. No group gives a general argument that four quantities cannot be sufficient.

Strategy B: The house of quadrilaterals as an example-provider

For checking potential congruence theorems, all groups have regularly used the usual special quadrilaterals. This is unsurprising because, as described above, there were tasks on definitions of general and special quadrilaterals before the congruence theorem task. The following example from G3 illustrates strategy B:

Clara: […] ...could apply to quadrilaterals. Yes, so SSSS in any case.

Alex: So, four sides don't work because of the rhombus. We can squeeze a square, and that's not the same, although the side lengths are all the same. [...]

[…]

Clara: Yes, that means it doesn't fit with the angles then, right? Or how?

Alex: Exactly. So at least one angle.

Clara: So, angle, angle, angle, angle would fit?

Alex: No.

Barbara: Nope.

Clara: Why this?

Barbara: Yes, square and rectangle.

Alex: Right […]

In the transcript, SSSS is refuted by square and rhombus and AAAA by square and rectangle. In addition, there is the idea of "squeezing" (see Alex) a figure. This is based on the idea that if a figure defined by certain quantities can be squeezed together, these quantities obviously cannot be a congruence theorem. This statement is correct and fits in with the approach to quadrilateral congruence chosen in the textbook excerpt in the introduction. However, there is a danger of equating the "congruence" of two figures and the "rigidity" of each figure constructed from the quantities. However, this is wrong, as the usual counterexample for a congruence theorem sSA shows: The two non-congruent triangles do not merge into each other by movement in this case.

Strategy C: Successive construction

This strategy describes a procedure used by all groups. It is used when, starting from a side length or an angle, an ordered chain of alternating sides and angles is to be specified by which the congruence of two quadrilaterals (i.e., SASAS and ASASA) is determined. The following excerpt from G1 illustrates this strategy:

Betty: [...] I'll start with one side. And I added an angle here. Done. And here's an angle. Done. Now I have another side. [sketches something on her pad]

Chris: Mhm. And which one?

Betty: Hm? This one, if you like. So that means we actually start with an angle that we have.

Chris: How do you want to?

Betty: An angle without a side.

Chris: You need two sides to make the angle.

Betty: Yes, but now you can just add an angle here. [sketches something on her pad]

Chris: Yes. And now, do you think you can draw them like this [demonstrates two rays meeting at a point with her arms] Draw the two rays like this and in …

Betty: If I draw them here like this… And then they intersect. And then I have the side and the. [points to the sketch on her pad] Right?

Anton: So, what do you mean now? Angle, side, angle, side, angle? Three angles, two sides?

Chris: It's a disaster [laughing].

Anton: It's not completely stupid.

[...]

Anton: These two angles tell you how long the side is, how long the side is, and how big the angle is. [points to the sketch on Chris's sheet]

In the transcript, the PSTs from G1 derive the congruence theorem ASASA. They start with the central angle, add a side to both rays and add another angle to the end of each. Chris and Anton then argue that the intersection of the free rays of this angle determines the remaining vertex and the angle at this vertex. This discussion clearly shows the mixing of congruence and construction already mentioned in the theory section. This is because the PSTs do not actually argue that two quadrilaterals that are congruent in these quantities are congruent but claim that one can construct exactly one quadrilateral from the given quantities in the given order starting from an angle. However, no mention is made of the fact that this construction does not work for every 5-tuple of sizes (e.g. if the sum of the three angles is already greater than 360°). This cannot be the case when comparing two existing quadrilaterals instead of construction one.

The construction strategy can also be found in our data in relation to SASAS and SSASS. Here, the PSTs argue that the fourth vertex results from the intersection of two circles. An example is the following quote:

> So, if you have the angle, that's what I just meant, then you have, if you still have all the side lengths, then you have two radii here, and this is where the intersection point must be. [sketches two sides with an angle between them and then draws two circles around the two vertices so that the point of intersection is the fourth vertex] This means that the angle in between is sufficient in any case if you know all the side lengths. There is no other possibility if you have all the side lengths. (Amy, G2).

The danger of this strategy becomes particularly clear. On the one hand, there are side lengths so that the circles do not intersect at all. On the other hand, and Amy overlooks this, the circles also intersect at another point, which would result in a non-convex triangle.

Summary and what does not appear

We have presented three main strategies that could be identified in all four groups. These are the recourse to the congruence theorems for triangles (strategy A), the use of the house of quadrilaterals as a pool for examples (strategy B), and the finding of congruence theorems via successive constructions, starting from an initial angle or side (strategy C). The observed strategies fit the anticipated prerequisites of the PSTs described in the theoretical background: The concept of congruence is strongly characterized by congruence theorems for triangles and constructions. The present study also shows that using constructions and rigidity arguments and thus the focus on one figure instead of the relation between two figures is obviously an attractive strategy for the PSTs when justifying congruence theorems.

It is interesting to note that although all groups referenced triangle congruence, no group came up with the idea of subdividing the quadrilaterals into triangles. On the one hand, this could have led to congruence theorems such as SSSSD and, on the other hand, would have provided a powerful horizon of justification for other congruence theorems.

In addition, the implicit underlying definition of quadrilaterals, particularly convexity, was not reflected at any point. Given that the study participants are no first-year students and have already completed a significant proportion of their university mathematics courses, this is worrying, as it raises the question of why the precise handling of definitions, so relevant to the pursuit of academic mathematics, is not a matter of course when dealing with a new piece of mathematics.

DISCUSSION AND OUTLOOK

In this paper, we have analyzed how PSTs proceed in developing and discarding potential congruence theorems for quadrilaterals. Of course, the results are not representative and do not show a complete picture. Nevertheless, they provide an interesting insight into possible approaches that can be built upon in the design of courses as well as in other studies. In particular, the study shows that the PSTs succeed in talking about quadrilateral congruence at a school mathematical level, but that further support is needed for precise mathematical clarification. This must then be explicitly addressed in treating the topic of congruence in the geometry course so that the PSTs can exploit the full potential of textbook excerpts, such as the one presented in the introduction (Fig. 1), in the classroom. This includes a proper reflection on the differences between congruence, constructability, and mobility/rigidity of a figure.

For the mathematical training of PSTs, this study once again shows the relevance of training in the precise use of definitions. This is in line with studies such as Salinas et al. (2014), which problematized the way teachers deal with definitions. To this end, it would make sense to revisit the question of quadrilateral congruence at a later point in the semester (after the PSTs have dealt with geometric definitions). Overall, however, this is a cross-sectional task of the university mathematics education of PSTs, which should perhaps be implemented even more as such across all courses.

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Conexiones entre las matemáticas universitarias y las matemáticas del bachillerato: La resolución general de las ecuaciones Diofánticas

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En una exploración de desarrollo profesional del profesorado del bachillerato sobre el establecimiento y desarrollo en la práctica de conexiones entre temas de matemáticas universitarias y las del bachillerato, diseñamos y montamos en plataforma Moodle un curso de 16 semanas para abordar el tema de la resolución general de ecuaciones Diofánticas. Este tema incluye propiedades básicas de los números enteros y divisibilidad, importantes para encontrar soluciones generales. Entre los resultados, resalta que el profesorado estableció conexiones en acuerdo con temas del currículum matemático del bachillerato, y también se avanzó en la implementación de las conexiones en la práctica.

Keywords: transition to, across, and from university mathematics, connections between university and secondary mathematics, teaching and learning of specific topics in university mathematics, general resolution of Diophantine equations.

INTRODUCCIÓN

En México no existen instituciones de nivel universitario (universidades o escuelas normales) encargadas de la formación inicial para el profesorado del nivel bachillerato, a diferencia de lo que sucede, por ejemplo, en los Estados Unidos de Norteamérica (USA, por sus siglas en inglés), o en Alemania. En estos países es en las universidades en donde se prepara al futuro profesorado del nivel secundario (ver, por ejemplo, Wasserman et al, 2023; Hanke et al., 2021; Murray et al., 2017). Ahí, los futuros docentes en el nivel educativo secundario siguen una serie de cursos que, en acuerdo con Wasserman (2023), usualmente son impartidos por profesores de tres departamentos o facultades universitarias distintas, a saber: (i) los cursos de matemáticas son impartidos por profesores del Departamento de Matemáticas de la universidad; (ii) los cursos de pedagogía general son impartidos por profesores del Departamento de Educación; y (iii) los cursos de didáctica específica (o educación matemática) son impartidos por el departamento o sección departamental especializada. Esta estructura común de funcionamiento en las universidades en los Estados Unidos de Norteamérica (Wasserman et al. (2023), o en Alemania (Hanke et al., 2021), se corresponde con objetivos especificados por cada uno de los departamentos que intervienen de manera separada o fragmentada en la formación inicial del profesorado de este nivel educativo (ver Figura 1). En el trabajo de Wasserman et al. (2023), se revisa dicha estructura separada de funcionamiento entre los tres diferentes departamentos universitarios (el de Matemáticas, el de Educación Matemática, y el de Educación) buscando dilucidar o encontrar respuestas a los cuestionamientos que en todos lados existen en cuanto a la utilidad de las matemáticas de la universidad para la enseñanza de las matemáticas escolares en la parte alta del nivel secundario (o bachillerato, como le decimos en México, con estudiantes de 15-17 años).

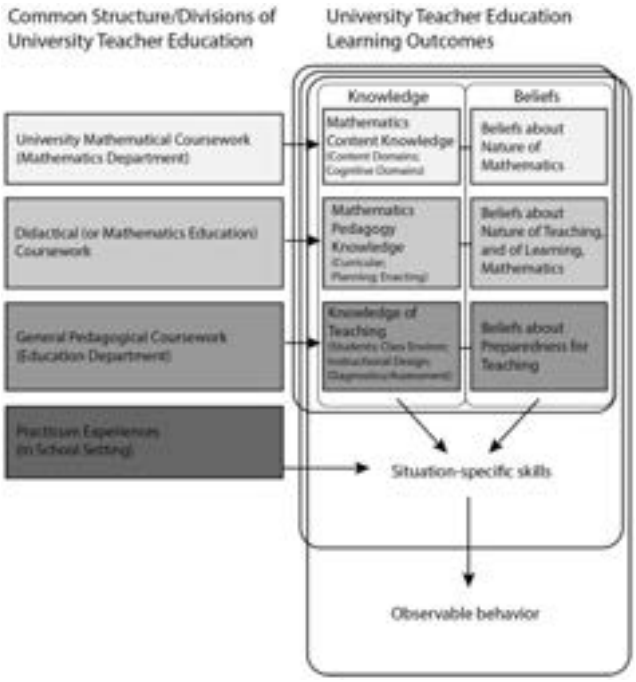

Figura 1: Estructura común, separada, entre la educación del profesorado en la universidad en USA y los objetivos de aprendizaje (Wasserman et al., 2023, p. 721).

Según estos autores, esta situación constituye un reto para la educación matemática del profesorado de este nivel educativo (p. 719).

En México, aparentemente estamos desconectados de tal problemática educativa. Sin embargo, no es así, pues a la pregunta: ¿quiénes o cuáles son las instituciones educativas encargadas de la formación inicial del profesorado de matemáticas del nivel medio-superior en México? La respuesta, en muchos casos, es que la formación en matemáticas que recibió el profesorado de matemáticas del nivel bachillerato es precisamente la que adquirieron durante sus estudios de licenciatura en la universidad. Esto es, en México también sucede con frecuencia que el conocimiento de las matemáticas del profesorado del bachillerato pasa por una *doble discontinuidad* término –descrito por Félix Klein (2016) desde el primer tercio del siglo pasado, y que Wasserman et al. (2023) identifican como parte de la problemática asociada al cuestionamiento de la utilidad de las matemáticas de la universidad para la enseñanza de las matemáticas escolares del nivel bachillerato. Además, también otros autores (ver Grenier-Boley, 2022; Biehler & Durand-Guerrier, 2020) han señalado esta doble discontinuidad como una avenida promisoria de investigación en muchos sentidos, entre ellos el de "encontrar tópicos relevantes con fundamentación epistemológica fuerte que les permita a los (futuros) profesores comprender de manera profunda vínculos cruciales entre las matemáticas de la universidad y las matemáticas del nivel secundario, desde una perspectiva profesional." (Grenier-Boley, 2022, p. 130).

El trabajo que aquí se presenta pretende contribuir en la línea de investigación sobre la utilidad de las matemáticas del nivel universitario en la enseñanza de las matemáticas escolares del nivel bachillerato, específicamente, por medio de la realización de un proyecto de exploración sobre el desarrollo profesional de profesorado del nivel secundario, para el que diseñamos y montamos en una plataforma *Moodle* un curso en línea de 16 semanas en el que participaron docentes en servicio del nivel educativo mencionado. El tema de matemáticas universitarias que elegimos abordar en el curso fue la resolución general de ecuaciones Diofánticas, tema que incluye propiedades básicas de los números enteros y de divisibilidad, ambos subtemas también forman parte de cursos universitarios de álgebra superior o de teoría de números. Además, en el diseño del curso se incorporó una revisión pedagógica del currículum del bachillerato, y la búsqueda de posibles vínculos de los temas de este currículum con el tema y/o los subtemas matemáticos del curso.

Las preguntas que nos interesó responder fueron: (a) ¿Cuáles son las posibles conexiones que generan o establecen los docentes entre una temática específica de las matemáticas universitarias con las matemáticas del currículum del nivel bachillerato? (b) ¿Cuál es una posible estrategia pedagógica efectiva, para que los docentes desarrollen en la práctica, con sus estudiantes, tales conexiones? (c) ¿Cuál sería una posible trayectoria de aprendizaje de esas conexiones, por parte de los estudiantes, en la práctica?

MARCO TEÓRICO, METODOLÓGICO, Y RESULTADOS

Enfoque en la práctica en el salón de clase y en la enseñanza conceptual de un tema matemático específico

En esta sección, se revisa principalmente el trabajo de Murray et al. (2017), quienes "se centran explícitamente en las implicaciones para la enseñanza que se derivan de la comprensión de los docentes de las conexiones entre las matemáticas del bachillerato y las [matemáticas] avanzadas" (p. 2), y en el cual se buscó indagar sobre el impacto en las prácticas de enseñanza de los conocimientos en las matemáticas universitarias aprendidas por los docentes en su paso por la universidad.

El enfoque en el impacto en las prácticas de enseñanza, en una indagación sobre desarrollo profesional de docentes, es reciente en el campo de la investigación en educación matemática. Incluso en el mismo trabajo de Murray et al., no se alcanzó a observar qué sucedía en la clase de matemáticas –y/o si el docente llevaba a la práctica las conexiones revisadas/encontradas en el curso universitario en donde tales conexiones habían llegado a establecerse. En su trabajo, Murray et al. (2017, pp. 2-3) discuten y llegan a identificar categorías que ejemplifican formas potenciales en que ciertas conexiones (entre matemáticas universitarias y matemáticas del bachillerato) potencialmente influyen en las prácticas de enseñanza. Así, uno de los objetivos del trabajo de Murray et al. (2017) fue "entender qué conexiones impactan la instrucción, como un medio para identificar y situar el conocimiento matemático universitario bajo una lente del conocimiento docente basado en la práctica." (p. 3). Finalmente, una de las contribuciones más importantes de Murray et al., en su indagación sobre el tema (de las conexiones entre las matemáticas universitarias y las del bachillerato), es que no se trata de "discusiones sobre las conexiones matemáticas per se, sino más bien de discusiones de formas específicas en las que el conocimiento de las conexiones matemáticas podría influir en las elecciones pedagógicas del profesorado de bachillerato, en el aula." (Murray et al., 2017, pp. 2-3)

Sobre el tipo de comprensiones clave en el desarrollo (KDU)

En la trayectoria conceptual que trazan Murray et al. (2017) en la concreción de su indagación, también aparece el constructo teórico de *comprensiones que son clave para el desarrollo* (KDU, por sus siglas en inglés). De acuerdo con Simon (2006), las *KDU* son conocimientos matemáticos (del contenido) críticos, necesarios para que se lleve a cabo el desarrollo matemático de los estudiantes. Según Simon (2006), las KDU implican un avance conceptual. Más aún, Murray et al. (2017) sugieren que las matemáticas avanzadas pueden ayudar a los profesores a desarrollar "conocimientos que respalden la enseñanza conceptual de un tema matemático particular" (Silverman & Thompson, 2008, p. 508. Citado en Murray et al., 2017, p. 3). Así, para que un tema de las matemáticas avanzadas les sea útil a los profesores, Murray et al. sostienen que tal tema tiene que servir como KDU para el contenido del bachillerato.

Elección de un contenido relevante al abordar la segunda discontinuidad de Klein

Con respecto a la búsqueda de "tópicos relevantes con fundamentación epistemológica fuerte que les permita a los (futuros) profesores comprender de manera profunda vínculos cruciales entre las matemáticas de la universidad y las matemáticas del nivel secundario, desde una perspectiva profesional", Grenier-Boley (2022) sugiere cuatro aspectos principales a considerar para que una noción sea relevante en [el aprendizaje de] los futuros docentes:

> (i) la elección de una definición y la articulación entre la definición y el proceso de prueba; (ii) el uso de herramientas específicas que arrojen luz en la resolución de un problema; (iii) la elección y uso de un registro apropiado de representación y la subyacente flexibilidad para mediar entre ellos; (iv) la importancia para investigar y para resolver inconsistencias y 'círculos viciosos' en la currícula. (Grenier-Boley, 2022, p. 131)

Características del curso, desarrollo de contenidos matemáticos y pedagógicos, y su relación con posibles conexiones entre temas de matemáticas universitarias y el currículum del bachillerato

En la exploración que aquí se presenta, sobre el establecimiento de conexiones entre el tema de la resolución general de las ecuaciones Diofánticas y los temas de matemáticas del bachillerato, diseñamos un curso en línea para el desarrollo profesional del profesorado de la Escuela Preparatoria Oficial (EPO) 171, con duración de 16 semanas, el cual se montó en una plataforma Moodle. Participaron los tres profesores de matemáticas encargados de todos los cursos de matemáticas de la EPO mencionada.

El curso, denominado "*Propiedades de los números y resolución de ecuaciones en números enteros*" se montó en una plataforma *Moodle* (ver http://pascal.ajusco.upn.mx/mpupn/course/view.php?id=33) de la Universidad Pedagógica Nacional (México). Como se sabe, una ventaja del uso de una plataforma de este tipo es la posibilidad de acceso a todos los materiales y actividades del curso desde cualquier lugar y en cualquier momento. Facilidad conveniente para el desarrollo profesional de los docentes en servicio. Además de que toda la actividad de los docentes participantes en torno de la realización de las tareas asignadas queda registrada en la plataforma.

Sobre el contenido matemático del curso, el tema de las ecuaciones Diofánticas es un tópico del álgebra superior o de la teoría de números que se aborda en la universidad, y raras veces se ha incorporado en la educación matemática del profesorado. Sin embargo, goza de un potencial natural para el establecimiento de conexiones matemáticas entre las propiedades fundamentales de los números enteros y de la divisibilidad de la teoría de números, subtemas que hace tiempo Healy & Hoyles (2000) usaron al indagar sobre la prueba con estudiantes del bachillerato. Cuestión por la que en particular es posible asociar estos subtemas con el primer aspecto que hace a una noción o tema relevante para el aprendizaje de los docentes, como señaló Grenier-Boley (2022). Por otro lado, la resolución general de las ecuaciones Diofánticas conecta con el currículum del nivel secundario, específicamente con la resolución de ecuaciones lineales en dos incógnitas y con las ecuaciones de segundo grado con dos incógnitas, incluidas las ecuaciones de las cónicas.

Así, las ecuaciones Diofánticas lineales son ecuaciones de primer grado en dos incógnitas con coeficientes enteros, forman parte de las ecuaciones conocidas como lineales en el currículum del bachillerato, y su gráfica es una línea recta. El problema general que se plantea en cualquier ecuación Diofántica, por ejemplo, $2x-y= 5$, es el de encontrar la expresión general (o fórmula) para calcular/encontrar todas las soluciones enteras de la ecuación dada. Así, en la ecuación $2x-y= 5$, una solución es $x=3$, $y=1$, pues $2(3)-1= 5$. Otra solución es $x=8$, y $y=11$, pues $2(8)-11= 5$. Y el problema que se plantea resolver en este tipo de ecuaciones es: ¿cuál es la fórmula general para todas las soluciones enteras de esta ecuación? El desarrollo del contenido matemático (el cual pertenece a un curso de álgebra superior universitaria) que planeamos para el curso de "Propiedades de los números…" fue comenzar abordando dos tratamientos teóricos distintos del tema: un tratamiento deductivo y un tratamiento inductivo (el cual parte de encontrar/tener una resolución entera para las ecuaciones). Para conocer/estudiar estos diferentes acercamientos, se hizo una revisión de dos textos distintos sobre el tema, a saber, Cárdenas et al. (1988) para el acercamiento deductivo, y Guelfond (1981) para el inductivo.

Esta idea de revisar el contenido matemático del curso mediante dos aproximaciones distintas (una deductiva y otra inductiva), la seguimos con base en la teoría de la variación (Marton, 2015), para distinguir o contrastar diferencias posibles que propician el discernimiento del pensamiento lógico y/o el pensamiento matemático. Es de resaltar que ya sea el pensamiento lógico y/o el matemático, se plantean como objetivos a alcanzar en la educación matemática de los estudiantes del Sistema de Educación Media Superior (SEMS) del bachillerato del Estado de México (ver SEP-SEMS, 2017), sistema educativo al que pertenece la Escuela Preparatoria Oficial 171 (EPO171), escuela preparatoria (o bachillerato) en la que labora el profesorado participante en el estudio.

En síntesis, se planeó desarrollar el curso "Propiedades de los números…" basados en la implementación de una pedagogía para el aprendizaje de los contenidos en donde media el discernimiento de elementos críticos que intervienen en los diferentes subtemas propuestos. Partimos de una contrastación inicial entre diferentes aproximaciones al tema de la resolución general de las ecuaciones Diofánticas (deductiva vs inductiva). En general, la pedagogía del discernimiento de elementos críticos para el aprendizaje de un tema está basada en la teoría de la variación, una teoría del aprendizaje desarrollada en occidente por Marton (2015). Sin embargo, es importante mencionar que en el desarrollo de las actividades del curso "Propiedades de los números…", en general nos restringimos a la recreación y/o utilización de solamente uno de los distintos patrones de discernimiento, a saber, el patrón de contrastación.

ANÁLISIS Y RESULTADOS

A. Uno de los objetivos principales del presente trabajo fue que el profesorado participante llegara a elegir subtemas matemáticos de su preferencia, de entre los subtemas incluidos en el proceso de obtención de las soluciones generales de las ecuaciones Diofánticas (ver Cárdenas et al, 1988; Guelfond, 1981), y en concordancia con el currículum de matemáticas del bachillerato. Más aún, esperábamos poder observar que tales elecciones llegasen a ser subtemas susceptibles de ser desarrollados en las clases de matemáticas del profesorado, con sus estudiantes.

En efecto esto se logró, y desde nuestro punto de vista, fue debido a las actividades de discernimiento (ver Marton, 2015) implementadas en el curso sobre las propiedades de los números y las ecuaciones diofánticas. En general, las actividades de discernimiento estuvieron vinculadas a sucesivas y diferentes contrastaciones propuestas en las distintas unidades del curso. La serie de contrastaciones (por ejemplo –ver Figura 2: entre razonamiento o procedimiento matemático deductivo vs razonamiento o procedimiento matemático inductivo; o entre acciones abductivas o inductivas/deductivas), se desplegaron a medida que se llevó a cabo la revisión de los materiales y la realización de tareas, y estuvo asociada a la revisión de los contenidos de las diferentes unidades de aprendizaje del curso. En particular, es de notar que se abordó la revisión de subtemas matemáticos y/o subtemas pedagógico-didácticos en el curso, en acuerdo con Murray et al. (2017).

En el esquema que se presenta a continuación (ver Figura 2) se muestran las sucesivas contrastaciones propuestas a lo largo del curso. Las contrastaciones abordadas fueron nucleares en la promoción de comprensiones claves en los docentes (KDU's, de acuerdo con Simon, 2006), para el desarrollo del aprendizaje de subtemas identificados como apropiados para ser desarrollados en sus clases de matemáticas.

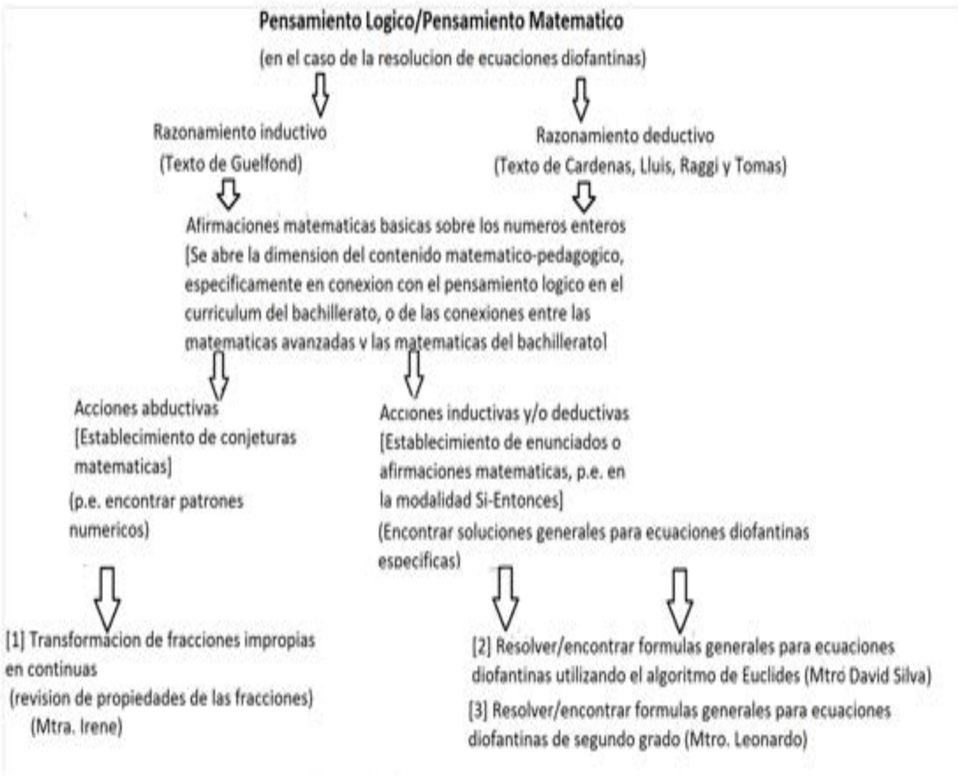

Figura 2: Esquema de diferentes fases de discernimiento y/o de contrastación en el curso "Propiedades de los números…" (elaboración propia)

Así, los temas de las matemáticas universitarias, sugeridos por el profesorado participante como susceptibles de ser abordados en sus clases, o como parte de los temas del currículum de matemáticas del bachillerato, fueron los siguientes:

1) Profesora Irene: La transformación de fracciones impropias en continuas.

2) Profesor David: Resolver/encontrar fórmulas generales para ecuaciones Diofánticas lineales.

3) Profesor Leonardo: Resolver/encontrar fórmulas generales para ecuaciones Diofánticas de segundo grado.

En otras palabras, los subtemas de las matemáticas universitarias enunciados por el profesorado en los puntos 1, 2, 3, son parte de las comprensiones claves que ellos desarrollaron en el curso, en acuerdo con Simon (2006), y/o en conexión con los temas de matemáticas del bachillerato, a saber, con la enseñanza de las fracciones (maestra Irene), la enseñanza de las ecuaciones lineales con dos incógnitas (maestro David), y con la enseñanza de las ecuaciones de segundo grado en dos incógnitas y/o de curvas y/o de las cónicas (maestro Leonardo).

B. Desde nuestro punto de vista, el desarrollo del tema de la solución general de las ecuaciones Diofánticas que presenta Gelfond (1981), hizo posible el uso de herramientas específicas que arrojaron luz en la resolución de un problema, en este caso, el de encontrar una fórmula o solución general (ver arriba punto ii de Grenier-Boley), y que además les "permitió a los (futuros) profesores comprender de manera profunda vínculos cruciales entre las matemáticas de la universidad y las matemáticas del nivel secundario, desde una perspectiva profesional", como lo muestran la resolución de sus tareas en el curso, vinculada con su elección de las tareas a resolver, y con los temas curriculares del bachillerato que en particular llamaron su atención, como de manera parcial se puede ver enseguida (en el ejemplo de resolución y de tarea elegida), por parte de la profesora IG:

> … Por lo tanto hay 3 clases de razonamiento: Abducción examina una masa de razonamiento necesario, la deducción también se le llama razonamiento necesario y solo es aplicable a un estado ideal de cosas o a un estado de cosas y puede conformarse con un ideal que da un nuevo aspecto a las premisas. En el caso de Inducción o investigación experimental, su procedimiento es cuando la abducción sugiere una teoría se emplea la deducción para deducir a partir de esa teoría ideal una (mezcla de manera confusa y heterogénea) variedad de consecuencias de tal manera que si realizamos ciertos actos nos encontramos a nosotros mismos enfrentándonos con ciertas experiencias cuando intentamos realizar esos experimentos (Peirce).
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> REFERENCIAS:
>
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>
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C. Finalmente, en seguimiento a las características de los cursos universitarios para la formación del profesorado del nivel secundario que se discutieron arriba, en la revisión del trabajo de Murray et al. (2017), y como parte de los resultados del presente proyecto de formación o desarrollo profesional de docentes del nivel secundario en un curso universitario que conjuntó matemáticas y pedagogía, reportamos lo siguiente: En adición a los puntos A y B, también fue posible observar el diseño y desarrollo de una secuencia de aprendizaje para estudiantes del grado 10 en la EPO171, por parte del maestro David (antes nombrado, en el punto 2) quien fue participante –y colaborador muy importante en la implementación del curso de propiedades de los números y ecuaciones diofánticas. Pero esa colaboración es motivo para la escritura y publicación de otro aspecto de este trabajo, el de los aprendizajes logrados por los estudiantes de este nivel educativo. Por ejemplo, en el trabajo de Silva-Bautista (2022) con los estudiantes del nivel secundario, él aplicó nuevamente la pedagogía de la variación específicamente a la enseñanza y aprendizaje de la resolución general de las ecuaciones Diofánticas lineales en la EPO171. El trabajo a profundidad desarrollado por Silva-Bautista (2022 y 2024) sugiere una renovación pedagógica del tema de resolución de ecuaciones lineales del bachillerato, cuestión que desde nuestro punto de vista resalta la pertinencia del tema de la resolución general de las ecuaciones Diofánticas, como se señaló arriba, "para resolver inconsistencias…en el currículo", en acuerdo con lo planteado por Grenier-Boley (2022).

CONCLUSIONES

Los resultados obtenidos en este estudio exploratorio están asociados a las respuestas a las preguntas de investigación que aquí se plantearon, a saber:

I. ¿Cuáles son las posibles conexiones que establecen los docentes entre una temática específica de las matemáticas del nivel avanzado con las matemáticas del currículum del nivel bachillerato?

La respuesta que aquí avanzamos es que el profesorado participante estableció (tres) conexiones diferentes entre el tema universitario de la resolución general de las ecuaciones Diofánticas y los temas de matemáticas del bachillerato: a) Conexión entre la transformación de fracciones impropias a continuas, mediante la revisión del tema de fracciones situado en el primer semestre del grado 10 en el bachillerato; b) Conexión entre la obtención de la fórmula general de las ecuaciones Diofánticas lineales, con la resolución de ecuaciones lineales con dos incógnitas, situada en el segundo semestre del grado 10 en el bachillerato; c) Conexión entre la obtención de la fórmula general para resolver ecuaciones Diofánticas de segundo grado, con la revisión de la resolución de ecuaciones de segundo grado con dos incógnitas, incluidas las ecuaciones de las cónicas, situada en el primer semestre del grado 11 en el bachillerato, o curso de geometría analítica.

II. ¿Cuál sería una posible estrategia pedagógica, por parte de los docentes, para el desarrollo de esas conexiones en la práctica?

Aquí se avanzó en una respuesta sugerida de manera muy breve, arriba, en el inciso C: Consistió en la aplicación de la teoría de la variación a la enseñanza-aprendizaje de las matemáticas, en el caso de la resolución de ecuaciones Diofánticas lineales, con un grupo de estudiantes del grado 10 de la EPO171 (ver Silva-Bautista, 2022).

III. ¿Cuál sería una posible trayectoria de aprendizaje de los estudiantes en el desarrollo de esas conexiones en la práctica?

En el desarrollo de este proyecto, solo fue posible observar el diseño e implementación de una de las conexiones en la práctica, la establecida por el maestro David. Y la respuesta sugerida mediante su trabajo de investigación (ver Silva-Bautista, 2022), es que los estudiantes siguen diferentes trayectorias de aprendizaje para llegar a la resolución o fórmula general de las ecuaciones Diofánticas lineales. Entre ellas, tal vez la más relevante, es la del equipo de estudiantes que logran llegar a la fórmula general utilizando las propiedades de la división y del algoritmo de Euclides, lo cual desde nuestro punto de vista muestra las posibilidades del desarrollo del pensamiento matemático de los estudiantes del grado 10 del bachillerato, uno de los objetivos del currículum de matemáticas de ese nivel educativo.

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Insights from prospective primary and secondary teachers' lesson preparation in the context of division with different operands

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The aim of this study is to gain insight into conditions affecting prospective primary and secondary teachers' lesson preparation during their student teaching practice under mentorship of practising teachers. We chose the context of division with different types of operands. Data for our analysis consisted of 14 lesson plans from students in their final years of study from two large public universities in Croatia. We carried out a qualitative analysis of the lesson plans by determining mathematical and didactical praxeologies. Results showed praxeologies were not connected and students' lesson plans correspond to the traditional teaching paradigm with teacher in the dominant role. Shaping prospective teachers' praxeological equipment requires redefining the roles of university, teacher educators, student teachers and practising teachers.

Keywords: Teachers' and students' practices at university level; Curricular and institutional issues concerning the teaching of mathematics at university level; Transition to, across and from university mathematics; Prospective teachers; Division.

INTRODUCTION

Initial teacher education holds great responsibility for future education. It should prepare prospective teachers to handle ever-changing challenges of contemporary education. Connecting their theoretical knowledge gained through subject and didactic courses with school practices is necessary to ease the transition from university to school and empower prospective teachers in their future profession.

Students first experience in teaching is during their student teaching practice organised by their teacher training institution and supervised by practising teachers. We chose to explore students' lesson preparation in that context as an initial exploration aimed at understanding influences on prospective teachers' professional development.

BACKGROUND OF THE STUDY

Theoretical framework

According to the Anthropological theory of the didactic (ATD), mathematical objects exist in different forms within specific institutions and their manifestations depend on conditions and constraints imposed from different sources, where conditions can originate at the following levels of the scale of didactic codeterminacy: Humankind ⇄ Civilisation ⇄ Society ⇄ School ⇄ Pedagogy ⇄ Discipline ⇄ Domain ⇄ Sector ⇄ Theme ⇄ Subject (Kidron et al., 2014; Otaki & Asami-Johansson, 2022). Curriculum requirements impose conditions at the Society, available tools at the School, and teaching paradigm at the Pedagogy level. Otaki and Asami-Johansson (2022) placed the didactic profession between Society and School levels. It engages actors, including teacher educators, in a paradidactic activity – studying the study process itself.

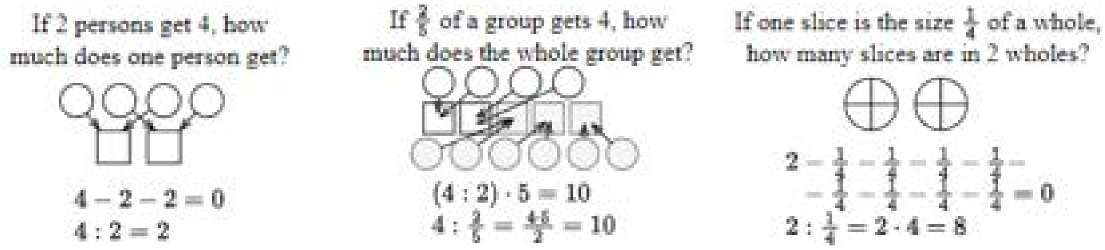
Study process can be described with mathematical and didactical praxeologies, where praxeology models any activity through *praxis* – the know-how and *logos* – the know-why. *Praxis* is determined with a task of some type and a corresponding technique of solving it, and *logos* with a technology supporting the technique and theory formally justifying the whole of a praxeology. Mathematical praxeology is the result of the study process, and didactic praxeology is the means to achieve the result (Barbé et al., 2005). Praxeologies manifest through activities with ostensives – perceptible objects such as words, visuals and symbols, representing certain idealised institutionally relevant object (Arzarello et al., 2008). There are six moments in the didactics process: the first encounter (with a notion), exploration of the type of tasks, work on a technique, construction of the discourse, institutionalization, and evaluation (Barbé et al., 2005).

ATD describes two teaching paradigms differing in the way mathematical object emerges within the study process (Jessen, 2022). The Visiting the Monuments (VM) paradigm corresponds to "traditional" methods where teacher introduces new object and then demonstrates and evaluates its application for solving tasks of some type. The Questioning the World (QW) paradigm places students at the core of study process as they explore questions, search for answers, pose new questions and find related objects along the path. The major difference between these paradigms is in roles of a teacher and students in their activities with media and within a milieu (Kidron et al., 2014).

Literature on division

Literature discerns several semantic structures in division contextual tasks, e.g. equal groups, share, comparison, and unit problems (see e.g. Lee, 2017). With respect to the referent unit-whole relationship, division problems can be partitive or quotative. In partitive division, dividend, quotient and remainder have the same type of unit. In quotative division, this holds for dividend, divisor and remainder. Partitive share problems appear appropriate for introducing whole number division, and quotative and partitive unit problems for fraction division (Kim & Pang, 2017). Relevant division properties include repeated addition and subtraction, division as the operation inverse of multiplication, and representation by iterating, grouping or partitioning using set, length, and area models, number line, and other. Depending on the type of operands, or even their values, different division strategies and algorithms apply, e.g. invert and multiply fraction division algorithm or dividing fractions by reducing them to common denominator, e.g. $\frac{2}{3}:\frac{5}{6}=\frac{4}{6}:\frac{5}{6}=\frac{4}{5}$ (Kim & Pang, 2017).

Context of the tasks affects creating appropriate representations and strategies (Lee, 2017). Repeated subtraction is commonly used for solving partitive whole number division problems, but in fraction division, it complies with quotative division problems, as in Figure 1 (Lee, 2017). There are benefits of using tasks of different, but also of the same semantic structure when sequencing types of operands, both while connecting context, representation and division strategies (Kim & Pang, 2017). Studies show that both primary and secondary prospective teachers (PT) struggle with division ideas and algorithms. They have difficulties explaining different division techniques (Lee, 2017; Sitrava, 2019) and representing and relating context with a division technique (Sitrava, 2019). They misinterpret fraction multiplication and division, as well as partitive and quotative division (Sitrava, 2019). However, there are few studies considering division with different types of operands although establishing connection between them is seen relevant and useful (Kim & Pang, 2017).



Task	Solve contextual task	Solve contextual task	Solve contextual task
Technique	Divide using repeated subtraction; Divide within 10×10 multiplication table	Divide using inverse operation; Divide using invert and multiply rule	Divide using repeated subtraction; Divide using invert and multiply rule
Technology	Relation between division and repeated subtraction	Relation between multiplication and division	Relation between division and repeated subtraction
Theory	Whole number division	Fraction division	Fraction division
Ostensives	Partitive division semantic structure; Set model representation	Partitive division semantic structure; Set model representation	Quotative division semantic structure; Area model representation

Figure 1: Example of division semantic structures, praxeologies and included ostensives

Research questions

The aim of this study is to gain insight into conditions and constraints affecting PTs' lesson preparation in the context of development of knowledge of division across primary and lower secondary education in Croatia. We chose to examine lesson plans from their student teaching practice since we find their work multifacetedly influenced by their available knowledge, knowledge taught in didactic of mathematics courses and when observing practising teachers' lessons. Teacher education institutions have means to set conditions on the levels below the School level on the scale of didactic codeterminacy (Jessen, 2022). By examining students' work, we opt for disclosing constraints from different levels of the scale that would hinder the transition to QW paradigm and development of knowledge of division. Recognising such conditions supports sustainable change in educational practices (Otaki & Asami-Johansson, 2022). For that reason, we posed the following research questions:

- What are available mathematical praxeologies and ostensives for division, in terms of semantic structure of contextual tasks, representations, and calculation strategies, in PTs' lesson plans across grades?
- How can PTs' teaching paradigm, in terms of didactic praxeologies and moments in the didactic process, be described related to QW paradigm?

METHOD

In Croatia, compulsory education is divided into primary education with class teachers for pupils aged 7-10 and lower secondary education with subject teachers for pupils aged 11-14. Prospective primary school class teachers (PPT) and prospective secondary school mathematics teachers (PST) are initially trained in different university institutions. Faculties of teacher education provide university education for PPTs, which includes psychology, pedagogy, didactics, subject matter and subject didactics courses. Faculties or departments of mathematics provide university education for PSTs with mathematics courses, courses in didactics of mathematics, and courses in psychology, pedagogy and general didactics. All teacher training programmes include student teaching practice in schools. It is placed after the subject didactics courses for both categories of PTs. Teacher training institution assigns PTs their mentors among collaborative experienced practising teachers. They provide support to their mentee in developing classroom management skills and advise them in creating and implementing activities for pupils and devising lesson plans. As exam, PT holds a lesson in their mentor's class, and their university teacher educator observes and assesses their lesson plan and its implementation.

Data for our analysis consisted of 14 lesson plans related to division that PTs prepared during their student teaching practice (see Table 1). Eight PPTs and six PSTs were in their final years of university study from two large public universities in Croatia. Following the curriculum requirements, we term division within 10×10 multiplication table, long division, signed integer division and invert and multiply fraction division as standard division techniques. Other techniques that rely on division properties, not necessarily unfamiliar to pupils, will be considered as non-standard.

Grade, age	*Lesson plan subject*	*Student*	*Number*
Second, 7-8	Division within 10×10 multiplication table	PPT	3
Third, 8-9	Long division of whole numbers up to 1000 by one-digit number	PPT	3
Fourth, 9-10	Long division of whole numbers up to million by two-digit number	PPT	2
Fifth, 10-11	Difference between partitive and quotative division	PST	1
Sixth, 11-12	Signed integer division	PST	3
Sixth, 11-12	Fraction division	PST	2

Table 1: Details of collected PTs' lesson plans

This study is a qualitative analysis of PTs' lesson plans, carried out by determining mathematical and didactical praxeologies appearing within different didactic moments of each collected lesson plan. When noting mathematical praxeologies, we searched for ostensives, i.e. objects that are directly perceptive: symbolic ostensives as the arithmetic in techniques performed to solve tasks and technology supporting them; visual ostensives as representations used in posing tasks of different type, in techniques of solving tasks, and in technology describing and validating techniques; and verbal ostensives in semantics of tasks posed as contextual problems. When observing didactic praxeologies, we sought to discern PTs' didactic intentions as lesson goals or tasks at hand, actions as means or techniques to achieve the goal, and reasons as technology originating from recommendations or beliefs coming from various levels of the didactic codeterminacy scale. Technology of a didactic praxeology points to a condition, as a discourse, stemming from particular level of didactic codeterminacy.

RESULTS

Mathematical praxeologies and ostensives

Symbolic ostensives. Figure 2 scarcely describes how division could be taught according to PTs' lesson plans in subsequent grades. In all lesson plans, numbers and arithmetic expressions were the symbolic ostensives for different division techniques across grades and types of operands. Introduction of the whole number division depended on familiar arithmetic operations; initial and consistent division techniques were the repeated subtraction and operation inverse of multiplication. The dominant technique for division beyond table was long division unrelated to techniques based on division properties. In the second and sixth grade, operation inverse of multiplication had the role of a division technique, e.g. $-16:2 = -8$ because $2 \cdot (-8) = -16$. Otherwise, it was used to verify the quotient obtained differently. E.g. following the context, pupils divided fractions by converting measurement units: since $7/2\ \text{kg} : 4/5\ \text{kg} = 350\ \text{dag} : 80\ \text{dag}$, then $7/2 : 4/5 = 35/8$, and verified the result by multiplying fractions. Invert and multiply technique for fraction division did not follow from the inverse operation relationship.

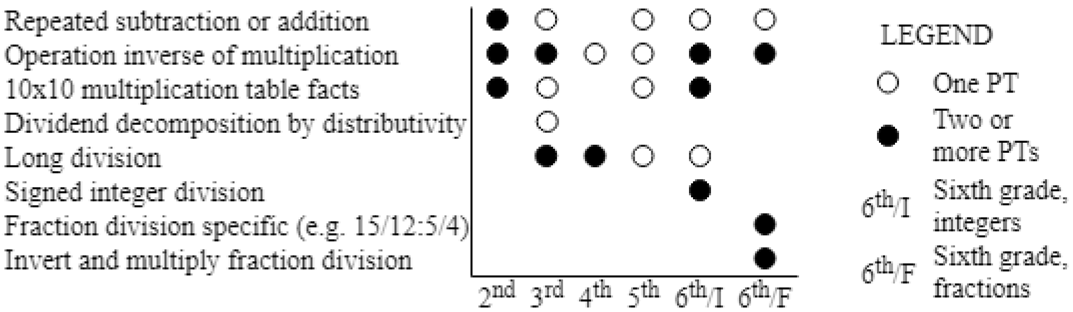


Figure 2: Division techniques across grades according to PTs' lesson plans

Visual ostensives. Manipulatives, models and diagrams were rarely included in PTs' lesson plans. In the second grade lesson plans, division was represented as partitioning of a set model, but also as partitioning of an area model, and as repeated subtraction on the number line. Third-grade pupils used money model and repeated addition to find the quotient of 153 and 3 (Figure 3.A). A set model and repeated subtraction were used in the fifth grade to differentiate between partitive (Figure 3.B) and quotative division. In the sixth grade, a bar model represented fraction division indirectly implying repeated addition and partition, e.g., $\frac{7}{2}$ divided into ten of $\frac{1}{3}$ and half of $\frac{1}{3}$ (Figure 3.C).

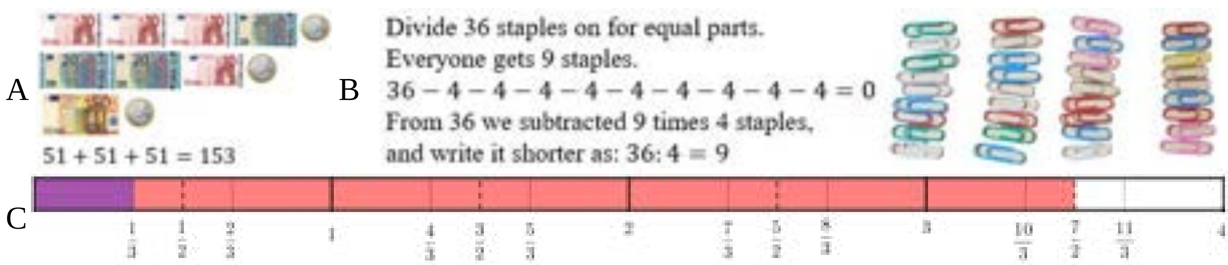


Figure 3: Visual ostensives observed in a PPT's (A) and PST's (B and C) lesson

Verbal ostensives. In lower grades, PPTs proposed solving contextual problems of different semantic structure, particularly arithmetic and multi-step problems. Partitive division problems were more common than quotative division ones and group problems were more frequent than unit problems (Figure 4). PSTs proposed problems of different semantic structures for signed integers division, two related to the depth of a lake as a negative length and one related to the negative temperature, without any visual representations. E.g. a quotative unit problem "How many years does it take to fill up a well of 27 m depth to the top if its depth decreases 9 m every year due to mud deposition?" was solved using symbolic ostensives and repeated subtraction division technique $-27-(-9)-(-9)-(-9)=0$. Quotative unit division motivational problem for fraction division asked "How many cakes can Maja make from 7/2 kg of strawberries if one cake requires 4/5 kg of strawberries?". Comparison problems suggested for fraction division required finding how many times one metric value is larger or smaller than other value of the same type.

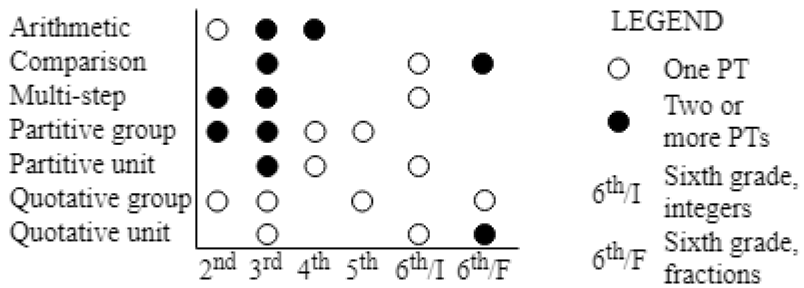


Figure 4: Semantics structures used across grades according to PTs' lesson plans

Teaching paradigms

Following and comparing individual prospective teachers' detailed lesson plans, we generalised how typical lesson plan would have been carried out in a classroom.

Typical PPT's lesson. PPTs started their lessons by recalling related objects of knowledge, often through game-like activities and digital quizzes. They motivated pupils by using kinaesthetic activities and posing contextual problems thereby purposefully concretising the problem. E.g. a student teacher called up six pupils in front of the class and encouraged them to group themselves into a given number (six, three, two, one) of groups. Acting as the provider of knowledge, the teacher then demonstrated and/or consolidated objects of knowledge pupils ought to know and use. E.g. the teacher said: "Watch and listen carefully as I am about to show you how to divide two-digit by one-digit number". Afterwards, pupils worked individually, in pairs or in groups, on solving tasks related to the presented object of knowledge in the textbook or worksheet. The prepared tasks were of different types: calculate, evaluate arithmetic expression, find unknown operand, interpret arithmetic or contextual problems. Tasks were mainly routine, but occasionally PPTs posed non-routine tasks; e.g., "pose contextual problems related to multiplication and division by 1" and "find mistakes in the following incorrect long division example". Both mentioned situations were planned as optional activities at the end of the lesson.

Typical PST's lesson. PSTs started their lessons by posing a motivational contextual problem with a familiar semantic structure. They expected pupils to recognise the need for a new, unfamiliar object of knowledge. E.g. in previously mentioned strawberry cake problem, pupils noticed they ought to divide 7/2 and 4/5 but they don't know how to perform fraction division. Teacher then provided pupils with worksheets containing selected types of tasks. Questions in worksheets were programmed to guide pupils through using available techniques and objects of knowledge, and making analogies and generalisations to disclose the new, unfamiliar object of knowledge. Teacher then constructed and institutionalised the new object of knowledge and validated it by applying it to solve the motivational problem. Pupils worked on applying the new content to appropriate types of tasks.

Mathematical and didactical praxeologies

According to the studies mentioned in the literature on division above, PTs chose appropriate semantics structures to interpret division with a particular type of operands. Partitive group division problems were dominant in grades focused on whole numbers division, while partitive unit, comparison and quotative division problems were posed in grades focused on integer and fraction division. The operation inverse of multiplication was a dominant technique for constructing and verifying standard division technique, repeated subtraction and addition were directly or indirectly merited across all grades, whereas visual representations were the least significant ostensives for division, and seldom used in the construction of knowledge. Mathematical praxeologies observed in PTs' lesson plans were mainly applications – apply division to solve word problem or apply the suggested algorithm for division calculation. No opportunities were given to exploring and connecting semantic structures, representations and strategies as grounds for constructing division algorithm. Though PTs used problems with different semantic structures, and, seldomly, non-standard techniques and representations, we discerned no connection to support the continuity in division of different types of operators.

Solving open-ended contextual problems, representing division with models, and dividing numbers using non-standard techniques appeared as secondary ideas in PPTs' lessons. They used these praxeologies with the didactic intention to `motivate´ pupils but not for exploration or construction of the standard division techniques. The student teacher presented the knowledge to be taught and had pupils work on applying it for solving different tasks. PSTs posed a problem with familiar semantic structure with the didactic intention to show the need for encountering a new object of knowledge. They had pupils solving a new set of tasks using familiar techniques. Pupils' explorations were intently guided with the didactic intention to generalise simplified available knowledge into the standard division technique. PSTs' have not planned to represent division with models or to use non-standard division techniques.

Both PPTs and PSTs chose to pose a contextual problem as a motivation, their didactic reason following the pedagogically accepted and common structure of lessons (Figure 5). PPTs tended more to concretisation, games and kinaesthetic, this being didactic activities professionally suggested for primary school education. They also included digital quizzes and self-assessment tables their didactic reason stemming from intensive propaganda from the Ministry of education during recent curricular reform. PSTs tended to connect mathematical objects within sector (division and other arithmetic operation) and theme (semantics of contextual problems) whereas PPTs focused on pedagogical means to encourage pupils to solve as many tasks. The main resource PTs used for lesson preparation were textbooks selected by their mentors. PPTs also referenced textbooks from other publishers and other curriculum materials, while PSTs referenced materials from their didactics of mathematics course.

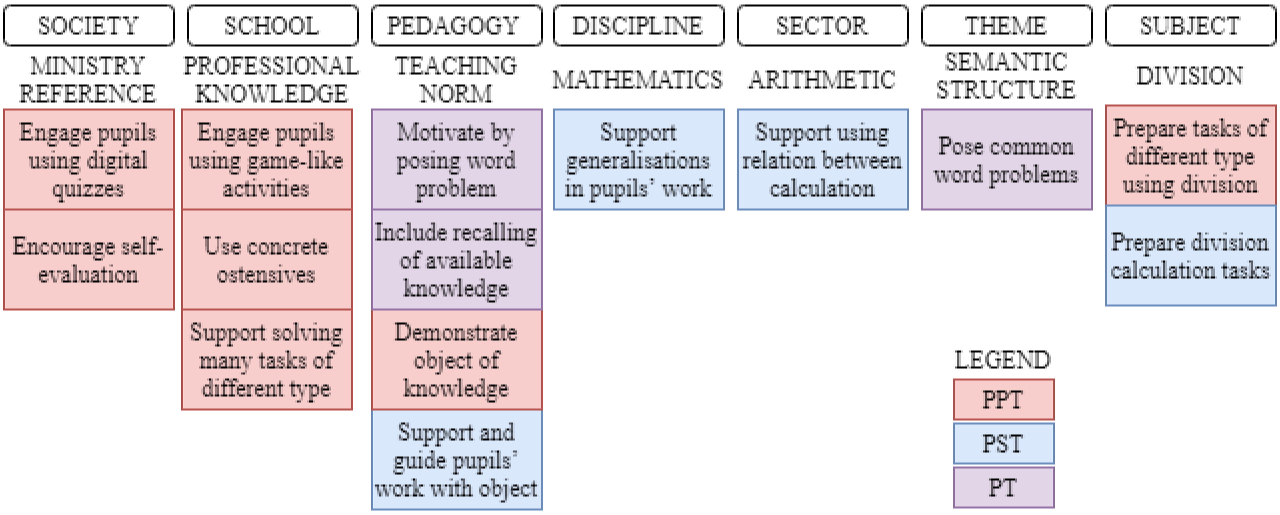


Figure 5: Activities and reasons from PPTs and PSTs didactic praxeologies on the levels of didactic codeterminacy acknowledging the national curriculum

DISCUSSION AND PERSPECTIVES

Observed PTs' lessons showed ample of division praxeologies but raised questions of their coherency. Praxeologies were not connected horizontally by means of different ostensives nor vertically by aligning praxeologies for division of different operands, although linking division of different types of operands by using similar verbal and visual ostensives might be useful, as suggested by Kim & Pang (2017).

Another mode of incoherency appears in the PTs' didactic praxeologies. Their lesson plans correspond to the VW paradigm with teacher as the dominant manager of the media and milieu (Kidron et al., 2014). In the PPTs' case, a teacher is a provider of knowledge, and learning mathematics means working on solving a lot of tasks of different type using the knowledge institutionalised by the teacher. Otaki and Asami-Johansson (2022) used the term paradidactic bipolarisation to describe such teachers' tendency to focus on general pedagogy and particular subject at stake. In the PSTs' case, a teacher is a guide to knowledge, but with very strict didactic rules of the discipline. For them, learning mathematics stems from making mathematical judgments with symbolic ostensives. PTs planned pupils' activities with objects existing in their milieu, but PPTs had no didactic intention to produce new objects, opposite to PSTs.

PTs' university education appeared to have equipped them with some mathematical praxeologies relevant for knowledge of division but their didactic praxeologies do not align with the constructivist QW paradigm. We acknowledge several constraints in initial teacher training to explain the phenomenon. In PPTs' university education in Croatia, courses in psychology, pedagogy and general didactics outweigh the subject matter courses thus their didactic choices revolve around pupils' participation. In PSTs' university education, mathematics courses prevail thus they focus on constructing new objects through analogy and generalisation as modes of mathematical reasoning. In PTs student teaching practice, they have to conform their mathematical and didactical praxeologies with the didactic contract their mentors hold in their classroom. The latter might be a major constraint for accommodating QW paradigm, assuming the influence of practising teachers' and traditional teaching paradigm prevails the influence of initial teacher education of PTs. Studies of teacher practices in Croatia showed they rely heavily on textbooks with clearly segmented units (Domović et al., 2012) and scarcely use manipulatives (Kišosondi et al., 2022). Shaping PTs' didactic praxeological equipment requires breaching the didactic contract and redefining the roles of university teachers, PTs and practising teachers.

Based on the observed lessons, we suggest focusing on following didactic choices:

- Posing open and rich questions and problems – A PPT posed an open-ended problem and planned to solve it frontally. By posing open-ended problems, a teacher provides opportunities for pupils to encounter relevant questions, engage in exploration by generating different ideas and techniques supported with various ostensives, and to construct new object of knowledge from their own ideas.
- Developing a coherent praxeological sequence – Teacher educators and practising teachers could use student teaching practice as a platform to design activities that would support utilising and connecting semantic structures, representation and strategies coherently across grades in primary and secondary education.
- Expanding media and milieu – A PST proposed manipulatives to interpret partitive and quotative division. Some PTs used digital resources to design closed-questions quizzes. By proposing and providing different resources, such as manipulatives, picture books, videos, etc., a teacher provides opportunities for pupils to choose, explore and consult with the sources of information and among themselves.
- Leveraging teacher's and pupils' responsibilities – Following the choices mentioned above, teachers' role inevitably changes as they transfer the responsibility for exploring, constructing and validating knowledge onto their pupils.

This study informed about constraints in PPT and PSTs' lesson preparation in the context of rational number division. To overcome these constraints requires developing mathematical praxeologies that promote vertical connections regarding models and techniques applicable for division of different types of operands, and didactic praxeologies that promote pupils' study and inquiry. Collaboration between university and practising teachers on developing such praxeologies while mentoring PTs in their professional training could merit all involved actors.

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The manipulation of a theorem to promote meaningful learning at the university level

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This paper reports on a study dealing with the role of a theorem as a cultural artefact to promote meaningful learning at the university level. The design builds on a problem-solving task obtained by manipulating the Weierstrass Theorem and addressed both to prospective teachers and to undergraduate students in mathematics to identify some critical learning signs and encourage reflections on how designing activities based on using or preventing them. The research has a dual final goal: to improve prospective teachers' knowledge and support students' meaningful learning. Results about the identification of the prospective teachers expected critical signs in undergraduate students' protocols are presented, and some reflections on the causes and on the intervention for future courses are proposed.

Keywords: university mathematics, artefact, meaningful learning, signs, theorem.

INTRODUCTION AND THEORETICAL BACKGROUND

Meaningful Learning (ML) is a process to construct *meaningful* knowledge that is linked to prior knowledge and focused on understanding:

> Meaningful learning as a process presupposes, in turn, both that learners employ a meaningful learning set and that the material they learn is potentially meaningful to them, that is, relatable to relevant anchoring ideas in their cognitive structures (Ausubel, 2012)

ML is based on constructivism, a perspective believing that an individual constructs his/her knowledge based on his or her experiences with related concepts. Knowledge is not "passively received, but rather actively constructed by an individual". Regarding mathematics education, Selden and Selden (2013) explain that "knowledge" refers to the mental structures that allow an individual to interpret the meaning of something, evoke ideas in their mind, or explore new mathematical problems effectively. Further, individuals can use this "old" knowledge to construct "new" knowledge. The construct of Meaningful Learning introduced by Jonassen, Peck and Wilson (1999) is based on a framework of five characteristics of meaningful learning. *Active engagement*: students shouldn't learn passively. Rather, they should use active learning strategies like experimenting, testing hypotheses, and inquiring; *relevance to prior knowledge*: students must build upon what they already know. Teachers should start a learning experience by identifying what the learners already know, and finding out how the new information relates to that; *integration with existing knowledge:* learning occurs when new knowledge is compared to existing knowledge; *elaboration and reflection:* elaboration means taking the new knowledge and seeing how they can use it in multiple different contexts; *personal* *significance***:** the information needs to have some point that makes sense to the student. The use of an artefact in teaching and learning activities creates conditions for a meaningful learning setting and encourages knowledge construction. Artefacts play an important role as instruments and mediators in the construction of mathematical knowledge. Following Vygotsky's seminal idea of semiotic mediation, the theoretical framework of Theory of Semiotic Mediation (TSM) was developed with the goal of providing a teaching and learning model that focuses on the semiotic processes associated with the use of cultural artefacts (Bartolini Bussi & Mariotti, 2008). Personal meanings are associated with the use of artefacts, particularly for the purpose of performing a task (instrumentation); on the other hand, mathematical meanings can be associated with the artefact and its use (instrumentalization) (Rabardel, 1995). This evolution is encouraged by the action of the teacher, who guides the process of production and development of signs centered on the use of an artefact: 'the teacher uses the artefact as a tool of semiotic mediation' (Bartolini & Bussi, 2008). Wartofsky (1979), identifies three types of artefact: *primary artefact*, externally oriented technical tool, directly used for intentional purposes (e.g., compasses, prospectographs,...); *secondary artefact*, inward-oriented psychological tool, used in the maintenance and transmission of specific acquired technical skills (e.g., writing, schematics, calculation techniques, ...) and *tertiary artefact*, a system of formal rules that have lost the practical aspect linked to the instrument (e.g. mathematical theories). Following Wartofsky' s broad perspective also a theorem in a theory is an artefact. We could call it *theoretical artefact.* Providing students with opportunities in which a theorem becomes an artefact to solve a task (instrument) and improve their learning (mediator) a semiotic mediation instrument, is a central rationale for this research. We consider "students as signs-producers" to be a powerful tool in promoting meaningful learning both in mathematics and in meta-mathematics, especially critical signs, i.e., signs that denote difficulties in understanding. In the latter situation, the construct of interpretative knowledge plays a crucial role. Ribeiro et al. (2016) introduce the construct of Interpretative Knowledge (IK). It refers to a deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge starting from their own reasoning and productions, without how not standard or incorrect they might be. IK completes the knowledge of typical errors or solution strategies with the knowledge of possible sources for errors and the knowledge of possible uses of errors. IK also includes the ability to develop specific feedback based on the sense given to the students' reasoning; therefore, it should allow them to exploit the potential of erroneous or unexpected strategies. Research agrees that concept understanding (Tall & Vinner, 1981) occurs through activities that involve the construction of examples (Watson & Mason, 2005). Such practices would also help in the management of a statement and its proof, or, in short, in overcoming difficulties in proof that are often also related to not knowing the meaning of proof (Weber, 2001). In our activities, once the difficulties foreseen by future teachers and those detected in the students' protocols have been examined, the study aims to reflect on the causes and on how to intervene at both levels to create learning opportunities aimed at building knowledge and meta-knowledge. We face the issue of promoting meaningful knowledge, starting with a theorem as a semiotic mediation tool and moving on to create conditions towards interpretative knowledge for prospective teachers and active engagement in problem-solving activities for undergraduate students. *RQ1: What is the role of the manipulation of a theorem in the development of meaningful knowledge for both future teachers and students of mathematics? RQ2: How to intervene on the expected critical signs to create learning occasions or on the possible causes that produced them to reduce them?*

METHODOLOGY

Designing choices.

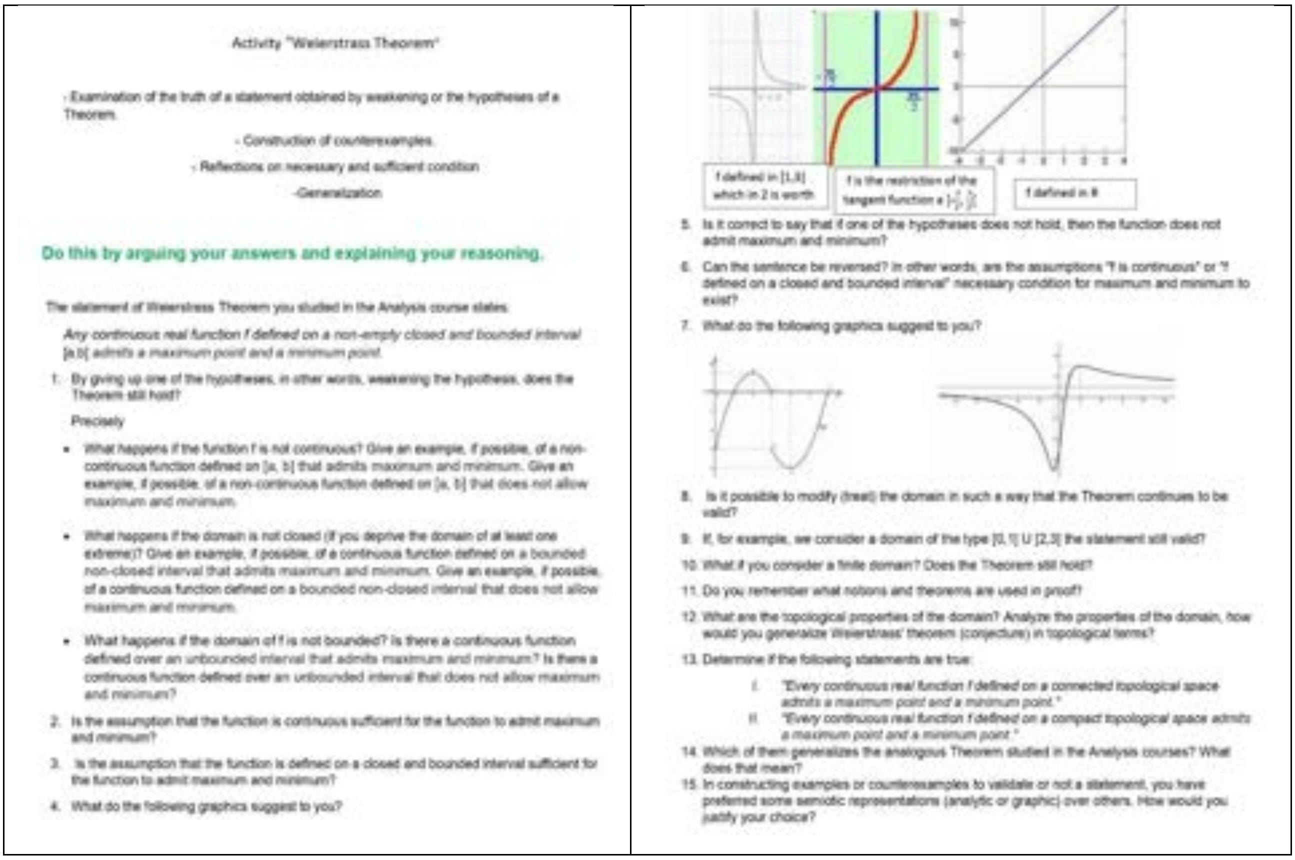
As both instruments and mediators, artefacts play an important role in the construction of mathematical knowledge. The notions of artefact and sign are central to the theoretical framework of semiotic mediation (TSM) developed by Bartolini Bussi and Mariotti (2008) from a Vygotskian perspective. Semiotic mediation is the theoretical framework to which we refer when designing the task and planning activities based on the produced critical signs. Through a semiotic lens, we compare critical signs expected by prospective teachers in mathematics and critical signs produced by undergraduate students and reflect with future teachers on the actions to take to make undergraduate students' personal meanings evolve towards the mathematical meanings. The reflection is not only directed on how to use the emergent critical signs but also on how the mathematical and meta-mathematical critical signs could be reduced by intervening on the hypothesized causes. We intentionally used the theorem as a semiotic mediation tool to design a task to mediate not only the mathematical contents included in the statement but also some meta-mathematical contents. We use the theoretical artefact first as a tool to identify critical signs of difficulties and then to reflect on how to plan activities transforming these signs into learning occasions, in the perspective of interpretative knowledge, and activities supporting students' meaningful learning. The lens we analysed the critical signs, mainly related to the difficulties with proof, refers to the research of Weber (2001, 2002), Watson and Mason (2005) and Tall and Vinner (1981).

Participants and context.

The experiments took place at two different times and were carried out at two different university levels. In 2021, 36 prospective teachers, attending a laboratory second-year master's course in mathematics, divided into 9 groups of 4 students, were engaged in an activity related to the manipulation of hypotheses and the generalization of Weierstrass' theorem, a theorem they already knew. Among the requests for the task, there was also that of indicating what could be the greatest difficulties encountered by second-year undergraduate students who would have been asked to carry out the same mathematical activity. In 2022, 38 second-year students enrolled in the master's degree of mathematics were invited to participate in activities aimed at improving their skills in solving mathematical problems. At the beginning, students filled out a questionnaire about their past experiences with mathematics, as well as their approach to the production of proofs and the strategies used up to that point to study and pass the exams. Subsequently, in the activities, students worked in groups and focused mainly on topology problems, with particular attention to those activating the generation of examples and generalisation processes. They were also assigned the task previously tackled by future teachers, except the teaching-oriented questions, to be carried out individually. Finally, a feedback questionnaire on the experience and how much it had been perceived as useful for their training.

Task.

Prospective teachers and undergraduates were asked to solve the following task. The aim of this first phase, bringing out the knowledge background that the participants had in the topic, was to begin modeling the knowledge necessary to predict and use any critical signs and errors that students with less experience might show or the knowledge needed to develop didactical actions to prevent them Ball et.al (2008).



Activity "Weierstrass Theorem"

- Examination of the truth of a statement obtained by weakening or the hypotheses of a Theorem.

- Construction of counterexamples.

- Reflections on necessary and sufficient condition

-Generalization

Do this by arguing your answers and explaining your reasoning.

The statement of Weierstrass Theorem you studied in the Analysis course states:

Any continuous real function f defined on a non-empty closed and bounded interval [a,b[admits a maximum point and a minimum point.

1. By giving up one of the hypotheses, in other words, weakening the hypothesis, does the Theorem still hold?

   Precisely

   - What happens if the function f is not continuous? Give an example, if possible, of a non-continuous function defined on [a, b] that admits maximum and minimum. Give an example, if possible, of a non-continuous function defined on [a, b] that does not allow maximum and minimum.
   - What happens if the domain is not closed (if you deprive the domain of at least one extreme)? Give an example, if possible, of a continuous function defined on a bounded non-closed interval that admits maximum and minimum. Give an example, if possible, of a continuous function defined on a bounded non-closed interval that does not allow maximum and minimum.
   - What happens if the domain of f is not bounded? Is there a continuous function defined over an unbounded interval that admits maximum and minimum? Is there a continuous function defined over an unbounded interval that does not allow maximum and minimum?

2. Is the assumption that the function is continuous sufficient for the function to admit maximum and minimum?
3. Is the assumption that the function is defined on a closed and bounded interval sufficient for the function to admit maximum and minimum?
4. What do the following graphics suggest to you?

f defined in [1,3] which in 2 is worth

f is the restriction of the tangent function a]$-\frac{\pi}{2}$, $\frac{\pi}{2}$[

f defined in R

5. Is it correct to say that if one of the hypotheses does not hold, then the function does not admit maximum and minimum?
6. Can the sentence be reversed? In other words, are the assumptions "f is continuous" or "f defined on a closed and bounded interval" necessary condition for maximum and minimum to exist?
7. What do the following graphics suggest to you?
8. Is it possible to modify (treat) the domain in such a way that the Theorem continues to be valid?
9. If, for example, we consider a domain of the type [0, 1] U [2,3] the statement still valid?
10. What if you consider a finite domain? Does the Theorem still hold?
11. Do you remember what notions and theorems are used in proof?
12. What are the topological properties of the domain? Analyze the properties of the domain, how would you generalize Weierstrass' theorem (conjecture) in topological terms?
13. Determine if the following statements are true:
    i. *"Every continuous real function f defined on a connected topological space admits a maximum point and a minimum point."*
    ii. *"Every continuous real function f defined on a compact topological space admits a maximum point and a minimum point."*
14. Which of them generalizes the analogous Theorem studied in the Analysis courses? What does that mean?
15. In constructing examples or counterexamples to validate or not a statement, you have preferred some semiotic representations (analytic or graphic) over others. How would you justify your choice?

In addition, they were asked to answer the questions:

16. In your opinion, what answers would the students of a Mathematics course in the first two years of university give? Imagine the possible wrong reasoning they might make. How would they justify their answers?
17. What do you think could be the difficulties that arise in such an activity? What are the causes?

Data collection and data analyses.

Our collected data, specifically prospective teachers previsions and students' solutions, were qualitatively and quantitatively analysed through the systematic and objective identification of some critical signs or difficulties, of imbalance, foreseen by the prospective teachers, which could be linked to some beliefs deriving from their training. In a first phase this was done classifying sentences relating to a single theme, and in a second phase looking at how these themes emerged in students' answers and labelling the signs. The analysis was developed by classifying the sentences related to a single critical point, for each group, and then looking at their interactions, and differences to understand how to intervene developing didactical actions to use or prevent errors and whether some practices foster meaningful learning processes. The critical signs emerged were: *Necessary and sufficient condition, constructing examples, domain analysis and manipulation, mastery and knowledge of basic concepts, hypothesis manipulation, examples seen as sufficient to prove a statement, theorem equivalent to existence of maximum and minimum.*

FINDINGS AND DISCUSSION

Prospective teachers.

The prospective teachers' predictions represented a significant element of study to evaluate whether an alternative educational path to the traditional one provides useful skills to improve mathematical skills and mastery of the concepts encountered. This can be done by comparing the beliefs of future teachers on the critical signs, and what really emerges from the work of the second-year undergraduate students participating in the experiment. We indicate a group with Gr i. Looking at the answers to questions 16. and 17, we classified with respect signs and selected a representative prevision.

Necessary and sufficient condition. In 8 out of 9 groups (Gr 1, Gr 2, Gr 3, Gr 4, Gr 5, Gr 7, Gr 8, Gr 9), the expectation is highlighted that the second-year student gets confused in relation to the notions of necessary and sufficient conditions:

> Gr 7: […] We expect students to confuse the necessary and sufficient conditions and to establish the necessity or sufficiency reason in the same way we did, that is, by looking for specific examples using the graphical representation.

Constructing Examples. In 4 out of 9 groups (Gr 3, Gr 4, Gr 6, Gr 9), it is highlighted the possibility that the second-year students will find it difficult to provide examples:

> Gr 3: The difficulties of the students that manifest themselves in this type of activity could be: the lack of understanding of the questions, the inability to show counterexamples and examples.

Domain analysis and manipulation. 4 out of 9 groups (Gr 1, Gr 3, Gr 4, Gr 8), the possibility that the second-year student will encounter difficulties related to the domain manipulation is highlighted:

> Gr 3: In point 8 the students' answer would be that the domain cannot be changed because otherwise the theorem would be invalidated. […]

Mastery and knowledge of basic concepts. In 3 out of 9 groups (Gr 1, Gr 4, Gr 5), the possibility is highlighted that students will meet difficulties due to the lack of mastery of basic concepts such as closed and limited interval, continuity of a function, definition of absolute maximum and minimum:

Gr 4: [...] Engaging in the verification of statements, having to validate or not hypotheses, is still a difficulty that can certainly vary depending on the student's mastery of concepts, as the continuity and the type of interval.

Hypothesis Manipulation In 3 out of 9 groups (Gr 2, Gr 3, Gr 5), it is highlighted the possibility that the second-year student will find it difficult to manipulate hypotheses due to the lack of habit to this type of approach, having for most of their life as students reproduced with confidence what is proposed by the teacher:

Gr 5: [...] This may be because in high school students don't encounter many proofs, so they may not be used to handling the hypotheses, theses, and notions necessary to carry out effective reasoning [...]

Examples seen as sufficient to prove a statement. Only 1 group out of 9 considers the possibility that providing examples can be a sufficient process for generalization:

Gr 6: [...] In a first two years of university, a wrong reasoning could be dictated by the fact that students believe that an example that verifies the thesis of the theorem can be sufficient to generalize it [...]

Theorem equivalent to existence of max and min. Only 1 group out of 9 considers that the validity of the theorem can be seen as equivalent to the existence of maxima and minima, as students might tend to affirm that maxima and minima do not exist in any case if one of the hypotheses is missing:

Gr 5: [...] A first mistake that students could make is to assert that in case of lack of one of the hypotheses then the non-existence of absolute maximum and minimum is certain (we have in fact seen that from the theorem if the hypotheses hold, their existence is assured. If one of these falls, there is no certainty of their presence, but these could still exist) [...]

Undergraduate students.

The second-year students tackled the activity individually. The answers to the questions were useful to make a comparison with the work of the future teachers, to observe if the critical signs were the same as those expected by the latter, and to evaluate whether the path undertaken during the entire course has allowed them to internalise the peculiar concepts (Tall & Vinner, 1981) that constitute a statement and its proof (mainly a necessary condition, sufficient condition and generalization). Let's start by looking at the answers to question Q1, which is basically about constructing examples (Watson and Mason, 2005). Of this, as well as the following ones, we consider only a representative of the overall situation, focusing on the type of error (if any) that occurs. We highlight how and where the expected signs appear.

Mastery and knowledge of basic concepts. In the following figure, when asked to provide an example of a discontinuous function that does not have a maximum and minimum, the error is due to a *lack of attention to the definition of function*, but essentially the sign is caused *by the incorrect Concept Image of continuity*.

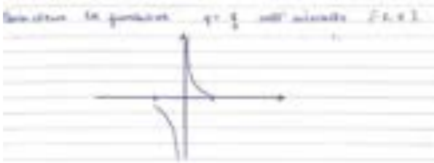


Fig.1

Let us consider the function $y = \frac{1}{x}$ *in the interval [-1,1].*

St 21 exhibits a "function" with a closed and bounded "domain" [-1, 1] that ***cannot be the domain of that function***, being the same undefined at a point within the interval.

Theorem equivalent to existence of max and min. We consider a representative answer in which the students mistakenly extended the sentence "the closure and boundedness of the domain are necessary for the existence of maxima and minima" as *if the hypothesis were equivalent to the existence of maxima and minima:*

St 18: The *theorem does not hold if one of the three hypotheses is renounced, so the given assumptions cannot be weakened.* If a hypothesis is not respected, the function may be devoid of a minimum or maximum in the given range. However, there are special cases in which the function is not continuous but admits maximum and minimum. For non-continuous functions, it is not possible to guarantee the existence. […] The function defined on the interval [0,1], is not continuous and has no absolute maximum; $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{otherwise} \end{cases}$ […]. *It follows that it is not possible to find a continuous function defined on a bounded but not closed interval in such a way that it admits maximum and minimum.* The function defined on the interval (0,1) is continuous but does not admit a maximum. Weierstrass' theorem does not work when the boundedness hypothesis fails: $f(x) = \frac{1}{x} [0, +\infty[$ *Therefore, there is no continuous function defined over an unbounded interval that admits a maximum and a minimum.*

Necessary and sufficient condition. Let us now look at the most significant answers to questions *2*, *3*, *5*, and *6*, which refer mainly to the concepts of necessary condition and sufficient condition. The errors are mainly due to *confusion about the proper meaning of the concepts of necessary condition and sufficient condition*.

St 13: The hypothesis of continuity is a sufficient condition for the theorem to continue to hold since a non-continuous function could still admit max and min (Q2); the assumption that the function is defined on a closed and bounded interval is not a sufficient condition for the theorem to continue to hold; in fact, a function defined on a bounded unclosed interval or on an unbounded interval does not admit max and min (Q3).

In the following, the critical points are mostly due to the problem of *seeing the validity of the theorem as equivalent to the existence of maxima and minima*.

St 18: If the hypothesis of closure or limit of the interval does not hold, it can be concluded that the function certainly does not admit maximum and minimum. If, on the other hand, the hypothesis of continuity of the function does not hold, it does not necessarily admit maximum and minimum (Q5); The assumption that f is defined on a closed and bounded interval is the only condition necessary for maximum and minimum to exist (Q6).

Domain analysis and manipulation. Looking at questions *8*, *9*, and *10*, which focus on domain manipulation and when and how this can affect the validity of the theorem, as seven students (St 6, St 8, St 17, St 20, St 24, St 26, St 27) St 38 say:

St 38: One might consider treating the domain as the intersection of closed and bounded sets. which, since they are compact, retain their closure and boundedness (Q8). Considering a finite domain, the theorem continues to hold when this is a point ($[a, a,]$). In other cases, the domain with a finite number of points is not a range, and we may have continuity issues (Q10).

We found an imbalance in relation to the *concept of continuity*. To highlight its weight and cause, we have reported two representative responses:

St 18: Yes, it is possible to treat the domain in such a way that the theorem continues to be valid. In fact, if I considered the domain as a finite union of closed and bounded intervals or as a finite set, the theorem would still hold (Q8); for real functions, we have defined continuity from intervals or finite unions of intervals. So, the theorem doesn't hold (Q10).

At this point we ask ourselves: *do the predictions agree with what happened? Or are there any major differences?* To answer these questions, we will first focus on the predictions and the possible reasons that may have led to them. Secondly, we will compare them with the critical issues encountered by students. The following Table 1 gives an idea of how widespread each critical sign is quantitatively among the minds of the various groups. It is immediately noticeable that most groups have imagined imbalances mainly related to the concepts of necessary condition and sufficient condition. This critical sign achieves a much higher percentage of expectation than all the others. The motivations are to be found primarily in the personal experiences of the individual members of the groups. This perceived greater effort may have led future teachers to foresee a similar difficulty in students with less experience. All this can also be linked to the memory of the first years at university: "*What was I like in the first two years of university? What difficulty was I having? Had I mastered this concept?*" The answers depend very much on the type of knowledge that led to the training of future teachers. They worked little autonomously, concentrating mainly on learning and understanding the statements and the related proofs, without delving deeply into the manipulation of hypotheses or, among other things, establishing whether a certain statement was a necessary or sufficient condition. Results in the table in relation to other critical points report lower percentages of prediction. It is therefore likely that, while some concepts give a greater sense of mastery, in some cases this feeling arises from erroneous beliefs. Their approach was for the most part reproductive, and therefore we wonder: *if future teachers who answered incorrectly* *previously had the occasion to reflect themselves on the use of definitions or on proofs, would they have made the same mistakes? Or would they have acquired and matured a greater mastery and a more solid knowledge of the concepts?* Before reaching certain conclusions, let's proceed by looking at second-year students in relation to the critical points expected by prospective teachers. In the first line, the 18% calculated corresponds to only 7 students out of 38, so there is not that strong an impact of this critical sign in question on the correctness of the answers. A percentage that is in line is that relating to the construction of examples, in which we record 47% of students showing difficulties. The generation of examples is very familiar to second-year students, and a certain sense of creativity also shines through in their works. This differs from the Gr 9 group's prediction that *"There could be a tendency to rely exclusively on the well-known elementary functions that are being studied or, in any case, without deviating too much from them, forgetting that one can also independently construct a function, even if defined by chances*". It is precisely in such a statement that the unfamiliarity of future teachers with self-engagement is highlighted: such phrases derive from their own personal experience, in which creativity has often been left apart. Where does that 47% next to the item "construction of examples" come from? From the analyses, it is easy to conclude that the percentage is closely related to the values 55% for "*domain analysis and manipulation*" and 58% for "*mastery and knowledge of basic concepts*".

Critical signs	***Expected by prospective teachers***	***Detected in undergraduates' protocols***
Necessary and sufficient condition	89%	18%
Constructing Examples	44%	47%
Domain analysis and manipulation	44%	55%
Mastery and knowledge of basic concepts	33%	58%
Hypothesis Manipulation	33%	13%
Examples Seen as Sufficient to Prove a thesis	11%	3%
Theorem equivalent to existence of max and min	11%	16%

Table 1: Percentages of critical signs expected and detected.

The critical issues that manifest themselves in lower percentages are those relating to the last three items. For the last two, we are in line with the previsions. It is then necessary to make some clarifications on the manipulation of hypotheses, intended as a starting point to reach the generalisation of the theorem object of the activity. The results of the survey show that 33% of prospective teachers predicted that students would have difficulty manipulating assumptions. However, only 13% of the students encountered such difficulties. Based on our research, the second-year students obtained good results, at least in line with or higher than those obtained by future teachers. In conclusion, from prospective teachers' discussion, it emerged that an excellent starting point for the development of effective strategic knowledge reducing critical signs is to intervene soon with activities stimulating students' active engagement (Miranda, 2023) and, when critical points occur, transforming them into learning occasions according to the IK model (Ribeiro et al., 2016).

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A path for addressing Klein's second discontinuity: Interplay between precision and approximation mathematics. The case of differential equations as mathematical models

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Mathematical modelling ability is one of the goals of the high school curriculum. Consequently, the need to train teachers to fulfill this goal arises. However, there is a scarcity of instructional proposals including mathematical modelling activities in extra-mathematical contexts facing this previous training. This is due to the complexity of the role of mathematical models in these contexts for mathematics teachers. This corresponds to Klein's second discontinuity. To overcome this complexity, it is necessary to analyse these contexts. Indeed, three praxeologies of mathematical modelling in the context of circuits were analysed. The analysis made tangible a dialectical relationship between precision and approximation mathematics, serving as an epistemological reference for designing an instructional proposal.

Keywords: Differential equations, electrical circuits, Klein's second discontinuity, preparation and training of university mathematics teachers, transition to, across and from university mathematics.

INTRODUCTION

Mathematics modelling plays a prominent role in transitioning from university to high school education, presenting mathematics teachers with a significant challenge: adapting the mathematical modelling learnt at the university level for high school teaching. However, university mathematics courses typically fail to equip teachers with the necessary tools to integrate modelling into their teaching practice, especially in non-mathematical contexts (Paz-Corrales et al., 2023). Thus, mathematics teachers often face "insufficient previous experience with modelling tasks and time limitations" (Sen Zeytun et al., 2023, p. 12), due to epistemological and systemic reasons (Frejd, 2013). This phenomenon is known as *Klein's second discontinuity* (Klein, 2016/1933; Wasserman et al., 2023; Weiss, 2023); nevertheless, in 1908, Klein published his work "*Elementarmathematik vom Höheren Standpunkte aus*" in three volumes. In the first two, he draws the attention of *Gymnasium* teachers to the importance of pure mathematics for their teaching practice. He outlines the development of mathematics, which, through its teaching, leads to two Plans: A and B. The former is characterised by rigour, and the latter by intuition. Particularly, Plan B bridges the gap between pure and applied mathematics: precision and approximation mathematics. Indeed, Klein shows in his Volumen III (Klein, 2016/1928) how mathematics arises from observation and then moves towards abstraction. He emphasises that the best way to think about the importance of mathematics is through its applications, denouncing the misdirected formalisation of traditional education (Moreno-Armella, 2014). Based on the above and considering that the *Universidad Pedagógica Nacional* (UPN) in Honduras is the institution alone dedicated to mathematics teachers' education, the following research question arises: What conditions and resources does teacher education offer for the integration of instructional proposals focused on mathematical modelling that allows *for addressing Klein's second discontinuity?* To address it, the analysis has focused on teaching mathematical modelling at the university level, particularly regarding the use of electrical circuits as an extra-mathematical context in teacher education for studying differential equations as mathematical models. This analysis will be a basis for designing an instructional proposal providing pre-service teachers with tools to integrate mathematical modelling in high school, where the same context is studied.

ELEMENTS OF THE ATD

To address this research problem from mathematical modelling, this study is based on the Anthropological Theory of the Didactic (ATD). This approach posits that mathematical activity and mathematical modelling activity are similar, as "doing mathematics mostly consists in producing, transforming, interpreting and developing mathematical models" (Barquero et al., 2019, p. 321). Since the inception of the ATD, mathematical modelling has been discussed in terms of intra-mathematical and extra-mathematical situations. Consideration has been given to investigating the conditions and constraints that may promote or hinder the implementation of modelling processes, referred to as the ecological dimension. Modelling proposed by this theory suggests that teaching mathematical modelling should be synonymous with functionally teaching mathematics rather than just teaching it formally. Mathematical modelling, can be analysed through a single model: *praxeology*. It has four elements: type of task, technique, technology, and theory. The first two elements form the technical-practical block or *know-how*, and the second elements form the technological-theoretical block or *knowledge*. Praxeologies can be created, taught, and used in any institution (e.g., university, high school, workplace) and circulate between them, undergoing transposition processes (Chevallard, 1999). According to Barquero et al. (2019), the process of didactic transposition (adaptations made on a praxeology to turn it into a teaching object) involves studying these institutions (e.g., mathematical discipline), the educational system (e.g., curriculum, educational model) and the classroom (e.g., mathematics courses).

DIDACTIC ENGINEERING

Didactic engineering is a research methodology used in designing and implementing instructional proposals for mathematical modelling (e.g., Ramírez-Sánchez et al., 2023). This paper only reports on the progress of the first of the four phases, i.e., preliminary analysis of the research.

Preliminary analysis of differential equations

This analysis can vary according to the interest of the study; for example, it can include: an epistemological analysis of the content taught, an institutional analysis, a didactic analysis and the difficulties encountered in the traditional teaching of the subject (Artigue et al., 1995). In this case, elements of the epistemological and didactic analyses of differential equations are presented, preceded by an institutional analysis of the mathematics teacher education at the university from Honduras: UPN.

Analysis of mathematics teacher education at the UPN

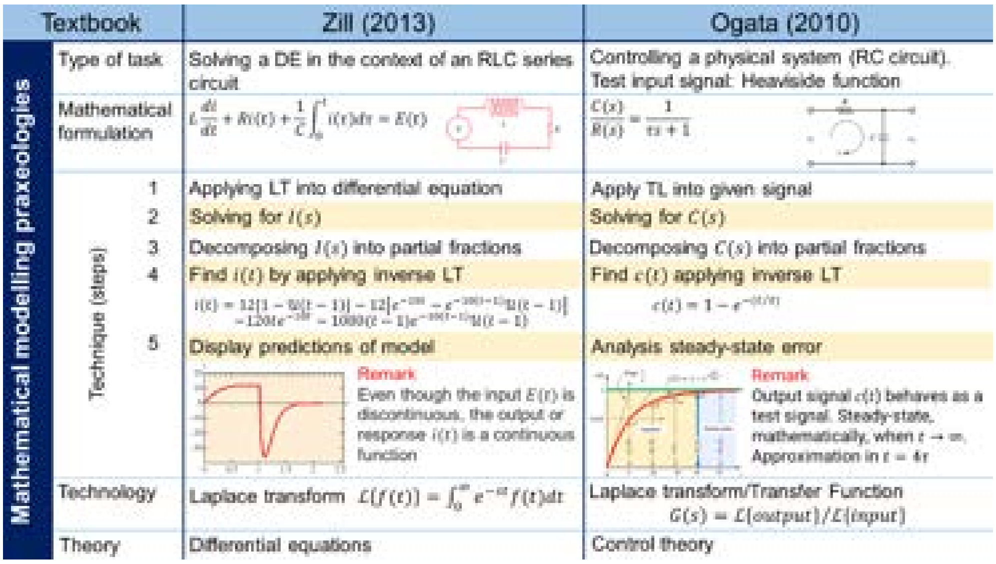
Mathematics teacher education in Honduras was established at the end of the 1960s. Teachers are qualified to teach mathematics at the secondary and high school levels. The different curricula have long provided three training areas: general, pedagogical, and specific. In 1989, an essential change in the curricula was the inclusion of a course on differential equations (Benavides-Cerrato et al., 2022). More recently, in 2008, a competence-based approach was adopted in this mathematics teacher education (UPNFM, 2008), which include the mathematical modelling competence.

Epistemological analysis of differential equations

The theory of differential equations has been developed since the foundations of calculus in the 17th century, and three approaches to its scientific development and operation can be distinguished (Artigue & Rogalski, 1990): algebraic, numerical, and geometric. The algebraic approach refers to the "exact" solution of equations by formulae; the numerical aims at the "approximate" numerical solution; and the geometric approach consists of the qualitative global study of the solution curves of the differential equation. The algebraic approach dominated this scientific field until the appearance of the geometric approach proposed by Poincaré at the end of the 19th century. The rapid development of dynamical systems and computational resources also brought changes favouring qualitative and numerical approaches.

Didactic analysis of differential equations

The teaching of differential equations has been dominated by the algebraic approach (Artigue et al., 1995), "which ignores the scientific development of this mathematical field" (Artigue & Rogalski, 1990, p. 113). This means that a course on differential equations has traditionally focused on quantitative methods. This approach is based on algorithms, as opposed to the geometric approach, because qualitative studies can lead to the development of methods, but cannot be transformed into algorithms. This second approach has been given an infra-mathematical status, i.e., the graphical solution is accepted only if it is accompanied by a formal solution (Artigue et al., 1995). In France, the change in the didactic perspective of the differential equation from a mathematical object to a tool is recognised (Douady, 1986). For example, the differential equation $y' = y$ with $y(0) = 1$ is used to introduce the exponential function through concrete situations. According to Klein "in Plan B these connections make their appearance quite intelligibly, and in accord with the significance of the functions" to the differential equations "which lie naturally at the basis of all those applications" (2016/1933, p. 83). In Honduras, the mathematics teacher education curriculum mentions that the differential equations course is an excellent way to "appreciate the relationship that exists between pure mathematics and the physical sciences through the execution of modelling projects, that is, the description of the behaviour of some real systems or phenomena in mathematical terms” (UPNFM, 2008, p. 171). This aligns with Klein’s intention in the third volume of his work (2016/1928). The main textbook for this course is Zill (2013), which covers various methods for solving differential equations (e.g., separation of variables, Laplace transform, Euler’s method). But why does this textbook dedicate an entire chapter to the study the method based on the Laplace transform? To answer this question, two praxeologies of mathematical modelling have been analysed. In Zill (2013), the first praxeology was identified, where the type of task is to solve a differential equation in a non-mathematical context. In Ogata (2010), a university textbook, a second praxeology was identified, where the Laplace transform plays a crucial role in control theory. Figure 1 shows the main aspects of the both praxeologies.



Mathematical modelling praxeologies	Textbook	Zill (2013)	Ogata (2010)
	Type of task	Solving a DE in the context of an RLC series circuit	Controlling a physical system (RC circuit). Test input signal: Heaviside function
	Mathematical formulation	$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$	$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$
Technique (steps)	1	Applying LT into differential equation	Apply TL into given signal
	2	Solving for $I(s)$	Solving for $C(s)$
	3	Decomposing $I(s)$ into partial fractions	Decomposing $C(s)$ into partial fractions
	4	Find $i(t)$ by applying inverse LT $i(t) = 12[1 - \mathcal{U}(t-1)] - 12[e^{-10t} - e^{-10(t-1)}\mathcal{U}(t-1)] - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}\mathcal{U}(t-1)$	Find $c(t)$ applying inverse LT $c(t) = 1 - e^{-(t/T)}$
	5	Display predictions of model Remark Even though the input $E(t)$ is discontinuous, the output or response $i(t)$ is a continuous function	Analysis steady-state error Remark Output signal $c(t)$ behaves as a test signal. Steady-state, mathematically, when $t \to \infty$. Approximation in $t = 4T$
	Technology	Laplace transform $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$	Laplace transform/Transfer Function $G(s) = \mathcal{L}[output]/\mathcal{L}[input]$
	Theory	Differential equations	Control theory

Figure 1: Mathematical modelling praxeologies in two university textbooks

In the first praxeology, the type of task is to determine the current of an RLC-series circuit, a phenomenon modelled by a second-order differential equation. This is done by substituting the given data into the mathematical model provided. The type of task in the second praxeology, identified in Ogata (2010), involves controlling a physical system by analysing the system response of an RC circuit to a test signal, the Heaviside function. The mathematical model in this praxeology is the transfer function, which is related to differential equations. Both praxeologies include a circuit diagram. The general technique in both local[1] praxeologies is based on the Laplace transform. In Zill’s (2013) praxeology, the Laplace transform theorem used to transform the voltage function depends on the form of this function in the differential equation. This is illustrated by five cases: Case 1. Polynomial, sinusoidal, or exponential function (polynomial, sinusoidal, or exponential theorem). Case 2. The product of the above two functions (first translation theorem). Case 3. Unit-step function (second translation

[1] It has only one technology that produces several techniques to solve different types of tasks. In this case, the technology is a Laplace transform (with various methods depending on the voltage function).

theorem). Case 4. Integro-differential equation (convolution theorem). Case 5. Dirac Delta function (unitary impulse transform theorem).

In contrast, in Ogata (2010), the voltage function corresponds to the input signal, and the technique also depends on the test signal, for each of which there is a corresponding theorem (typical test signals (functions): unit-step, ramp, acceleration, impulse, sinusoidal and white noise). However, only the test signals: unit-step, ramp, acceleration and impulse signals have been used. Both textbooks provide a list of Laplace transform theorems and their inverse. The first praxeology includes in its technique a step that is uncommon in other types of task analysed, called: displays predictions of the model. This step considers what information the geometry of the curve provide. The graph of the solution is shown and the author remarks that although the input function in the differential equation is discontinuous, the solution is a continuous function. This step is also observed in the second praxeology, where relative stability and steady-state are analysed, and a graph of the solution is presented. The author highlights several aspects: the output signal behaves like the input signal, specifically the unit-step function; stability is achieved mathematically in infinite time but in practice a reasonable estimate is made for a time instant $t = 4\tau$ (a more detailed analysis of both praxeologies can be found in Paz-Corrales et al., 2023).

Praxeology	Zill (2013)	Ogata (2010)	Silva (2018)
Process	*Plan A*	$\rightarrow$*Plan B*	*Plan B* ?
Mathematics	*Precision*	*Approximation*	*Precision* $\leftrightarrow$ *Approximation*
Approach	*Algebraic*	*Algebraic/Geometric*	*Algebraic/Geometric*
Use of context	*Level I*	*Level II*	*Level III*

Figure 2: Analysis from Klein's perspective of the three praxeologies

The first praxeology identified leads to teaching Plan A (Klein, 2016/1933) because the emphasis is on procedural fluency, i.e., the quantitative approach obtains the exact solution of the differential equation. Although the graphical solution is shown, this step is not as common as the numerical approach, which is not dealt with in other tasks analysed in the Laplace transform chapter. The modelling treatment is intra-mathematical, and the extra-mathematical context is only addressed when the elements of an electric circuit replace the data of the situation. The second one could be a candidate for teaching Plan B (see Figure 2) because it aims to show the relationship between mathematics (differential equations) and physics (control of an electrical system). In this case, what Klein mentions is recognised. The idea is that Plan B does not replace Plan A, but rather that they meet at some point. The "exact" solution of the equation, the output signal, is obtained. However, the real analysis came from the qualitative approach. With the graph, it was possible to know approximately at what time interval the system would stabilise. With this in mind, the analysis of a third praxeology was considered. Ramírez-Sánchez et al. (2023) point out three criteria for its selection: the mathematical models used can be related to those of the curriculum (e.g., differential equation models); its development does not require highly specialised engineering knowledge or extensive experience in solving this type of tasks or machinery; and its approach is possible through simulation programmes.

Praxeological analysis of defibrillator design

It is important to mention that the third praxeology, like the other two, is found in the context of electrical circuits. This praxeology is based on the study of Silva (2018). Its elements are highlighted in Figure 3. The type of task involves modelling the operation of a defibrillator. The defibrillator works by delivering an electric shock to a person to restore the heart's rhythm. The technique consists of four steps: establishing assumptions and hypotheses through enquiry (e.g., assuming that the average human heart has a high heart rate and the average human's heart has a resistance between 75 and 150 ohms); formulating the mathematical model, a first-order differential equation (with the status of a tool, considering the defibrillator as an application of RC series circuits, two physical laws governing the circuits are used: Kirchhoff's and Ohm's law), and solving the differential equation using the Laplace transform; calculating the values of the electronic components (e.g., capacitor, resistor); simulating the RC circuit in a software and validating the values obtained in the previous step. This step allows comparing the calculated values with those displayed by the simulator, as well as with the values with the values of a capacitor available in the industry. The previous steps of this technique are base on Laplace transform theorems, as described in the praxeology of Zill (2013). However, some steps are justified by non-mathematical technologies. For example, the resistance interval of the human heart is established by a cardiological institution, and electronics determine the values of a capacitor. In this situation, mathematics and other scientific scientific disciplines work in synergy. Could this be a praxeology of mathematical modelling that leads to the teaching of Klein's Plan B?

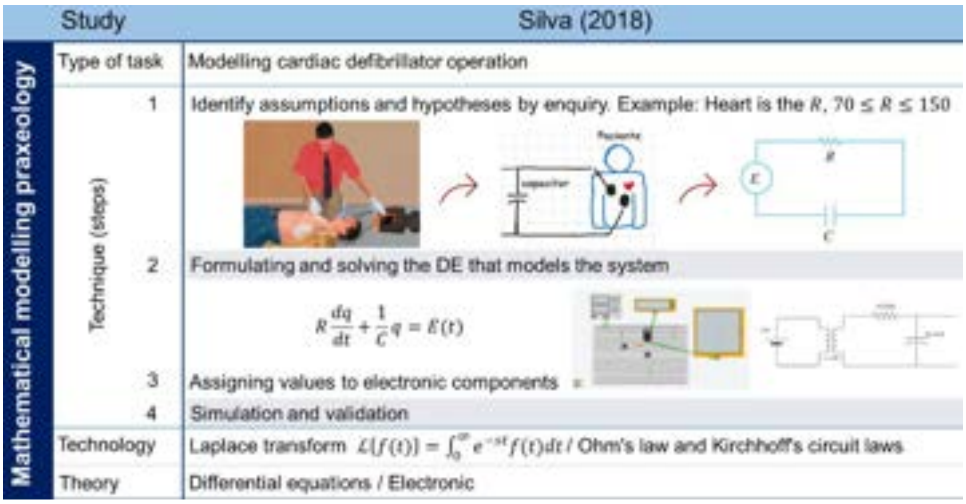


Figure 3: Analysis from Klein's perspective of the praxeology "Defibrillator design"

RESULTS

The analysis of the three praxeologies led to their classification into three levels, according to their use of the extra-mathematical context (with level I making less use than level III). In the case of the praxeology in Zill (2013), it is classified as level I, which refers to the language and diagram of the context, but extra-mathematical knowledge is not required for its development. For example, if $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$ were expressed as: $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, what is the difference? The emphasis is on the Laplace transform as another way of solving differential equations. It has been observed that the mathematical model is always given, and the emphasis is on *solving* the differential equation. Furthermore, in this type of task the technique is different as the voltage function changes (e.g., polynomial, sinusoidal, or exponential functions). There is a specific Laplace transform to transform the voltage function. Moreover, according to Klein's classification, it corresponds to precision mathematics, where an exact answer is obtained (Artigue et al., 1995). It is commendable that Zill (2013) shows different extra-mathematical contexts in which differential equations are used and shows a modelling process for solving a differential equation, described in four steps: assumptions and hypotheses, mathematical formulation, solving the differential equation, and displaying the predictions of the model. Unfortunately, the first two are not done in the type of tasks analysed in the chapter aon the Laplace transform in this textbook. Although it is mentioned that there are three methods for solving a differential equation (analytical method, qualitative analysis, and numerical method), corresponding to the three approaches identified by Artigue and Rogalski (1990) and described in the preliminary analysis of this study, the analytical method is favoured, while qualitative analysis is relegated. In the praxeology identified by Ogata (2010), it is observed that the extra-mathematical context is more influential than in the first praxeology (level II). Circuit diagrams and a language specific to electrical circuits appear, and the control of the system is analysed with the information provided by the geometry of the solution curve. It is a mathematics of approximation, which in practice consists of determining reasonable values for the time at which the physical system is stabilised. In the type of task analysed in Ogata (2010), the differential equation changes structure only because of the test signal (e.g. impulse, unit step). However, looking the technique in this praxeology, its development requires very solid engineering knowledge. This is because elements such as block diagrams and the transfer function are not typically found in mathematics teacher education. The third praxeology (Silva, 2018) is classified at level III (see Figure 2), due to its use of the extra-mathematical context. This praxeology identifies steps that are common to the first two praxeologies analysed. According to the first step of the modelling process described by Zill (2013), the assumptions and hypotheses are made through an exploration of the phenomenon. As for the mathematical formulation, the mathematical model is formulated using circuit laws, ignoring some variables of the phenomenon by considering only the human heart as the resistance in the RC-serie circuit.. Like the second praxeology, this approach involves a mathematics of approximation that allows the differential equation to be seen as a tool and then to pass to the status of a mathematical object. Finally, by simulating and validating the values of the components, it is possible to display the predictions of the model. Considering praxeologies such as the second and third analysed could suggest a shift in university teaching as proposed by Klein (2016/1933), specifically modelling in the university, by emphasising "the heuristic value of the applied sciences as an aid to discovering new truths in mathematics" (Klein, 1894, p. 46).

CONCLUSIONS

The didactic analysis showed that a differential equation such as $2.5y' + \frac{1}{0.08}y = 5$, belonging to precision mathematics (it is insisted that the equation is exactly satisfied), can capture and model diverse phenomena such as RC series circuits with $2.5\ \Omega$, $0.08\ f$ and $5\ v$, where y is the charge function, respectively. But it could also be an equation that models other applications such as population growth or cake cooling. Klein (2016/1933) argues that "mathematics of approximation alone plays a role in applications" (p. 38). Zill (2013) points out: "we have seen that a single differential equation can serve as a mathematical model for diverse physical systems. For this reason, we examine just one application" (p. 192). However, based on the analysis of the three praxeologies, the complexity of choosing one context over another and the adaptations that the contexts imply are recognised. So, is there a separation between pure and applied mathematics, as Klein discusses? Is the lack of relationship between the two mathematics structural or didactic? Klein (2016/1933) already argues that the separation is in harmony with human perception. For example, he mentions the theorem of mathematical induction, which, according to him, has an intuitive origin, and leads beyond the limit where perception fails. Based on the analyses presented, one of our interpretations is that, instead of a separation, there is a dialectical relationship between what is so-called pure and applied mathematics, metaphorically akin to a Möbius strip. In this regard, Thanheiser (2023) states that if we follow this premise: abstract and applied are separated, then we accept that context can be separated from mathematics. Does it make sense to focus on abstract systems in high school education, and then have students learn to apply them in different contexts? From the notion of praxeology, with technique in the technical-practical block and technology in the technological-theoretical block, indicates a movement of a back-and-forth between pure and applied. However, the differential equations studied in university education are used as mathematical models of physical, biological, and chemical phenomena and are ideal models whose solutions are exact. Zill (2013) mentions that the construction of a mathematical model begins with the identification of the variables responsible for the changes in the system. One may choose not to include all of these variables in the model, which involves determining the level of resolution of the model. For some models, it may be perfectly reasonable to settle for low-resolution models. This raises intriguing questions: Is a good model one that leaves some uncertainty?, is this type of model real or an illusion?, how accurate or approximate should solutions be in a mathematical modelling situation?

It is considered that the design of an instructional proposal for mathematical modelling, whose epistemological referent is shaped by these three praxeologies, would make it possible to balance the dialectical movement of Zill's textbook (2013) towards Klein's longed-for Plan B. An instructional proposal that goes beyond Plan A would assign a significant role to the Laplace transform in extra-mathematical situations and would advocate a qualitative and numerical approach to the solution of the differential equation, as suggested by systems associated with an electrical circuit studied in teacher education and that also appear in high school education. And in turn, look again at the *missing link* between university and high school mathematics.

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AN EXPERIMENTATION TO ADDRESS KLEIN'S DISCONTINUITY IN THE CASE OF THE INTEGRAL

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In this article, we present an experiment based on the modeling of the second Klein discontinuity developed by Carl Winsløw within the framework of the anthropological theory of the Didactic. This experiment was conducted in the institutional context of secondary teacher training in France, involving a population of students holding a bachelor's degree in mathematics (L3). The case study focuses on the connections between the integral taught in high school in France, introduced as the area under the curve, and its relationship with the Riemann integral and measure theory taught at the university.

Keywords: Teaching and learning of analysis and calculus; Transition to, across and from university mathematics; Klein second discontinuity; Integral.

INTRODUCTION

The problem of the second discontinuity was formalized as early as 1908 by Felix Klein. This second discontinuity occurs when a student leaves university to become a secondary school mathematics teacher, while the first discontinuity happens upon entering university. To address this issue, Klein set out to present mathematics in a series of books based on three principles: emphasizing connections between mathematical domains, demonstrating how academic mathematics relates to school mathematics, and highlighting the links between mathematics and real-world applications. These three principles constitute his so-called Plan B for mathematics education.

The second discontinuity appears to persist today (Wasserman, 2018), and students still struggle to perceive the connections between university-level mathematics and the mathematics to be taught in secondary education. Recent empirical results (Hoth et al., 2020) illustrate that the transfer of knowledge from academic to school mathematics is not automatic.

What mathematical knowledge is useful for a future teacher? What types of connections need to be developed and strengthened between university-level mathematics and school mathematics in training programs to promote the professional development of teachers? These are ongoing debates that currently make the second discontinuity of Klein a vibrant question in mathematics education research and a significant challenge for the teaching and training profession. In this direction, new tools have recently been introduced (Winsløw and Grønbæk, 2014; Winsløw 2020) by addressing the issue of Klein's second discontinuity with the Anthropological Theory of Didactics (ATD; Chevallard and Bosch, 2020). Winsløw employs the concept of an individual's relation to an object of knowledge within an institution, using the Anthropological Theory of the Didactic. He distinguishes between high school (*HS*) and university (U) institutions, as well as three different institutional positions: high school student (s), university student (σ), and high school teacher (t). An object of knowledge (in the case of this article, taking the integral as an example), which exists across both institutions, is denoted as "o" in high school and as "ω" when it pertains to a theory of integration (Riemann or Lebesgue, linked to the general theory of measure) taught at the university. Winsløw (2014) then proposes the following modeling of discontinuities:

$$R_{HS}(s, o) \rightarrow R_U(\sigma, \omega) \rightarrow R_{HS}(t, o)$$

where where $R_U(\sigma, \omega)$ reads “the relation of a university student sigma to the object of knowledge omega within the institution “University”. Klein's response to the transfer problem consists of establishing a connection $R_U^*(\sigma, \omega)$ weaving connections between o and ω in the light of the change in position indicated by the last arrow. In a subsequent modeling, Winsløw (2020) denotes $R_U(\sigma, o \cup \omega)$ this new integrator relationship, and it is this notation that we will retain. How can we construct a sequence in teacher training to enable the development of this new relationship? This article introduces such a sequence, experimented within in the MEEF master's program[1] in France in 2020.

THEORETICAL FRAMEWORK

This research is anchored in the Anthropological Theory of the Didactic (ATD). Firstly, it offers a language for modeling Klein's double discontinuity, as explained in the introduction. Secondly, the theory of praxeologies plays a crucial role in bridging the gap between High School (HS) and University (U) knowledge. ATD emphasizes the relative nature of knowledge objects (o) in relation to the institutions (I) that develop, standardize, and transmit them, as well as the individuals (x) subjected to these institutions (Chevallard & Bosch, 2020). Therefore, ATD focuses more on the generic positions (p) individuals occupy, such as teacher (t) or student (s), rather than the individuals themselves. The study aims to examining the institutional relations $\left(R_I(p, o)\right)$ of individuals within the institution (I), in their respective positions (p), with regards to the knowledge object (o). The arrow diagram presented by Winsløw and Grønbæk (2014) in the introduction summarizes the various institutions, institutional positions, and knowledge objects involved in Klein's double discontinuity. Our research primarily addresses institutions where mathematics is taught.

Praxeologies form the core of ATD, emphasizing the analysis of human activities. A praxeology (P) comprises both a praxis Π and a discourse Λ on that praxis. ATD

[1] MEEF : Métier de l’Enseignement, de l’Education et de la Formation. The MEEF master's program prepares students for careers in education and teaching.

suggests that the relationship $(R_I(p,o))$ arises from praxeologies in which the object (o) is involved, operating at various levels within the praxeology: the type of task (T), the technique (τ) employed to solve tasks of this type (t), the technology (θ) supporting the technique, or the theory (Θ) providing the ultimate basis for the praxis.This set of praxeologies can be described in the form of a structured model that is called a reference praxeological model (RPM; Florensa et al., 2015). RPMs are reconstructions of the knowledge to be taught, obtained by considering different levels of the didactic transposition (via historical epistemology, official programs, textbooks, and teaching materials).

METHODOLOGY

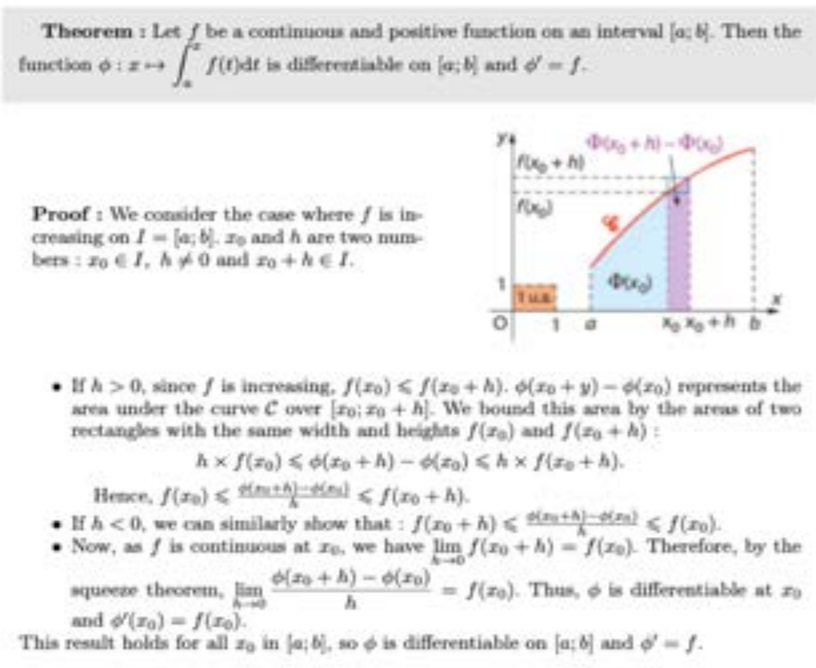
The experimented sequence is intended for students undergoing teacher training as part of the MEEF master's program at the University of Montpellier, who hold a bachelor's degree in mathematics from the University of Montpellier (other profiles are enrolled in this master's program but are not subjects of the study). To conduct this study, we chose a subject of study: the integral. We hypothesize that students' praxeological equipment regarding integration after their bachelor's degree corresponds to what is expected by the university institution, in particular that they have studied the Riemann integral and the Lebesgue integral with measure theory. Thus, the design of our experimentation is based on the description of $R_U(\sigma,\omega)$ using a RPM. We thus constructed a dominant praxeological model, which will be our RPM for teaching integration at the University of Montpellier. Our modeling reveals two sectors: the first sector is related to the Riemann integral and contains five local mathematical organizations (integrability, properties of the integral, primitives, integration calculations, Riemann and Darboux sums). The second sector pertains to measure theory and the Lebesgue integral, revealing four regional mathematical organizations around the general theory of measure, the general theory of integration, image and product measures, and finally L^p spaces. On the other hand, since $R_{HS}(t,o)$ contains $R_{HS}(s,o)$ we have also constructed a RPM for the integral at the high school level. The Winsløw's model highlights the need to create a new relationship $R_U(\sigma,o\cup\omega)$ to facilitate the transfer of advanced knowledge, where "o" designates the high school integral, and ω designates the university integral. From our RPM, we will then formalize a mathematical organization which, in our view, realizes the Klein plan for integration in the sense that it will mobilize elements of logic and praxis from praxeological models related to high school integration, Riemann, and Lebesgue. To carry out the Klein plan and strengthen the connections between high school mathematics on integration and university mathematics on integration, we based our construction on the proof of a manual of the fundamental theorem of analysis (Figure 1.), as it is required in high school. The task corresponding to the RPM related to the high school integral is t_{FTC} "Demonstrate that if f a non negative, increasing and continuous function on $[a;b]$, with $a<b$, then the function $\phi: x \mapsto \int_a^x f(t)\,\mathrm{d}t$ is differentiable and the derivative function is f". The technique employed (see below), as described in Figure 1, involves the intuitive notion of area and some of its properties:

The area under the curve of a continuous (even just increasing) and positive function admits an area; the area is additive; the area of a segment is 0. Our objective is therefore to construct a logic that allows justifying, within the rigorous standard of the university, the various steps of the proof of this theorem. Our epistemological investigation has identified the Jordan measure theory of quareable sets (see below) as underlying the theory of areas. Thus, our project is to use this transitional element, the Jordan measure of squarable sets, to highlight the connections between the different theories. In the following section, we present the various tasks proposed to students aimed at reconstructing this logic, and then, based on a priori analysis, the praxeologies that are targeted. We will thus observe the development of two types of praxeologies: the first type, denoted as P^*, represents a praxeology stemming from the dominant praxeological model of the University institution but whose engineering work has modified certain components in order to establish connections between o and ω (which will subsequently appear in the study process). The second type, which we denote as $P^{\sim}$, corresponds to praxeologies originating from the High School institution but which, during the study process, are enriched by praxeological elements related to ω.

PRESENTATION OF THE SEQUENCE

In our praxeological study (Planchon, 2022), we have identified a task, denoted as t_{FTC}, that is found in different institutions (HS and U).

In the institution “high school”, the technique to be implemented consists of recognizing that, for $x_0 \in [a;b]$ and $h > 0$, then $\phi(x_0+h) - \phi(x_0)$ represents the area under the curve, between the lines $x = x_0$ and $x = x_0 + h$. One can then bound this area with the area of two rectangles and conclude. The proof, required for high



Theorem : Let f be a continuous and positive function on an interval $[a;b]$. Then the function $\phi : x \mapsto \int_a^x f(t)\mathrm{d}t$ is differentiable on $[a;b]$ and $\phi' = f$.

Proof : We consider the case where f is increasing on $I = [a;b]$. x_0 and h are two numbers : $x_0 \in I$, $h \neq 0$ and $x_0 + h \in I$.

- If $h > 0$, since f is increasing, $f(x_0) \leqslant f(x_0+h)$. $\phi(x_0+y) - \phi(x_0)$ represents the area under the curve $\mathcal{C}$ over $[x_0; x_0+h]$. We bound this area by the areas of two rectangles with the same width and heights $f(x_0)$ and $f(x_0+h)$:
$$h \times f(x_0) \leqslant \phi(x_0+h) - \phi(x_0) \leqslant h \times f(x_0+h).$$
Hence, $f(x_0) \leqslant \frac{\phi(x_0+h)-\phi(x_0)}{h} \leqslant f(x_0+h)$.
- If $h < 0$, we can similarly show that : $f(x_0+h) \leqslant \frac{\phi(x_0+h)-\phi(x_0)}{h} \leqslant f(x_0)$.
- Now, as f is continuous at x_0, we have $\lim_{h\to 0} f(x_0+h) = f(x_0)$. Therefore, by the squeeze theorem, $\lim_{h\to 0} \frac{\phi(x_0+h)-\phi(x_0)}{h} = f(x_0)$. Thus, ϕ is differentiable at x_0 and $\phi'(x_0) = f(x_0)$.

This result holds for all x_0 in $[a;b]$, so ϕ is differentiable on $[a;b]$ and $\phi' = f$.

Figure 1. Proof of the FTC (translated from textbook “Transmath”, (Bonneval et al., 2012))

school students, is written below (Figure 1) and is found in various textbooks at this level. The technique is justified here by the definition of the derivative, but also by various properties related to the concept of area: the area of a rectangle, the growth of the area, themselves justified by the intuitive notion of area, as stated in the official curriculum. Although the concept of area is related to the concept of measure, our RPM does not mention an explicit link between measure and the concept of area in the tasks proposed to students in the undergraduate program. In the University institution, this task denoted as t_U is also present, but the technique to be implemented is different: for $x_0 \in [a; b]$ and $h > 0$, we have:

$$|\phi(x_0 + h) - \phi(x_0) - hf(x_0)| = \int_{x_0}^{x_0+h} |f(x) - f(x_0)| \mathrm{d}x$$

The continuity of f ensures that, then, ϕ is differentiable in x_0 and $\phi'(x_0) = f(x_0)$. This technique is justified by the definition of the derivative, but also by the various properties of the Riemann integral, themselves justified by the theory of the Riemann integral (Planchon, 2022) . Later, in the third year of university, we encounter the same task within the framework of Lebesgue integral theory and measure theory, but the technique is limited to noting that if a positive function f is continuous, then it is Riemann integrable, and thus we reduce it to the case of the Riemann integral. Thus, the praxeological equipment of students at the end of the third year at the mathematic university includes the praxeologies related to the type of task of which t_{FTC} is an instantiation. In the perspective of generating a new relationship, we started by describing a theory, a new logos Λ_M^*, that justifies the technique implemented in high school for the execution of t_{FTC}. Daubelcour (1998), Douady (1987), Perrin-Glorian (1999) emphasized the Jordan measure for the measurement of areas. The theory that will form the basis of our mathematical organization is therefore the measure of measurable sets, as presented by Lebesgue (1975). This measure μ enjoys various properties that are mobilized in the proof of the fundamental theorem of analysis at the high school level: it is simply additive[2], invariant under isometry[3], the measure of the unit square is 1, the measure of a segment is 0, and the measure is increasing.

In the following, we provide an explanation of the document given to the students, particularly outlining the various tasks they will be required to complete. The first part of the guide introduces a concept, called area measure, defined axiomatically as follows: *We assume that there exists a subset Q of the set of subsets of* $\mathbb{R}^2$ *containing points, segments, interiors of polygons, stable under finite intersection and union. An area measure is a function μ defined on Q with values in $\mathbb{R}_+$, simply additive, invariant under isometry, and such that the measure of the square* $[0; 1[\times [0; 1[$ *is 1.* In this first part, we do not provide more details about what the set is. Three tasks are then assigned:

[2] If $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$

[3] If s is an isometry, then $\mu(s(A)) = \mu(A)$

t_1: Show that the area measure is a diffuse measure[4] ;

t_2 : Determine the area measure of a rectangle in terms of its dimensions with justification;
t_3: Drawing on the area measure, rewrite the proof of the Fundamental Theorem of Analysis extracted from the textbook with the rigor standard of the university (see Figure 1).

The choice of tasks t_1 and t_2 is based on our RPM related to the Lebesgue integral at the university. Indeed, with the aim of designing tasks that generate the new integrator relationship, $R_U(\sigma, o \cup \omega)$, we wanted to enable students to make connections with their previous knowledge, which had been seen in measure theory. In particular, we chose to use the term "diffuse" in the description of task t_1 to explicitly refer to a task already encountered by students (showing that a translation-invariant measure on $\mathbb{R}$ is diffuse).

Regarding task t_1, two techniques can be employed: either a proof by contradiction or a direct proof. In both cases, the growth of the measure is a key point and must be proven by the students. The techniques are theoretically known to the students. Therefore, the task involves adapting proofs already studied in the third year of the undergraduate program to the specific context of the theory of areas.

For task t_2, the goal is to distinguish the set of measures of rectangles. It is possible to limit ourselves to rectangles parallel to the axes (due to invariance under isometry). When the measures of the sides of the rectangles are integers, it suffices to mobilize the additivity of the area. This is also the case for rational dimensions. For the case where the measures of the sides are positive real numbers, which are non-rational, it is necessary to mobilize the density of $\mathbb{Q}$ in $\mathbb{R}$ and again mobilize the growth of μ. The praxeological elements to be mobilized here theoretically form part of the praxeological equipment of students who have studied in a mathematics undergraduate program. We model these two tasks as instantiations of two types of tasks from the RPM relative to measure theory at the university (demonstrating a property of a measure, determining the measure of a set for a given measure), but with a logo block modified, as it is not from measure theory.

Finally, for task t_3, the goal is to formalize a proof from the textbook. The choice to introduce a school textbook, through the proof of the fundamental theorem of analysis, highlights the relevance of the work done previously to analyze material from the school environment. The task to be performed here can be considered as a professional task, which distinguishes it from tasks t_1 and t_2. Here, the task corresponds to a task type originating from the RPM related to high school integration. But here, the technique to be implemented requires an adapted logo block, which contains the elements of Λ_M^*. But a question must emerge: Is the area bounded by a continuous

[4] the measure of points is 0

function curve an element of Q? The study of this issue is the subject of the second part of the activity, developed below.

The second part of the activity introduces the definition proposed by Lebesgue of measurable sets in the plane: For any natural number i, we call a level i grid the grid Q_i in the Euclidean plane $\mathbb{R}^2$, referred to an orthonormal coordinate system, whose vertices are the points with coordinates $\left(\frac{a}{10^i}, \frac{b}{10^i}\right)$, where a and b are integers. A closed square surface (in the sense of the usual topology of the plane) of level i is a set of the form $\{(x, y) \in \mathbb{R}^2, \frac{a}{10^i} \leqslant x \leqslant \frac{b}{10^i}, \frac{a}{10^i} \leqslant y \leqslant \frac{b}{10^i}\}$. If S is a bounded subset of the plane (i.e., contained in a square surface), we consider the set of closed square surfaces of level i contained in S and denote by s_i their union and by n_i their number. Then $s_i \subset S$. Similarly, we consider the closed square surfaces of level i that intersect S, denote by S_i their union, and by N_i their number, so that $S \subset S_i$. Finally, we define $u_i = \frac{n_i}{100^i}$ and $v_i = \frac{N_i}{100^i}$. We say that S is *quarrable* when $\lim\limits_{i \to +\infty} u_i = \lim\limits_{i \to +\infty} v_i$. The common limit is then denoted by $\mu(S)$ and called the area of the surface S (in the sense of Lebesgue).

Three tasks are then assigned:

t_4: Demonstrate that the semi-open unit square, $C = [0; 1[\times [0; 1[$, is quarrable and has a Lebesgue area equal to 1, and that the function μ also satisfies the additivity axiom.

t_5: What are the properties of μ that are mobilized in the reasoning presented in Figure 2? Relate these properties to the axioms or to general propositions that, if necessary, can be considered as new axioms.

In his treatise (around 250 BC), Archimede demonstrates that the disk has the same area as the right-angled triangle whose legs are respectively the radius and the circumference of this circle. Here is an interpretation of his approach with a contemporary reformulation, particularly incorporating the concept of limit and the properties of trigonometric functions:
Consider a circle with a radius of 1, and let p be the length of its semi-perimeter. For any non-zero integer n, denote P_n and Q_n as the regular polygons with n sides inscribed and circumscribed to the circle. The polygon P_n is the union of n isometric triangles at OAB, and the polygon Q_n is the union of n isometric triangles at OGF.
The Lebesgue areas of these polygons are $n\cos(p_n)\sin(p_n)$ and $n\tan(p_n)$. The difference between the areas of Q_n and P_n tends toward 0, so the disk is quarrable, and its area is given by $\lim\limits_{n \to +\infty} \mu(Q_n) = p_n$.

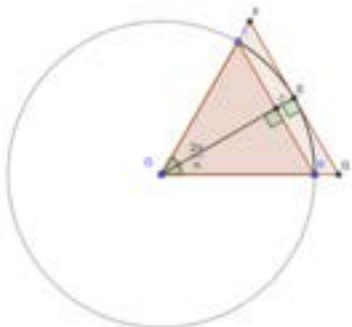


Figure 2. Area of the disk

t_6: Show that the area bounded by the curve of an increasing function is measurable.

Task t_4 is divided into two sub-tasks: sub-task $t_{4,1}$ is « demonstrate that the square $[0; 1[\times [0; 1[$ is quarrable with a Lebesgue area equal to 1, i.e., that the normalization axiom is satisfied », and sub-task $t_{4,2}$ is « demonstrate that μ satisfies the additivity axiom ». For sub-task $t_{4,1}$, taking the notations from the statement, we have $n_i =$

$100^i - (10^i + 10^i - 1)$ and $N_i = n_i + (10^i + 10^i - 1) + (10^i + 10^i + 1)$, and then $u_i = 1 - \left(\frac{2}{10^i} - \frac{1}{100^i}\right)$ and $v_i = u_i + \frac{4}{10^i}$. The sequences (u_i) and (v_i) are indeed sequences that converge to 1. Finally, the expected formalization for sub-task $t_{4,2}$ consists of introducing, at a fixed i, the number of level-i squares that are included in A, in B, and in $A \cup B$, as well as the number of level-i squares that intersect A, in B, and $A \cup B$. Denoting a_i, b_i, d_i as the number of squares in *A,B,D* and A_i, B_i, D_i as the number that intersect *A, B, D*, we have $a_i + b_i \leqslant d_i \leqslant D_i \leqslant A_i + B_i$. Finally, it is found that the sequences $\left(\frac{d_i}{100^i}\right)$ and $\left(\frac{D_i}{100^i}\right)$ are adjacent, so they converge to the same limit. This limit is the limit of $\frac{a_i}{100^i} + \frac{b_i}{100^i}$, so $\mu(A \cup B) = \mu(A) + \mu(B)$. The completion of this task involves elements of praxeologies in analysis related to the concept of the limit of a sequence, especially on adjacent sequences. Again, we model these task t_4 as instantiation of a type of task from our RPM relatives to the measure theory at the University (showing that an application is a measure). The task t_5 was constructed based on a singular task found in the RPM at the high school level, which can be modeled as « find an approximation of π based on the area of the disk ». This task involves, once again, formalizing reasoning that can be found in school textbooks. It is of the same type as task t_3, it means a task of a type encountered in the RPM related to high school integration, but with the logo block enriched by mathematical elements from Λ_M^*. From the study of this task, the logo is enriched by the following proposition: A surface S is quarrable if and only if there exist two sequences of polygons (P_n) and (Q_n) such that, for every n, $P_n \subset S \subset Q_n$ and $\lim\limits_{n\to+\infty} \mu(Q_n) - \mu(P_n) = 0$. This proposition can be used in the completion of the last task t_6, which is to demonstrate that the set bounded by the curve of an increasing and positive function, and then the one bounded by the curve of a positive continuous function, is a quarrable set.

For this, students are asked to consider, when *f* is an increasing function on $[a; b]$, the set $\Omega = \{(x; y) \in \mathbb{R}^2, a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$.

To complete this task, it is necessary to adapt the technique used in the task "show that an increasing function on an interval is Riemann-integrable over that interval", which is a task encountered by students in the second year of their undergraduate studies. Thus, the technique to be implemented is: for every *n* in $\mathbb{N}^*$ consider for $k \in \{0, \dots, n-1\}$, the rectangles:

$$r_k = \{(x, y) \in \mathbb{R}^2, a + k\frac{b-a}{n} \leqslant a + (k+1)\frac{b-a}{n}, 0 \leqslant y \leqslant f\left(a + k\frac{b-a}{n}\right)\} \text{ and}$$

$$R_k = \{(x, y) \in \mathbb{R}^2, a + k\frac{b-a}{n} \leqslant a + (k+1)\frac{b-a}{n}, 0 \leqslant y \leqslant f\left(a + (k+1)\frac{b-a}{n}\right)\}$$

By defining $P_n = \bigcup\limits_{k=0}^{n-1} r_k$ and $Q_n = \bigcup\limits_{k=0}^{n-1} R_k$ as described, we indeed have $P_n \subset \Omega \subset Q_n$ (due to the growth of *f*), and $\mu(Q_n) - \mu(P_n) = \frac{(b-a)(f(b)-f(a))}{n}$ tends to 0. Therefore,

Ω is quarrable. Here, we encounter a task that belongs to a type typically found in the domain related to measure theory (demonstrating measurability of a set), with once again a modified logo block.

CONCLUSION AND RESULTS

In this article, we have presented our experimentation as a response to Klein's problem for the integral, within the institutional context of secondary teacher training in France. The sequence presented here aims to approach secondary school mathematical concepts with the perspective of undergraduate knowledge. The development of the sequence was based on RPM related to the integral at the high school and university levels. These models were complemented by a mathematical organization constructed from our epistemological study. In the six tasks proposed here, tasks t_1, t_2, t_4, t_6 are derived from praxeologies whose praxis originates from the dominant praxeological model related to the integral at the university level (this means that the task proposed is of a type encountered at the university). The technique to be implemented requires confronting the technique developed at the university, adapting it to the specific situation, which necessitates questioning the discourse of the praxeology. The new type of praxeology we are modelling here is called P^*. The completion of these tasks, which thus mobilize university-level mathematics, aims to provide students with theoretical elements to justify, with the rigor of the university, the proof of the fundamental theorem of analysis. This work of justification is the focus of tasks t_3 and t_5. Here, the task originates from a type encountered in the high school institution. The new type of praxeology we are modelling here is called $P^{\sim}$. This formalization brings out the notion of Kleinian praxeologies (Planchon, 2022), the development of which generates the new relation. Thus, an important result of this study is to produce a proposal that responds to Winslow's formalization of generating a new relation. The analysis of student activity can highlight the links we aim to construct and may ultimately make it possible to detect the obstacles to the development of the targeted praxeologies.

The methodology described here can also be used, by mobilizing objects other than the integral, to construct learning situations that encourage the development of Kleinian praxeologies, of the P^* and the $P^{\sim}$ type.

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Interdisciplinarity and second transition: novices' attention towards examples in mathematics and physics

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This study concerns pre-service teacher education at university level, with a special focus on interdisciplinarity. The process of becoming teachers tackles the issue of the second transition from university to teaching at school, where a discontinuity can occur. Such discontinuity usually concerns university students who cannot connect university knowledge of mathematics to knowledge useful to teach. We reflect on the second transition for interdisciplinary teachers starting from the analysis of a discussion with pre-service mathematics and physics teachers about two physics textbooks during a university course in mathematics education. The analysis is centred on their structure of attention with respect to examples and exemplification.

Keywords: Transition to, across and from university mathematics, Teaching and learning of mathematics in other disciplines, teacher education, mathematics and physics, structure of attention.

INTRODUCTION

This contribution is framed in a wider study concerning pre-service teacher education, with a focus on interdisciplinarity between mathematics and physics (Branchetti et al., 2023). Reflecting on the interplay between mathematics and other disciplines from a didactical point of view is relevant for university mathematics education since one of main challenges at tertiary level is teaching non-mathematicians in such a way that mathematics is perceived as meaningful to other disciplines and professional fields and that the constraints and needs of other disciplines are considered (Gueudet et al., 2023).

The main issue to face is that people with a strong preparation in mathematics might not be aware of the nature of this interplay and not to be able to identify the critical issues that characterize it. University students usually have no opportunities to reflect deeply on these aspects. Here we focus on pre-service teacher education in university contexts and to second transition, from university into upper secondary school teaching (Gueudet et al., 2016). At its core, the second discontinuity concerns the detachment between the university mathematics and teaching mathematics at school. The issue is particularly critical when teacher educators face the challenge to train pre-service mathematics teachers to deal with interdisciplinarity between mathematics and physics. Indeed, the corresponding academic knowledge is expected to be developed both in advanced mathematics, physics or even mathematical physics courses, but to show the connections between different disciplines is not usually within the scopes of such courses. In the Italian context, this issue is very relevant since a master's degree in mathematics allows university students to become secondary mathematics and physics teachers, and the national curriculum requires explicitly to show the interdisciplinary connections between mathematics and physics (Bagaglini et al., 2021).

In this paper, we interpret the second transition as part of the long-term process of (mathematics and physics) teacher education. We focus on examples and exemplification in textbooks since dealing with them is crucial both in mathematics (Watson & Chick, 2011) and physics (Kuo, 2023), thus this is a good example of critical interdisciplinary issue from a didactical point of view at secondary level that is not addressed in an explicit way in university courses.

THEORETICAL FRAMEWORK

Mathematics pre-service teacher education

By teacher education, we mean the process of promoting the shift from novice to expert (Mason, 1998), where "The expert differs from the novice in the form and structure of their attention" (p. 243). According to Mason, the structure of attention "encompasses the locus, focus and form of attention moment by moment" (p. 250). We operationalize this structure as follows: the *locus of the attention* is a specific semiotic element in the textbook (for example, a word, a sentence, a figure, or a table). It is studied by examining what semiotic elements are referred to in the comments. Concerning the *form*, the attention towards a locus may be focused or diffuse, "centered on a single domain or else either flit between or simultaneously locate you in different worlds" (ibid.). Concerning the *focus*, it is the comment performed around the locus, including the reason why the locus is addressed. It is studied by examining the comments performed on specific aspects of the textbooks, and the extent to which these reasons are explicitly stated.

According to Viennot and Décamp (2018), a significant feature of expert teachers is the *critical attitude*, which means engaging in intellectual dynamics in first person and being able to discuss critically the knowledge at stake. They also state the worthiness of examining this faculty when pre-service teachers are confronted by any textual resource designed by a person whose aim is to explain something. Moreover, Viennot (2001) introduces the notion of *critical detail* to identify those aspects that have the potential to trigger, and deserve, a critical discussion. From Viennot and Décamp (2018)'s perspective, the recognition and properly articulated analysis of such critical details in textual resources might support the enactment of a *critical attitude*. Furthermore, we claim that enacting a critical attitude can be of support for the second transition, since critical discussing the knowledge at stake, from a teaching point of view, includes the ability of discerning connections between university and school knowledge. From Mason (1998)'s perspective, this coherently suggests examining the structure of attention while teachers comment on textbooks and interact with teacher educators.

Interdisciplinarity between mathematics and physics

We adopt a theoretical perspective on interdisciplinarity between mathematics and physics which derives from the European Erasmus+ project IDENTITIES (n. 2019-1-IT02-KA203-063184). Two concepts are crucial: *boundary* (Akkerman & Bakker, 2011) and *epistemic core of a discipline* (Erduran & Dagher, 2014).

We describe *discipline-based communities* as communities characterized by expertise in a discipline (e.g., scientists or mathematics teacher educators), which constitutes the *boundary of a community* (Akkerman & Bakker, 2011). Specifically, "a boundary can be seen as a sociocultural difference leading to discontinuity in action or interaction" (p. 133). Simultaneously belonging to both worlds and neither, boundaries embody not only ambiguities and risks but also opportunities. The two worlds can be bridged by artifacts, called *boundary objects*. Furthermore, *boundary people* are those subjects that are recognized and recognize themselves as members of a community but whose knowledge, practices, and interests do not belong only to that community. They indeed can move outside and bring into the community new languages and challenges (ibid.). Coherently, the *boundary between disciplines* is a metaphorical space of encounter and interaction where members of a discipline-based community deal with (boundary) objects produced by members of another (e.g., physics textbooks for mathematics pre-service teachers, as in Pollani & Branchetti, 2022).

Even with interdisciplinarity, it is still important to reflect on the single disciplines. Nevertheless, separating what belongs to one discipline and what does not is considered a problematic approach (Erduran & Dagher, 2014; Pollani & Branchetti, 2022; Satanassi et al., 2023). As proposed in Satanassi et al. (2023), we adopt the Family Resemblance Approach (FRA), as reconceptualized in Erduran and Dagher (2014). In the FRA, the *epistemic core of a discipline* consists of *aims and values* (like abstraction, objectivity, consistency, etc.), *practices* (like collecting data, making experiments, proving, etc.), *methods and methodological rules* (like inductive/deductive reasoning), and *knowledge* (like the laws of dynamics, the theorem of mean value, etc.) (ibid.).

The desired expertise concerning examples and exemplification

We claim that interdisciplinary teachers should become able to interact successfully with boundary objects at the boundary between disciplines. Expert interdisciplinary teachers should be able to enact a *critical attitude* by structuring their attention to identify, and critically discuss, critical details in textbooks that are relevant from an interdisciplinary point of view. Consistently, their structure of attention should be localized, and focused, on details that mirror the relationship between disciplines, respect their epistemic cores, reveal the resemblances between disciplines or the specificities with respect to their values and methods (Erduran & Dagher, 2014).

In this contribution, we focus on the critical attitude related to examples and exemplification. Bills et al. (2006) argue that analyzing examples offers "both a practically useful and an important theoretical perspective […] on the professional development of mathematics teachers” (p. 126). Watson and Chick (2011) argue that examples may act as *examples-of*, and the learner must be able to recognize variation of dimensions and at the same retain the essential properties. Examples may also act as *examples-for,* promoting the understanding of a new concept by means of generalization. The authors describe the following actions on examples: *analysis* (searching for relation between elements of an example), *generalization* (finding and describing similarities among examples), and *abstraction* (classifying similar examples and identifying them as a concept). We claim that it is important to move from considering the example per se to finding relations between the elements of the example (*analysis*), finding and describing similarities between examples (*generalizing*), classifying similar examples (*abstracting*). Bills and colleagues also refer to *generic example* as “something specific is being offered to represent a general class” (2006, p. 127).

In physics education, worked examples are of interest for their use in the processes of problem-solving and self-explanation (Kuo, 2023). Another topic of interest is the notion of *real-world example*, that is used to pursue the goal to show the relevance of physics in everyday life. This aim might become controversial if we consider naïve epistemological positions directly connecting the real-world examples and theoretical laws, not deepening the role of models and experiments (Tasquier et al., 2016).

The ambiguity of examples, which can play a role of *examples-for* and *examples-of*, *worked example*, *generic example*, *real-world example* can be relevant for interdisciplinary teaching, in the sense that they might function as (sources of) critical details that attract the attention of the reader. Indeed, they allow to deepen the issue of bridging the concrete and the ideal and conceptual world, that is at the core of interdisciplinarity between mathematics and physics.

Research question

From the perspective of the second transition and of the design of a teacher education activity that aim to smooth the second discontinuity, we aim to investigate what teacher education activities might trigger the enactment of the *critical attitude* in the sense of Viennot and Décamp (2018). In this paper, we start addressing this research issue by analyzing novice pre-service mathematics and physics teachers’ structure of attention emerged during a teaching activity based on secondary physics textbooks analysis. This analysis concerns the attention posed to interdisciplinary aspects related to exemplification. We ask ourselves: *what structure of attention can pre-service teachers have as novice interdisciplinary teachers commenting on examples in secondary school physics textbooks?*

METHODS

The data collection was carried out during a two-hour lesson of a university course in Mathematics Education at the Department of Mathematics of the University of Genoa in the academic year 2022-23. It involved five first-year master students who attended the course, three in presence and two remotely. All of them hold a bachelor's degree in mathematics that encompasses compulsory courses in physics (about classical mechanics, thermodynamics, electromagnetism, and modern physics), and chose a curriculum for a master's degree that encompasses courses in Mathematics Education. In this paper, we will refer to them as pre-service teachers since they are following a curriculum focused on teaching, their project is to become (secondary) teachers, and they have little or no previous teaching experience. The author FM was the teacher educator of the course, and the authors LB and LP attended the session and acted as teacher educators for the specific topic. All the authors were present in person. Participants were presented two excerpts from secondary school physics textbooks about the motion of a projectile (Ruffo, 2014, pp. 96–97; Walker, 2010, pp. 79–80). The topic was chosen for its relevance both for the national curriculum and for the co-evolution of mathematics and physics (Branchetti et al., 2022). Walker (2010) was chosen for its epistemological richness (Bagaglini et al., 2021), while Ruffo (2014) was chosen to push details forward by creating a background and a foreground. Pre-service teachers were asked to read the excerpts and to answer the following questions, first individually, then in small groups: "Is there any aspect that stands out for you? Why? Which one of the texts would you use in your class? Why? Which one of the texts do you feel most comfortable with? Why?". The choice of presenting secondary textbooks at university can create an opportunity for reflecting on second transition, fostered by these couples of questions: indeed, the first and the third can elicit pre-service teachers' mention of past experiences as students (or teachers, if any), while the second explicitly recalls their (future) teaching practice. After the small groups work, there was a discussion led by the three authors. Later, participants could make comments relying on the construct of Habermas' rationality (Habermas, 2003; for more details, see Branchetti et al., 2023) and on a third extract from Amaldi (2011, p. 298). The whole collective discussion was video recorded and transcribed. Pre-service teachers were pseudonymized. We carried out a qualitative analysis by coding the transcript with the above operationalization of the structure of attention (namely, locus, form – diffused or focused –, and focus). We coded separately and then confronted our analyses to reach a mutual agreement. We selected all the excerpts where pre-service teachers explicitly mention examples. The extracts from the textbooks are provided as supplementary materials.

ANALYSIS OF SELECTED EXCERPTS

As a premise, we briefly sketch some relevant aspects of the desired expertise concerning examples and exemplification in relation to the chosen excerpts. Text1, Text2 and Text 3 stand respectively for Walker (2010, p. 79–80), Ruffo (2014, p. 96–97) and Amaldi (2011, p. 298). The term "example" appears explicitly five times: once in both Text2 and Text3, and three times in Text1. In Text2 it stands for a *worked example*, showing a substitution of quantities in the formulas, while the others can be framed as *generic examples* (Bills et al., 2006) with respect to the sentence preceding the one containing "example". From an expert perspective, we note an epistemological ambiguity (are they *example-of*, *example-for*, *worked*, *generic or real-world examples*?) that could foster reflections on crucial interdisciplinary issues such as the relation between observations, experiments and measures, and different levels of modeling (also in relation to provided representations).

We now analyze a selection of excerpts from the discussion that contain pre-service teachers' references to examples revealing their structure of attention. First, we observe that what they refer to as "example" includes what textbooks present as such, but they used this term also to refer to physical phenomena or models (e.g., the motion of a ball falling from a table). They did not provide any explicit characterization of their meaning of the term "example".

***One* case and *the* case: the place of examples**

Amelia: The second page *[of Text1]* […] immediately brings an example […] in our opinion a little bit thrown in there, while […], when it *[Text2]* starts with the case of horizontal velocity, first it explains it generically, and then with… anyway the case of the ball but keeping, let's say, the letters… and then, right at the last, it says okay let's substitute two numbers, gives the little, tiny example and shows two things, however it stands very much on the generic […]. We *[Alice and herself]* like it anyway: instead of dwelling so much on one case, as in the first book, here it talks just about the case of horizontal velocity.

The locus of Amelia's attention is the example of the turtle and of the ball. Her form of attention is focused on the comparison between these two examples, contained in the two textbooks (Text1 and Text2, respectively). She observes that both are shown at the beginning ("*immediately*", "*starts*") and she reflects on their nature: the example of the turtle is perceived as very specific, while the example of the ball is appreciated for being more general ("*keeping… the letters*"). Only at the end Text2 "*gives the tiny example*", referring to the worked example where it considers and substitutes two numbers. Amelia criticizes the initial example of the turtle, for it does not fit in, it has numbers, and it comes first. For Amelia, numbers are discriminant between Text2's "*little, tiny example*", *example-of* (Watson & Chick, 2011), coming "*at the last*", and Text2's initial example of the ball, "the *case*", *example-for* (Watson & Chick, 2011), coming first. Amelia appreciates the approach of Text2 which seems to be frameable as *generic example* (Bills et al., 2006).

The difference in physicality of examples

Olivia: It *[Text3]* presents three examples of… always of the motion of projectiles, but it presents a different physicality […] one is a motion that is vertic… that I see it going up and down […] they are all motion of projectiles, but they are a little different… I don't know how to say, how to explain myself better: it is true that the model to be used for all is that of the motion of projectiles and the equations are the same, but they may seem to be three different motions, and they are indeed in the reality! […]

LB: I was interested in the sense of physicality that you mentioned earlier […] You see it physical, don't you? That kind of incipit gives you this sense.

Olivia: Yes… In the sense of reality […] recalling something you can do, I mean, I can pick up a ball and decide to throw it, in that sense.

LB: And there is an evaluation of what is the phenomenon before systematizing with trajectory, vectors, etc. […]

Olivia: Then whatever, I would put a blowgun in the hands of the students in the classroom.

The locus of Olivia's attention is the three *examples-of* from Text3. Her form of attention is focused on the comparison between these examples of the motion of projectiles. On the one hand, she points out their mutual diversity: "*they may seem to be three different motions* and they indeed are", they have a "*different physicality*", "*in the sense of reality*". This can be explained as a possible (inter)action, "I can *pick up a ball* and decide *to throw it*", and as a different look, "*one* […] I see *it going* up and down". On the other hand, Olivia recognizes that one similarity is that "*the model* to be used for all *is that of the motion of projectiles*". This sentence can be further explored for reflecting on the interesting relation between the physics phenomenon and *a* model, and the role of contextual factors. Finally, it is remarkable her mention of how she would act in classroom putting "*a blowgun in the hands of the students*".

Examples in physics, and the mathematization of examples

Albert: Deciding to start with an example, at least in physics, can be in my opinion a winning strategy, which you must distinguish it from mathematics by the fact that in physics… what is explained and illustrated has been proved by practical experience. So, I am pro-examples at the level of exposition […].

Alice: But the second text also starts with a real situation like the ball falling off the table, I find very similar such example to the ones in the third book […]

LB: Do you see any similarity and difference between the two types of examples *[in Text2 and Text3]* […]?

Olivia: To me it seems different because these ones here *[Text1 and Text2]* […] are very mathematized already, they are not real, they have been modeled... while instead those of the Text3's […] keep having […] a much more physical dimension, not in the sense of matter […], much more real than the former, although it is true that they are all examples.

LB: That is, you're saying that they're not already [...] neither modeled nor even described by perhaps a language of the discipline, not even of physics. […]

Emily: Even here *[Text2]* it is said "it *[the ball]* leaves it *[the table]* with a horizontal velocity" and not "it falls", that is closer to one could say. It is already being framed in a more… mathematical context.

Albert's first sentence is not localized in a specific element of textbooks but rather is on the general habit of starting with an example in physics. Here it emerges an epistemic and epistemological position of the role of "*practical experience*" in validating knowledge. Nevertheless, his appreciation of the strategy is on a communicative level, rather than an epistemological one. Afterwards, Alice localizes her attention on the beginning of the second text, with a focus on the similarity between Text2 and Text3 in the use of a real situation. The locus of Olivia's attention is the example from all the three texts. Her form of attention is focused on their comparison, and she stresses that in Text1 and Text2 examples have been "*very mathematized*", so partially losing a "*physical dimension*" and reality. After the intervention of LB, Emily strengthens Olivia's comment by localizing her attention on a sentence of Text2, "*it leaves it with a horizontal velocity*". Her focus is on the comparison between this expression and the alternative, "*it falls*", which is not already in a mathematical context but rather is closer to the natural language.

Examples in mathematics, in physics and past experiences

Albert: In mathematics textbooks you hardly ever see examples, also because finding in mathematics examples related to everyday life maybe is even a little bit more difficult? But for domain like probability […] it is easier to introduce the chapter with an example. Physics, which is based on empirical laws, in my opinion requires this kind of approach. Otherwise, it could be interpreted by students as the bad relative of mathematics […] I was lucky because I liked physics, in high school they introduced it to me using Amaldi […] I am more satisfied what I did in physics than what I did in mathematics. But there it's the fault of the school reforms… and not professor's fault.

Albert's sentence is not localized in a specific element of textbooks but rather is on the general occurrence of examples in mathematical textbooks. His focus is again on how "*to introduce the chapter*", assuming a didactical perspective. First, he compares mathematics domains, mentioning probability as an easier domain to find everyday-life examples. Then, he compares physics and mathematics with a focus on making students not interpreting physics as "*the bad relative of mathematics*". Again, it emerges the epistemological position of physics as empirical. Albert mentions his experience of learning physics and mathematics at school (but not at university), and the impact of school reforms.

DISCUSSION AND CONCLUSIONS

We addressed the following question: *what structure of attention can pre-service mathematics teachers have as novice interdisciplinary teachers commenting on examples in secondary school physics textbooks?* Concerning the *foci* of their attention, several lines of reflection emerged: the relation between examples and generality (the presence of numbers, the place of examples), examples in mathematics and in physics, the “mathematization” and the “physicality”. These aspects could be further developed to promote the enactment of a critical attitude. We observed that pre-service teachers, although not providing any explicit characterization of the term “example”, labelled as “examples” the situations presented in the excerpts. We may note that in literature on physics education the term “example” is much little frequent, apart for referring to worked example. The frequent use of the label “example” by pre-service teachers can be related to the phenomenon of “disciplinary capture” (Frodeman et al., 2012), which means not considering disciplines other than that of one own’s. Concerning the *loci*, we observed that detaching from textbooks here occurs with stereotypical conceptions of physics as empirical and mathematics as less related to everyday life. Moreover, pre-service teachers do not mention university knowledge. All this suggests a reflection on the second transition with respect to interdisciplinarity between mathematics and physics. During their university instruction, pre-service teachers attended courses both in physics and in mathematics. Nevertheless, they seem not able to reflect and relate the epistemologies of the two disciplines. The case of examples paves the way for reflecting on university teaching in a second transition perspective: can we say that pre-service did not acquire an “academic knowledge” of physics and of relating mathematics and physics epistemologies during their university studies? Or should we say they acquired it but were not able to use it when commenting on physics textbooks? The present study suggests that university courses about physics contents alone did not promote pre-service teachers awareness of interdisciplinary issues. The answers to the previous questions are worth exploring in future research to inform a design of teacher education activities.

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Designing inquiry teaching by pre-service teachers: Analysing their views of the lesson

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This paper focuses on what are their difficulties in the design process of pre-service teachers' inquiry teaching. This research adopts the context of Study and Research Paths for Teacher Education based on the Anthropological Theory of the Didactic. The research question is: "what constraints the students' views of teaching are on the design of SRPs". For this, their own inquiry and their design processes are analysed and the didactic design praxeologies of the students on lesson design are compared. The results of the research reveal that the students' views on teaching are influenced to a large extent by their experiences and knowledge, which can be attributed to the differences between the old and new paradigms for inquiry teaching.

Keywords: Teachers' and students' practices at university level, Novel approaches to teaching, Teacher education, Inquiry teaching, Lesson design.

INTRODUCTION

The scope and focus of research on mathematics teacher education and professional development is generally claimed on teachers' professional growth; professional knowledge, beliefs, reflections, and noticing; and the frameworks, models, and practices, i.e., content, methods, tools, and related impacts. On the other hand, difficulties in this research area are also discussed. These can be pointed to as methodological and/or outcome conflicts of *situatedness vs. generalisability* (cf. Helliwell, 2023).

However, teachers' actual lesson design activities have not always been clarified so far. One of the reasons for this may be related to the above-mentioned issue of situatedness vs. generalisability. The lesson in each country or culture presents diverse aspects, not only in terms of views, but also of specific practices and teacher-student relationships.

In particular, with regard to much of the recent discussion surrounding inquiry, a discrepancy between practising inquiry activities and designing inquiry teaching can be identified. These include a lack of shared meaning of inquiry and, particularly in Japan, the traditional framework of problem-solving lesson, which seems to be both a major conditions and constraints not only for practising lessons but also for their design.

In Japan, mathematics lessons have generally taken the format of problem-solving lesson since the 1980s, and this still dominates in schools today (Mizoguchi & Bosch, 2023). Problem-solving learning/lessons are not in themselves to be denied in any way. Rather, it is a basic format of learning that has supported mathematics education practice in Japan up to the present, and has been recognised internationally as Structured Problem-Solving (Stigler & Hiebert, 1999; hereafter SPS). In contrast, a form of learning known as inquiry or inquiry-based learning has recently been attracting attention. The first issue that needs to be addressed is whether the two forms of learning can coexist, and if not, what differentiates them. For this reason, in some of the lesson studies in which the author has participated, whether pre-service or in-service, many teachers have experienced difficulties in designing inquiry teaching.

Therefore, the purpose of this study is to identify what are their difficulties in the design process of pre-service teachers' inquiry teaching and what is needed to improve it.

THEORETICAL FRAMEWORKS AND RESEARCH QUESTIONS

The main frameworks of this research are the Paradigm of Questioning the World and the Study and Research Paths. In describing these, the Anthropological Theory of the Didactic (hereafter ATD) are the background.

Inquiry-based learning and pedagogical paradigms

In ATD, the pedagogical paradigm of the past and today is called the 'Paradigm of Visiting Works' (hereafter PVW). In other words, based on pre-defined curriculum contents, the teacher's main teaching action is recognised as guiding learners to better learn (visit) these knowledge (works). This often results in issues on the side of the learner such as why it is necessary to learn the knowledge at stake (*raison d'être* of knowledge) and, on the side of the teacher, even though there are efforts on the part of the teacher to motivate the learner in the act of teaching, it is often the case that the learner is not aware of need and necessity of the knowledge, and sometimes even the teacher him/herself is not aware of them. In other words, what is required of learners is 'what they know' (Chevallard, 2019). Based on this perception and reflection on the current situation, Inquiry-Based Learning (hereafter IBL) has been attracting attention (Artigue & Blomhøj, 2013).

However, Japanese SPS is essentially in PVW and may be conceded to be the most sophisticated form of learning in PVW, so to speak. The author does not see IBL as a complement to or a replacement for SPS. Rather, it is recognised as being based on a shift in the pedagogical paradigm itself, rather than a mere transformation of the form of learning. In other words, the shift from conventional PVW to the Paradigm of Questioning the World (hereafter PQW), which is oriented towards 'what can be learnt and how it can be learnt', inevitably calls for a change in the form of learning. In PQW, we aim for an attitude of inquiry, which is considered to be the attitude of the scientist or mathematician. In other words, the need for any knowledge is determined according to the interests of the inquirer and its worth is determined through its function in the inquiry by the inquirer him/herself.

At present, daily teaching is developed under the PVW by both teachers and students who participate in it. On the other hand, the activities in PQW are expected to be essentially different (even if they include some of the activities of PVW). What differences characterise these activities and how teachers and student institutions demand such characteristics is expected to have a significant influence on the design of lessons.

Study and research paths

The ATD proposes Study and Research Paths (hereafter SRPs) as a model of inquiry in PQW (Bosch, 2018). SRPs are represented through the following Herbartian schema: $[S(X; Y; Q_0) \Rightarrow \{A_m^\diamond, W_n, Q_p, D_q\}] \Rightarrow A^\heartsuit$ (Chevallard, 2019). In this schema, $S(X; Y; Q_0)$ is the didactic system formed by a group of students X and a group of teachers Y addressing an open question Q_0. Q_0 is at the origin of the inquiry process, and the main aim is to collectively elaborate a final answer $A^\heartsuit$. During the inquiry process, X and Y raise derived questions Q_p, search and use bodies of knowledge or 'labelled answers' $A_m^\diamond$ they make available, together with empirical data D_q and other works W_n. These are called as didactic milieu.

What distinguishes SRPs from other inquiry-based teaching is their undetermined nature. That is, the questions approached remain the main objective of inquiry throughout, rather than the introduction of new concepts, systems of knowledge or the necessary skills and tools in themselves.

SRPs are not necessarily a theoretical product, but rather a model of the teacher's (or professor's) job. In other words, articulating the didactic milieu in the process of inquiry is important for teachers to design and supervise the learning process.

Questions-answers map

A summary of the inquiry process followed in an SRP can be described only focusing on the questions and answers that appear, thus generating a *questions-answers map* (QA map, Figure 2) of the inquiry process (Winsløw et al., 2013; Florensa, et al., 2021).

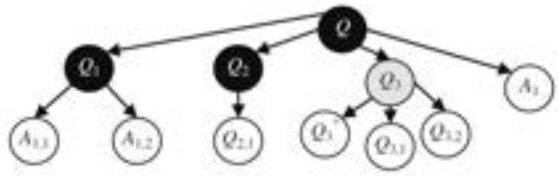

Figure 1: QA map (cited from Winsløw et al., 2013)

In this research, not only the process of students' inquiries but also the process of their lesson designs are described by the students themselves using QA maps.

Didactic design praxeology

Praxeology is one of the notions of ATD and describes the human knowledge-related activities. It is usually denoted as $\wp = [\mathrm{T}/\tau/\theta/\Theta]$ (Chevallard, 2019). In this research, the didactic design praxeology (hereafter DDP) $\wp_{dd}$ is a tool for analyzing and comparing pre-service teachers' lesson design, where T is the type of tasks that arise in the lesson design process, τ is the didactic technique used to resolve T, θ is the didactic perspective that justifies such a τ, and Θ is the pedagogical paradigm encompassing each θ, i.e., $\Theta =$ {PVW, PQW} here.

Research questions

Based on the above theoretical framework, the research question is: *What constraints do students' (pre-service teachers') views of the lesson have on the design of PQW-oriented SRPs?*

RESEARCH CONTEXT

Study and Research Paths for Teacher Education

The context addressed in this research is essentially Study and Research Paths for Teacher Education (hereafter SRP-TE, Figure 2). This is implemented in the class 'Design of Mathematics Lesson' offered at the author's university. The participants are seven second-year engineering students[1], divided into two groups of four (Group A) and three (Group B).

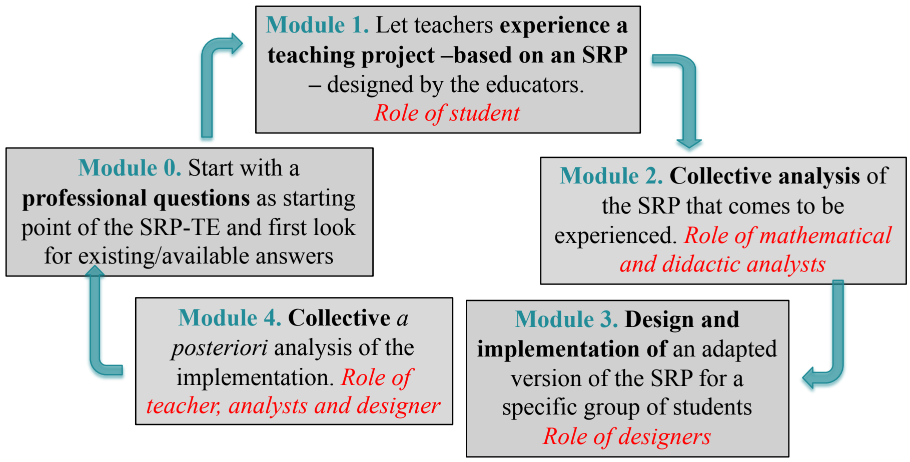

Figure 2: Module's structure of an SRP-TE

The question in Module 0 (Q_{0T}) is "How to teach inquiry?" As in the case of Mizoguchi, et al. (2024), students are already somewhat familiar with the QA map through different classes. For this reason, Modules 1 and 2 were not explicitly separated and were practised by the students themselves in a blended manner, without instructions from the teacher. It is also part of the design of Module 3 that is dealt with in this paper. (This is discussed below.) The class is currently in progress, and in the final session part of the class, a series of lessons designed by the students will be implemented in actual classroom in a school. The tasks for the students from the teacher (author) in each session are described in the respective sections below.

SRPs on padlocks

The SRPs addressed in this research relate to the security of padlocks. This is based on what is shown in Vásquez et al. (2019). However, both groups of students modify the original Q_0 at the beginning of their own inquiries. Subsequent inquiry activities, and the design activities based on them, are based on each modified initial question.

Data gathering

The data used in this study are basically observational field notes of students' activities in class and different types of QA maps made by students[2].

INQUIRIES BY STUDENT GROUPS

The actual padlocks used in the inquiry were of five types, shown in Figure 3. P1, P2 and P4 are dial type with 3 digits × 10, 4 digits × 10 and 5 digits × 10 respectively; P3 is a push type with 5 digits from 0~9 (once pushed, the number cannot be used); and P5 is a disc type with 3 times right (first number from 0~39) + 1 time left (second number from 0~39) + 1 time right (third number from 0~39).

When targeting these five types, the original Q_0 of these SRPs – "Which padlock is the most secure?" was too easy for the students, although Q_0 in Vásquez et al. (2019) is "How long would it take to open each padlock?". This may have been due, among other things, to the fact that all dial padlocks are comparable in 10^n ways.

Figure 3: Padlocks used in SRPs

Therefore, both groups posed new Q_0s as follows respectively:

Q_{0A}: How can the difficulty of unlocking each padlock be compared and how can this be increased? (what combinations of padlocks are available?);

Q_{0B}: How secure are the (prepared) padlocks relative to each other?

The inquiry activities of the student groups were very fine. As mentioned above, the students have already had experience in producing QA maps in other classes. Each inquiry process was described by the students themselves as the following QA map (Figure 4 and 5).

Group A began with Q_{0A} and focused on the difficulty of unlocking each padlock (Q_1) and how to increase the difficulty (Q_2) as the core of the paths, particularly how to combine several padlocks (A_{22}).

Group B began with Q_{0B} and focused on finding the probability of unlocking each padlock (Q_1) and how to represent them (Q_2) as the core of the paths, with particular attention to the unlocking time (under certain conditions) and how this was represented graphically (A_{23}). Furthermore, the psychological aspects of determining the unlocking code were also noted and the calculation of the unlocking time was improved based on a previous research paper as reference.

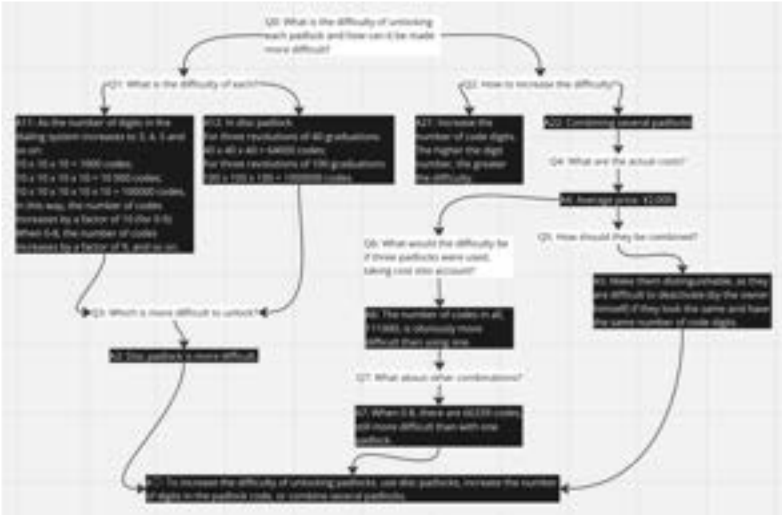

Figure 4: QA map of inquiry by Group A

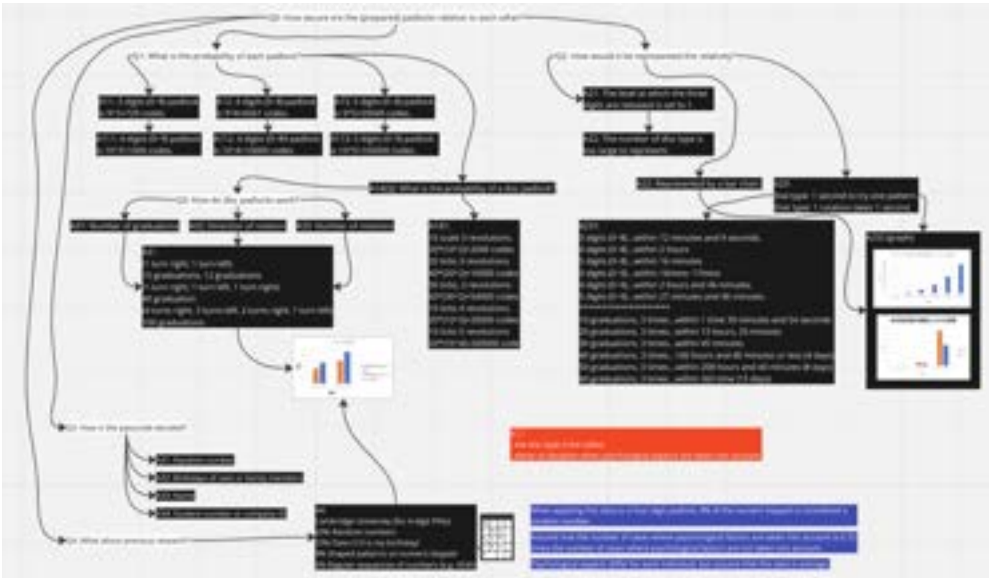

Figure 5: QA map of inquiry by Group B

Both groups did not formulate algebraic formulas for the number of events or probability in their inquiry processes, because they themselves already have the mathematical knowledge required.

DESIGNING INQUIRY TEACHING BY STUDENT GROUPS

First design

The task presented by the teacher to the student groups (Q_{3T}) was: "How do we design lessons based on our own inquiry process (QA map)?" It was confirmed in advance that the actual teaching would be implemented for 7th grade students, and the lessons were to be conceived under these conditions. Therefore, the problem here is to envision approximately how many lessons (one lesson hour is 50 minutes) will be needed when based on their own inquiry process (Q_{31T}). (At this moment, a detailed design of each hour was not necessary.) The student groups' initial approach to this task was as follows.

Group A allocated lessons for each sub-question path based on their QA map, and determined that approximately two lessons were required (for the inquiry itself). The initial question used in the lesson was their own Q_{0A}, but the actual structure of the lessons differed from their inquiry process and involved an initial whole-class exercise on the number of events and probability, which was then applied to finding the difficulty of unlocking each padlock (Lesson 1), followed by the question "How can we make it more difficult to unlock the padlock?", which was followed by pre-set tasks of (1) increasing the number of codes, and (2) combining padlocks, to find the final answer ($A^{\heartsuit}$) (Lesson 2).

Group B set up three lessons, which were envisaged as follows. In the first lesson, they asked the initial question, "Which seems more secure, P2 or P3?" and then "What if each padlock could be a simple case (P2 is a combination of three numbers 0~2, P3 is three numbers 0~4)?" to make them think in an easier way. In the second lesson, the students were asked to find the rules by inductively increasing the number of combinations based on the simple case in the first lesson, and in the third lesson, they were asked to find out how many different codes there are for P2 and P3.

Students' reflections on their own views of lesson

As will be discussed in the next section, it can be seen that the initial designs of both groups were highly influenced by the students' experience and knowledge of 'lesson'. Therefore, the teacher asked the students to reflect on their initial design and to describe what they themselves implicitly thought about the lesson. Students' descriptions (excerpts) consequently were as Table 1.

Describing the design process with QA map

In addition, the teacher asked each group of students to describe their initial design process on a QA map, regarding in itself an inquiry activity. They were then asked to identify where and how their unconscious thoughts influenced in the QA map they made. The QA map by Group A is shown below (Figure 6).

- There is a set timetable for each day of the week and time is divided for each subject
- Reviewing previous content at the beginning and summarising at the end of the lesson
- Need to be able to solve problems / It is important that everyone solves the problems and improves their scores in the examinations
- Textbook-based teaching progression
- The teacher leads the whole class in a style that conveys knowledge / The basic flow is 'teacher's explanation → examples → practice exercises by imitating the examples' / Students follow the progression of the lesson as organised by the teacher
- Need to assign appropriate tasks

- The whole class progresses at the same pace / A fixed amount of learning has to be done in a fixed amount of time, which means that some areas cannot be studied in depth - Despite the variety of ideas, other answers are not tolerated

Table 1: Students' descriptions about their existing views of the lesson

Describing the design process with QA map

In addition, the teacher asked each group of students to describe their initial design process on a QA map, regarding in itself an inquiry activity. They were then asked to identify where and how their unconscious thoughts influenced in the QA map they made. The QA map by Group A is shown below (Figure 6).

Figure 6: QA map of the first design process by Group A

Modified design

As above, the students reflected on their initial designs and modified these. Notably, rather than assuming a definite time (50 minutes), both groups began to consider the inquiry activity with the questions as key components. There was a shift in perception from which questions the activities would take and how long they would take, to rather that it would be difficult to simply define a time.

However, as the current constraints in the school, with its set timetable, could not be changed, it was decided to conceive of the time needed to allow for these by considering the pathways of activities. This was a common trend observed in both groups.

Furthermore, the structuring of the lessons by the two groups became consequently similar. Although the initial question was different for each group, it was conceived as Session 1 (two lessons + one lesson for presentation), which required an understanding of each padlock by Q1, and Session 2 (one lesson), which explored each group's specific question in Q2.

DISUCUSSION AND CONCLUSION

When teachers, not only students, design their ordinary lessons, they are likely to be influenced by the views they hold about the lesson. If these views of the lesson correspond to the lesson implemented, there will be no particular problems and the design process may not be difficult. In the design of inquiry teaching, i.e. SRPs, by the students in this paper, their reflections on their views of the lesson provided an opportunity for them to reconsider what inquiry learning was intended to be.

The intention of the teachers' Q_{31T} was to ask how much time (T_1) it would take for the students to actually do the inquiry undertaken by the pre-service teachers. Although the teachers' explanation was somewhat insufficient, their answer to Q_{31T} was, in fact, that the lesson would be reached to a certain completeness for each (τ_1). This reflects their view of the lessons, which are presented in Table 1 (in particular, the first two items). This means that in today's schools, each subject lesson is organised as a timetable (θ_{11}) and that each hour of lesson has a certain modality (θ_{12}). Thus, their first DDP is denoted as $\wp_{dd-1} = [\mathrm{T}_1/\tau_1/\theta_{11}, \theta_{12}/\Theta_1]$, where Θ_1 is the PVW. In addition, the Design QA map of Group A shown in Figure 6 describes their awareness of further details, which can principally be reduced to $\wp_{dd-1}$. Reflecting on the initial design, the answer to T_1 shifts to making the question a key component (τ_2). This automatically abandons θ_{12}, but θ_{11} is still a condition for the lesson. Therefore, Θ_1 also remains unchanged (to Θ_2 = PQW). So, their modified DDP is denoted as $\wp_{dd-2} = [\mathrm{T}_1/\tau_2/\theta_{11}/\Theta_1]$.

SRPs usually envisage a medium- to long-term inquiry process. For this reason, how to organise individual lessons or sessions is also an open question. This problem cannot be avoided when developing SRPs, at least within the constraints of the current school institution. Barquero (2018) refers to the relativity of conditions and constraints in discussing the ecology of implementing SRPs as mathematical modelling for university students. The discussion in this paper argues for similar issues in SRP-TE.

Furthermore, in the modified design by the two student groups in this paper, a change to a lesson structure with the questions in inquiry as key components was observed. However, how to describe this issue as a so-called 'lesson plan' is still an open question.

NOTES

1. This is because of the accreditation of the secondary mathematics teacher course in the Faculty of Engineering.

2. QA maps were originally written in Japanese and translated into English by the author

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Interdisciplinarity in pre-service teacher education: a physics and statistics approach to the falling balls

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Keywords: Teaching and learning of mathematics in other disciplines, Curricular and institutional issues concerning the teaching of mathematics at university level, Teacher education, Study and research paths, Interdisciplinarity.

In recent decades, there have been important movements in favour of a change in pedagogical paradigm towards more inquiry-based methodologies, breaking the strong barriers between disciplines. In the case of university teaching, this pedagogical paradigm shift has materialised through multiple initiatives for mathematics and science (Furtak et al., 2012). Teacher education at university is a fundamental step, as teachers are decisive agents in supporting and disseminating educational research advances. This poster presents an empirical study, supported by a research project developed at the Universitat de Barcelona, over two academic years 2022-23 and 2023-24, which involved a team of researchers in mathematics and science education. The multidisciplinary team worked together on the design, implementation, and analysis of *study and research paths* (SRPs) as proposed in the framework of the Anthropological Theory of the Didactic (ATD) (Bosch, 2018), and with special attention to *interdisciplinary* SRP (*i*-SRP).

QUESTIONS ABOUT INTERDISCIPLINARITY IN TEACHER EDUCATION

Our research addresses different aims about interdisciplinarity in teacher education. First, it focuses on creating conditions for the co-design of *i*-SRP within the collaboration of specialists of the different disciplines, more concretely, between mathematics and physics educators and researchers. Second, the selection of teacher education courses aims to create favourable conditions for implementation of *i*-SRP for the collaboration in pre-service teacher education at university. Third, it aims to analyse the instructional proposals, comparing the conditions created and the constraints that emerge in the dialogue and interaction between the respective scientific disciplines. Consequently, we inquire into what conditions can facilitate, and through which didactic devices, the collaborative design and implementation of instructional proposals for interdisciplinary (involving mathematics and natural sciences effectively) in pre-service teacher education?

SELECTED CASE STUDY AND RESULTS FROM AN *i*-SRP

Our research examines two mandatory courses for Primary School Teachers' degree at the Universitat of Barcelona: 'Didactics of Matter, Energy, and Interaction' (DMEI) with five groups of 30 students each, and 'Didactics of Mathematics' (DM) with one group of 50 students. Both courses, taught by the authors of this poster, were conducted concurrently during the 2nd semester with 2nd-year students. In this context, two different *i*-SRP have been experienced. The first about what does classifying mean? following the designs initiated by Lerma et al. (2021). And a second *i*-SRP about the physics and statistics approach to the falling balls. We focus on this second case, as the interaction and complementarity between the mathematical and physical work developed, thanks to the courses' interaction, is worthy of recognition. As shown in the following images, in the DMEI course students realized the experiment and collect measurements of the falling balls. It was in the DM course where the organisation, data cleaning, detection of measurements' errors, and the statistical and physical interpretations of the relation among variables were made. Students and the lecturers finished the experience recognising that they had learnt and taught more physics than expected, thanks to the statistical inquiry. And, reciprocally, more functional mathematics had been made visible thanks to the real experiment with the falling balls.

	Bola M1					
		altura	mitjana	mediana	dispersió	Diferència de dx
	GRUP			dm	ds	entre m1 i m2
1	181	26	75,38	74,25	4,56674	75,38
2	181	19,5	80,65	78,25	5,258485	80,65
3	182	33	5,555556	5,5		
4	182	15	88,0375	88,7		
5	183	30	51,98889	51,8		
6	183	15	50,5	55,5		
7	184	24	30,65	76,45		
8	184	15,7	19,88	66		
9	185	19	69,95	70,5		
10	185	26	71,8	76,15		

The relevance of this work lies on the way this *interaction provides new knowledge* about the disciplines themselves and *how* it can be enriched from this dialogue. Furthermore, it sheds light on the conditions required for future designs of *i*-SRP in other university contexts, in teacher education and beyond.

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New advances in teaching mathematics with Digital Interactive Mathematical Maps (DIMM)

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Keywords: University mathematics education, Digital resources, History, Transition, Defragmentation.

RESEARCH TOPIC AND MOTIVATION

In this poster, we present new advances in the Design-Based Research process of a digital tool for teaching of several areas of mathematics: Digital Interactive Mathematical Maps (DIMM). The idea behind the concept of the DIMM is amongst others to overcome historically developed fragmentation of mathematical contents in the curriculum of higher secondary schools and to offer an integration of the world of higher mathematics in the sense of Felix Klein (Klein, 1924/1932: "double discontinuity"), which is still a problem today (Winsløw & Grønbæk, 2014). The double discontinuity of Klein implies that teachers often base their school teaching on their own pre-university experiences instead of their academic knowledge gained at university. Mathematics teaching becomes thus fragmented, and students have fewer opportunities to notice connections and develop conceptual understanding (ibid.).

RESEARCH QUESTION

In order to address the discontinuity, Kilpatrick (2019) refers to Klein (1924/1932, pp. 1-2): "His goal was to show the *mutual connection between problems in the various fields,* a thing which is not brought out sufficiently in the usual lecture course, and more especially to emphasize the relations of these problems to those of school mathematics" (Kilpatrick, 2019, p. 217). In this context, a central research question for us is: How can the double discontinuity of Klein be addressed by the development of a visualizing digital tool showing historic and thematic connections and interdependencies?

RESEARCH PROCESS AND DESIGN RESULTS

The DIMM have been developed in an iterative Design-Based-Research process (Bikner-Ahsbahs et al., 2015; Przybilla et al., 2022), and their perceived usefulness and ease of use have been evaluated by the Technology Acceptance Model of Davis (1985).

Based on first ideas in Brandl (2009) the DIMM address the inner-mathematical fragmentation by integrating the historical origin of mathematical concepts as well as interdependencies between them. The tool is freely available at the web address https://math-map.fim.uni-passau.de/. It contains three-dimensional maps for the areas of Geometry, Algebra, Calculus (new version) and now also Stochastics, chronological timelines with information on selected mathematical results and concepts with links to suitable internet content such as texts and YouTube videos, and sometimes suitable problems from mathematical competitions. The maps exist in English, German, and now also in Spanish and Ukrainian. Illustrating examples and screenshots will be presented on the poster. We also shortly report how the DIMM may support the fostering of mathematically gifted students, in particular by using so-called vertical cuts and ways through the 3d-map with problems from mathematical competitions.

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Didactical engineering: an approach for carrying out an epistemological analysis from research problems in mathematics

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Keywords: preparation and training of university mathematics teachers, teaching and learning of number theory and discrete mathematics, concept of problem, epistemological analysis, didactical engineering.

INTRODUCTION AND RESEARCH QUESTION

The Theory of Didactical Situations is one of the important references in French didactics (Brousseau, 1997). This theoretical framework can be used to construct of teaching and learning situations, with a focus on didactical engineering as a research methodology (Artigue, 2015). This approach involves several stages: "preliminary analysis, conception and *a priori* analysis, realization, observation and data collection, *a posteriori* analysis and validation" (Artigue, 2015, p. 471). She emphasizes the importance of an epistemological analysis in the preliminary analysis "to support the search for mathematical situations representative of the knowledge […]" (p. 472). However, no method is specified for carrying out this type of epistemological analysis to determine consistent mathematical situations that are representative of the aimed knowledge and know-how. Therefore, *how can we carry out an epistemological (mathematical) analysis of a problem to identify mathematical situations that are relevant for university students?* To that end, we propose a method of epistemological analysis for designing didactical engineering for learning and teaching mathematics at university. The poster will illustrate this method using a discrete mathematics problem.

THEORITICAL FRAMEWORK

We define the concept of problem from the *syntactic* and *semantic* aspects (Da Ronch, 2022). For the syntactic aspect, a mathematical problem must be formulated as a set of instances and a general question (Garey & Johnson, 1979). For the semantics aspect, we are based on the concept of problem described by Giroud (2011) and in particular the notion of problem-space or universe of problems to characterise the scope of a given problem, and to study its ramifications and its proximity to other underlying mathematical problems by modifying the values of its instances and/or the scope of its question (e.g., Da Ronch, 2022). This will enable us to determine whether the problem holds a significant *epistemological quantity*. This quantity will be judged to be all the more significant if its problem-space or universe contains a significant number of problems in its neighbourhood, and that these problems are linked by relationships (partial sufficiency relationship, sufficiency relationship, necessary relationship and equivalent relationship between problems), based on the proximity of the questions, instances and also the invariants of the proofs used to solve them. The *zoom* concept will allow us to look at this space at different levels of granularity (Da Ronch, 2022).

METHOD FOR CARRYING OUT AN ESPISTEMOLOGICAL ANALYSIS BASED ON A MATHEMATICAL RESEARCH PROBLEM

We describe the universe of mathematical problems as an infinite space composed of problem sets such as $\boldsymbol{\Omega}_{\mathbb{P}} \coloneqq \{\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_i, \dots, \mathbb{P}_j, \dots\}$. Each of these problem sets falls into different branches of mathematics. Thus, during the epistemological analysis of a given problem $\boldsymbol{\mathcal{P}}$, when we wish to identify whether this problem is semantically interesting (not isolated), its universe $\boldsymbol{\Omega}_{\boldsymbol{\mathcal{P}}}$ is initially limited to sets of neighbouring problems $\mathbb{P}$ to which $\boldsymbol{\mathcal{P}}$ is related by neighbourhood relationships (linked to the question, instances and invariants of the proofs). Here, "neighbourhood" defines, by extension, a "metric" that can be used to determine the proximity (or distance) between problems. Thus, we need to focus on some of these problem sets that are judged to be significantly close to $\boldsymbol{\mathcal{P}}$. In this poster, we will illustrate our points with a contemporary research problem in discrete mathematics: the *Domino Problem* (e.g., Da Ronch, 2022). This problem allows to work on different concepts as the decidability in the algorithmic sense (computability), algorithmic complexity, finding of paths and circuits in a digraph $\boldsymbol{G}$, *etc*. Once these problem sets have been determined, we use the notion of zoom, which allows us to examine the problems in the universe $\boldsymbol{\Omega}_{\boldsymbol{\mathcal{P}}}$ that belong to the problem sets on which we have focused with an enlargement factor. The choice of problems from these sets is always determined by the epistemological study of $\boldsymbol{\mathcal{P}}$, which makes it possible to establish neighbourhood relationships between the problems of $\boldsymbol{\Omega}_{\boldsymbol{\mathcal{P}}}$, thus giving meaning to $\boldsymbol{\mathcal{P}}$. Thus, the richer the universe $\boldsymbol{\Omega}_{\boldsymbol{\mathcal{P}}}$ of problems with neighbourhood relationships, the more $\boldsymbol{\mathcal{P}}$ is a semantically interesting problem to study, since it is not isolated, and which, moreover, has a significant epistemological quantity. The epistemological analysis of the problem may be refinement according to the objectives and the target audience. In this way, the process of zoom can be carried out as many times as necessary, depending on requirements.

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Praxeological needs and Klein's double discontinuity: the case of friezes

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Keywords: Transition to, across and from university mathematics, curricular and institutional issues concerning the teaching of mathematics at university level, friezes, group theory, geometry.

WHAT EXACTLY IS A FRIEZE?

The teaching of geometry in high school (HS) in France has recently evolved to include the study of friezes. The exploration of textbooks and other institutional resources shows a variety of viewpoints on friezes, lacking uniformity in the definition of these objects. In particular, some objects may be considered as friezes that would not be acknowledged as such, for instance according to university (U) definitions of friezes. Such an example is given by the "frieze" shown in Figure 1. This "frieze" is not admissible as a frieze since it is not invariant under translations – which is a paramount property of friezes. We observe that, in HS, definitions of friezes are often inadequate and that the variety in the types of friezes studied is rather poor. These difficulties of the profession of mathematics teacher in high school are related to *praxeological needs*: a suitable mathematical framework for friezes is needed, the current relation $R_{\mathrm{HS}}(t,$ Friezes$)$ of high school (HS) teachers (t) to friezes is lacunary.

Figure 1. A "frieze"

The lacking of such a framework may be put into relation with the fact that, at University, students (s) rarely study friezes – their relation $R_{\mathrm{U}}(s,$ Friezes$)$ is almost empty –, so that beginning teachers have a relation to friezes $R_{\mathrm{HS}}(t,$ Friezes$)$ that is roughly similar to the one they had as high school students. This fact is an illustration of the well-known phenomenon coined as *Klein's double discontinuity* (Winsløw & Grønbæk, 2014): the first discontinuity is seen in the passing from the high school student position to the university student position, the second in the passing from the university student position to the high school teacher position (Figure 2).

$$\boxed{R_S(s, \text{Friezes}) \rightarrow R_U(s, \text{Friezes}) \approx \varnothing \rightarrow R_{HS}(t, \text{Friezes})}$$

Figure 2. Klein's double discontinuity for Friezes

A "DESIRABLE" RELATION TO FRIEZES

We propose a mathematical setting for friezes, which gives a constructive definition of friezes that is equivalent to the formal, usual university definition. It is also close to the unformal (and often lacking of rigour) definition given in high school. At University, friezes are seldom studied. When they are, they are roughly defined as follows: a subset F of the plane is said to be a frieze if the intersection of the isometry group of F and the group of translations is isomorphic to $\mathbb{Z}$: $\mathrm{Isom}(F) \cap \mathcal{T} \cong \mathbb{Z}$ (Tauvel, 1995). In high school, as can be found in institutional resources (*Ressources d'accompagnement du programme de mathématiques au cycle 4*, 2016), a frieze is a strip of plane in which a pattern is regularly repeated by means of translations. A pattern associated with the shortest possible translation (translation represented by the shortest possible vector) is a *basic pattern*; in turn, this pattern can be obtained from an *elementary pattern*, reproduced by other transformations (symmetries, rotations). Seeking for means to decrease the gap between the two above definitions, we propose the following definition: a subset F of the plane is a frieze if there exists a subset M included in a rectangle $ABCD$ such that F is the union of the translated images of M by the translations associated to integer multiples of vector $\overrightarrow{AB}$, and if there exists a non-zero vector $\vec{u}$ of minimal norm which is associated to a translation that stabilizes F.

Moreover, the introduction of the new object "frieze" leads naturally to the question of the classification of the friezes. The study of the isometries which leave invariant a frieze F, is the suitable tool to classify friezes. This study could be lead in teachers' training at University and up to a certain point in high school.

$$\boxed{R_S(s, \text{Friezes}) \rightarrow R_U({*s}, \text{Friezes}) \rightarrow R_{HS}({*t}, \text{Friezes})}$$

Figure 3. A desirable relationship to Friezes

The relation that we have just described would require a renewed position of student **s*, and is denoted $R_U(*s, \text{Friezes})$. The training of teachers based on this student's relation could lead to an enrichment of teachers praxeological equipment relative to friezes, hence to a modification of position t into a new position $*t$. Klein's double discontinuity would hopefully be reduced by considering the HS-U-HS transitions as depicted in Figure 3.

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Solving techniques of future mathematics teachers: the case of reflection

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Keywords: transition to, across and from university, teaching and learning of specific topics in university mathematics, reflection, praxeologies, semiotic analysis

INTRODUCTION

The work presented in this poster is part of a research project aimed at equal access to quality learning opportunities for all learners, particularly those defined as having special educational needs, in line with the principles of inclusive schooling promoted by UNESCO (2017). Two secondary school mathematics teachers taking part in this research designed and implemented a mathematical task in their class (grade 6, 11-12 years old). This task involves using prescribed materials to construct an instrument that can be used to determine the symmetric of a given point with respect to a straight line. We proposed the same task to first-year and second-year Master's students intending to teach mathematics in secondary schools in order to compare their knowledge involved in solving the task with that of the pupils.

We position our study in relation to Klein's second transition (Winsløw, 2020), which occurs for future mathematics teachers between their university studies and the time when they obtain a teaching position in secondary education. To quote Felix Klein (2008), an important aim of didactic research dedicated to this transition is to enable future teachers to adopt “a higher standpoint on mathematics”. At the very least, the aim is to enable them to see to what extent the mathematical knowledge they have acquired at university may (or may not) be relevant to their future profession, in dedicated courses – called “capstone courses” – such as geometry (Hoffmann & Biehler, 2020). In our case, these objectives are relevant, since on the one hand the notion of reflection is present at primary school and during the first years of secondary education in France, and on the other hand students have approached symmetries (reflection or not) or isometries from different points of view (affine space, vector space, complex numbers) in the course of their university studies.

MAIN ISSUE AND THEORETICAL FRAMEWORK

In order to compare them, we have chosen to model the mathematical activity of pupils and students in terms of praxeologies (Bosch et al, 2019), based on the process of didactic transposition (Chevallard & Bosch, 2014) linked to the secondary and higher education curricula.

In this poster, our main issue is to determine the (components of) praxeologies that are mobilised by pupils and students in carrying out the task under consideration.

More specifically, to what extent do the students supplement the pupils' praxeologies with elements of a technological or even theoretical nature?

In addition to a more precise presentation of the context and process, the aim of the poster will be to present the results of our study (in progress) with respect to this main issue. In order to enable a more detailed comparison of the activities of pupils and students, we will highlight the identification of possible mute, weak or strong praxeologies among students (Barquero et al, 2019), and we will report on the semiotic analyses of their activities (Petitfour & Houdement, 2022).

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Tertiary Mathematics for Prospective Secondary Teachers: The Case of Combinatorics

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Keywords: Teaching and learning of number theory and discrete mathematics, preparation of mathematics teachers, combinatorics, Anthropological Theory of the Didactic, tertiary mathematics

INTRODUCTION

The training of pre-service teachers is a topic of interest in mathematics education. In particular, the contribution of tertiary mathematics courses to the training of prospective secondary mathematics teachers (PSMTs) is currently attracting more attention from researchers and is under increased scrutiny (Wasserman et al., 2023). In parallel, research in discrete mathematics education is also growing; the inclusion of such content in both PSMT training and secondary mathematics programs seems beneficial for teachers and their students (Sandefur et al., 2022), assuming that teachers are adequately prepared. Our research is situated at the intersection of these two topics.

This poster proposal presents data from the first author's master's thesis, a large study which addresses the following question: *How do university mathematics courses related to combinatorics contribute to the training of prospective secondary mathematics teachers?* This research question is further broken down into three sub-questions: 1) In what ways does combinatorics content in tertiary courses relate to content in secondary school? 2) How do PSMT education programs address the various aspects of teacher knowledge regarding combinatorics (mathematical, curricular, pedagogical content knowledge)? 3) What issues do PSMTs face in preparing to teach combinatorics content? This poster focuses on the first sub-question.

THEORETICAL FRAMEWORK

We rely on Chevallard's (1999) Anthropological Theory of the Didactic (ATD) as a theoretical framework for this research. ATD claims that knowledge is institutionally situated, which means that any knowledge learned is dependent on the characteristics of the institutional context in which this learning takes place. Hence, ATD allows for the analysis of practices situated in different institutions; in our case, secondary and tertiary mathematics courses. For each of these institutions, we study the types of combinatorics tasks and the techniques that are available to solve these tasks. We also compare rationales on combinatorics at both levels of mathematics teaching: is combinatorics in secondary school and at university supported by similar or disconnected rationales (technologies and theories in ATD)? By identifying the praxis (tasks and techniques) and logos (rationales) of combinatorics in secondary and tertiary mathematics, we seek to understand whether this field may yield another example of tertiary-to-secondary mathematics discontinuity for teachers (Klein, 1908/2016).

METHODOLOGY

Our study is based in the province of Quebec, in Canada. We analyse the most popular secondary mathematics textbooks used in Quebec and review the lesson plans and material of two university mathematics courses: Discrete Mathematics and Probability. We also intend to distribute a survey and interview PSMTs in order to study their competencies in combinatorics and in the teaching of combinatorics, as well as their sense of readiness for teaching such content. We plan to collect our data at the Université de Montréal, where PSMTs attend tertiary mathematics courses taught in and by the Department of Mathematics and Statistics. It is important to note that combinatorics is not explicitly included in the didactics courses offered to PSMTs. Therefore, prospective teachers' understanding of combinatorics and their ability to teach such content strongly depend on their tertiary mathematics education.

RESULTS AND DISCUSSION

We present partial results (as this study is still ongoing) related to sub-question 1. We have analysed curricular documents and classified the types of combinatorics tasks presented in the selected secondary textbooks. Our results indicate that in Quebec's secondary mathematics curriculum, combinatorics is introduced exclusively in the probability chapter as a means of counting all possible outcomes of a probability experiment. However, the problems that include combinatorics do so without the explicit use of formulas; rather, it is suggested that students reason about combinatorics with the help of sketches, graphs or tables. In comparison, while there are direct links to secondary school mathematics in the tertiary probability course, combinatorics problems in this course are almost always solved using formulas (factorials, permutations, combinations, etc.). We intend to explore this further via the survey and interviews with PSMTs: is the disconnection between the techniques (and therefore in the logos) used in combinatorics in secondary and tertiary mathematics an issue that impacts the teaching abilities of PSMTs?

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Uncovering group theory in French secondary school curricula

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Keywords: transition from university mathematics, teaching and learning of abstract algebra, teaching and learning of group theory in university mathematics.

INTRODUCTION

In France, future mathematics teachers usually study group theory. This advanced knowledge is not only source of difficulties, it may also seem irrelevant with regard to the content they will have to teach in secondary school. This has been coined by Klein (1908/2016) as a "second transition".

The aim of my master's thesis (Rolland, 2023), supervised by Nicolas Grenier-Boley, was to uncover the links between group theory and French secondary school (*collège*, age 11-12 to 14-15 and *lycée*, age 15-16 to 17-18). These links are not explicit in the curricula, and require a thorough investigation to be brought to light. This poster endeavours to present a visual summary of this work.

The research question addressed in this poster is: In which ways is group theory linked to secondary school mathematics in the official curricula?

METHODOLOGY AND THEORETICAL FRAMEWORK

Wasserman thoroughly studied the content of the *Common Core Mathematics Standards from the United States* (CCSS-M), and identified associations between school mathematics and abstract algebra that could be relevant for teachers (2016). Building on his work, my aim was to uncover all the associations between abstract algebra and school mathematics, not only those deemed useful for teachers.

The theoretical framework used in this study is the Anthropological Theory of the Didactic (ATD). My study was focused on external didactic transposition, from scholarly knowledge to knowledge to be taught (Chevallard & Johsua, 1991). I used the 2022 *collège* and *lycée général* curricula. My work was focused on group theory, so whenever I identified a connection with a ring or a field, I only considered its underlying group(s).

A praxeological study of the curricula's items allowed me to gauge whether each item was backed-up by group theory notions (Chevallard, 1992). I treated each item as a type of task, and determined a corresponding technique. I then assessed the presence of group theory notions in the technology justifying the technique, or in the theory justifying the technology. An example of praxeology built on the type of task "add two fractions with different denominators" will be provided in the poster.

RESULTS

I identified the same four content areas as Wasserman ("Arithmetic properties", "Inverses", "Solving equations", "Structure of sets"). The latter originally encompasses structures that gradually get enriched (for example ($\mathbb{N}$, +) gets enlarged to ($\mathbb{Z}$, +) in middle school), but since I focus only on groups, in my analysis it means abstract groups, without a specified set or law.

I identified three extra content areas. The area “Geometric transformations” encompasses the Euclidean plane isometries (translation, rotation, symmetries), which, endowed with the composition, is a group. “Group homomorphism” includes exponential and logarithm functions. This area could be linked to “Structure of sets”, but since this notion plays an important role in group theory, I isolated it. The area “Iterated elements and order” gathers the operations which, iterated a given number of time, give the original element back. For example, the proof of Fermat’s little theorem relies on the notion of order of an element.

CONCLUSION

The study of the French secondary school curricula showed that group theory is implicitly involved in seven content areas. These links offer a potential for a capstone course designed for prospective teachers. The aim of this course would be to draw their attention on some concepts of group theory, work on them to fix potential misconceptions, and show of these concepts could be beneficial for teaching.

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TWG5: Teachers' and students' practices and experience

Believability in Mathematical Conditionals:

A Comparative Judgement Study

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Mathematical conditionals – interpreted as generalized conditionals with implicit universal quantification – are either true or false. Nevertheless, we conjecture that, like everyday causal conditionals, they vary in believability. This paper tests this conjecture using comparative judgement. We asked mathematics education researchers to judge pairs of mathematical conditionals, stating which of each pair is more believable; we then used these judgements to generate believability scores via the Bradley-Terry model. We report that this yielded reliable scores, meaning that the researchers broadly agreed about which conditionals are more believable. We also report that believability is imperfectly related to truth. Throughout, we relate this work to our broader aim, which is to design a mathematical conditional inference task.

Keywords: believability, comparative judgement, conditional, logic, reasoning.

INTRODUCTION AND THEORETICAL BACKGROUND

Conditionals and Conditional Inference

Conditionals are central to mathematics, especially at the undergraduate level, with its focus on proof. Many statements of interest can be expressed as conditionals, and textbooks (e.g., Houston, 2009) explain that mathematical conditionals are interpreted according to a *material* or *truth-functional* interpretation: the conditional 'if A then B' is considered true unless its antecedent, A, is true and its consequent, B, is false.

Conditionals are central in everyday life too, and have attracted extensive interest in cognitive psychology (e.g., Evans & Over, 2004). There, they are often investigated using *conditional inference tasks* with either abstract or everyday content (Oaksford & Chater, 2020). Abstract content commonly uses imaginary letter-number pairs; everyday content commonly uses causal conditionals (Evans et al., 2010). In both cases, task items correspond to inferences from a conditional together with affirmation or denial of its antecedent or consequent, as illustrated below.

Modus Ponens (MP)
If the letter is A then the number is 3.
The letter is A.
Therefore, the number is 3.

Affirmation of the Consequent (AC)
If the letter is A then the number is 3.
The number is 3.
Therefore, the letter is A.

Denial of the Antecedent (DA)
If the letter is A then the number is 3.
The letter is not A.
Therefore, the number is not 3.

Modus Tollens (MT)
If the letter is A then the number is 3.
The number is not 3.
Therefore, the letter is not A.

Modus Ponens (MP)
If John studies hard, then he does well on the test.
John studies hard.
Therefore, John does well on the test.

Affirmation of the Consequent (AC)
If John studies hard, then he does well on the test.
John does well on the test.
Therefore, John studied hard.

Denial of the Antecedent (DA)
If John studies hard, then he does well on the test.
John does not study hard.
Therefore, John does not do well on the test.

Modus Tollens (MT)
If John studies hard, then he does well on the test.
John does not do well on the test.
Therefore, John did not study hard.

By the logic of the material conditional, *modus ponens* and *modus tollens* inferences are valid, *denial of the antecedent* and *affirmation of the consequent* inferences are invalid (Evans & Over, 2004). The cognitive psychology literature, however, shows that people do not typically respond in line with this normative interpretation. For both abstract and causal conditionals, educated adults typically accept almost all valid MP inferences. But they also accept many invalid DA and AC inferences, and reject many valid MT inferences. For example, Evans et al. (2007) reported 98% acceptance for MP inferences with abstract conditionals, 47% for DA, 74% for AC and 50% for MT. For causal conditionals, acceptance rates are broadly similar but moderated by believability, so some people accept more inferences from more believable conditionals. Evans et al. (2010), for instance, found that participants with higher vs. lower general intelligence scores accepted 93% vs. 82% of MP inferences, 41% vs. 57% of DA, and 41% vs. 62% of AC, and 47 vs. 55% of MT.

In mathematics education, research has drawn on this work, but only to a small extent. It has been found that studying mathematics intensively at age 16-18 leads to more normative performance on standard abstract conditional inference tasks, primarily through improved rejection of invalid inferences (e.g., Attridge, Doritou & Inglis, 2015). Conversely, more normative performance in abstract conditional inference tasks predicts better performance in undergraduate mathematics courses (e.g., Alcock & Attridge, 2023). This is as we might expect: mathematics both trains and rewards success in distinguishing valid from invalid inferences.

Mathematics education research has not systematically used standard tasks with everyday content, perhaps because mathematics is seen as an abstract subject. However, mathematics is not abstract in the sense of content-free. On the contrary, its content is extremely meaningful to those with sufficient expertise. This raises the possibility that reasoning about mathematical conditionals might be more like reasoning about everyday causal conditionals, so that inference acceptance is affected by believability. To understand how everyday reasoning affects mathematical reasoning, it would be useful to investigate this possibility.

Conditionals in Mathematics

Mathematical conditionals are, of course, not entirely like everyday causal conditionals. First, unlike many causal conditionals, they do not involve temporal relationships between causes and effects. Second, where everyday claims like 'If John studies hard, then he does well on the test' are treated as meaningful without specifying which John or test is under discussion, a mathematical conditional like 'If $x < 2$ then $x < 5$' is a predicate that has no truth value without quantification (Durrand-Guerrier, 2003). Third, such predicates *are* commonly interpreted as meaningful generalised conditionals when the intended scope is clear (e.g. Houston, 2009), but, where everyday causal conditionals are typically reasonable claims that admit exceptions (Cummins et al., 1991), mathematical conditionals are either true or false.

However, mathematical conditionals are like everyday causal conditionals in that they express one-way inferential relationships. They should not be taken as biconditionals and, although any conditional with true antecedent and consequent is true, we are usually only interested in those in which the consequent can be inferred from the antecedent. Moreover, despite their being technically true or false, it is plausible that mathematical conditionals vary in believability. Indeed, it would be surprising if they did *not*. It is well known that individuals' concept images (Tall & Vinner, 1981) or personal example spaces (Sinclair et al., 2011) do not precisely match defined concepts, so less familiar examples might be overlooked, making a conditional seem somewhat believable when it is false or somewhat unbelievable when it is true.

This idea is consistent with findings on mathematical conditional inference. Durand-Guerrier (2003), for instance, reported that students' acceptance of conditional inferences was sensitive to mathematical content. But direct evidence is thin because research in mathematics education has usually not addressed all four inferences and/or not used standard phrasing across tasks. Durand-Guerrier (2003) did consider all four inferences for some items, but took conditionals from textbooks so that phrasing varied (e.g., 'For a function to be integrable on an interval I, it is sufficient that it is continuous on this interval'). Stylianides et al. (2004) used one everyday MT and one everyday DA item, with a mathematical contraposition on the validity of 'If $x = y \Rightarrow x^2 = y^2$' as a proof for 'If $x^2 \neq y^2$ then $x \neq y$ (for $x, y \in \mathbb{N}$)'. Case and Speer (2021) used all four inferences with conditionals with explicit universal quantification ('For all functions f, if f is differentiable at a point $x = c$, then f is also continuous at the point $x = c$'); also, their categorical premises and conclusions applied universal instantiation ('Suppose h is a function that is continuous at $x = 7$'). Universal instantiation is mathematically routine, but it adds an extra layer of complexity. There is, to date, no mathematical version of a standard conditional inference task.

We, therefore, aim to construct a mathematical conditional inference task and to use it – alongside tasks with abstract and everyday causal content – to investigate conditional inference among mathematics undergraduates and experts. We first seek to establish whether it is possible to construct a task that parallels those using everyday causal conditionals by systematically varying believability. In the study reported here, we address two research questions:

1) Can believability for mathematical conditionals be reliably measured?
2) How is believability related to truth?

METHODOLOGY AND METHOD

Believability and Comparative Judgement

The literature above points to the notion of believability as potentially important, but the methods typically used to measure believability do not transfer well to mathematics. One method is to use a pre-test in which participants generate distinct counterexamples (e.g., Cummins et al., 1991). Some researchers have asked one group of participants to generate counterexamples before another group completes a conditional inference task (e.g., Cummins, 1995); others have used within-subjects designs in which a single group generates counterexamples then later completes a conditional inference task (e.g., De Neys et al., 2003). Both approaches are ill-suited to mathematics because counterexamples are often singular ('zero') or in infinitely large classes ('the negative numbers'). In the first case, one cannot produce distinct counterexamples; in the second, producing a list is an artificial task. A second method is to ask participants to quantify belief in conditionals directly using measurement scales (e.g., Evans et al., 2010). This, too, is ill-suited to mathematics, because participants will likely be aware that a conditional must, in fact, have an agreed truth value. Again, this renders the task artificial.

In our study, we therefore ask not for counterexamples or absolute believability, but for judgements about *relative* believability. Specifically, we use *comparative judgement* (Jones & Davies, 2023), asking multiple judges to compare multiple pairs of conditionals, for each pair judging which is more believable. We use these judgements, via the Bradley-Terry model (Bradley & Terry, 1952), to construct a scaled rank order of believability scores. We then evaluate these scores for reliability.

Comparative judgement has several advantages for our situation. First, it does not require detailed assessment criteria (Bisson et al., 2016); judges can use their knowledge to make holistic judgements (Verhavert et al., 2019). Second, judges need not give absolute scores; they can focus on relative believability for just two conditionals at a time, which is an easier task (cf. Thurstone, 1994). Third, asking for relative judgements circumvents the problem that different judges might have different absolute standards, which could make it appear that people disagree about believability when in fact they agree (cf. Sa et al., 2023). These features are useful for constructs that resist rubric-based assessment (Jones & Alcock, 2014), and our work thus parallels earlier comparative judgement studies on conceptual understanding (Bisson et al., 2016), problem-solving ability (Jones & Inglis, 2015), mathematical beauty (Sa et al., 2023), and conceptions of proof (Davies et al., 2021).

Collecting Conditionals

To collect mathematical conditionals that potentially varied in believability, we asked eight mathematics education researchers to generate five conditionals each, according to the criteria that these should:

- Cover a range of mathematical topics;
- Have plausibly related antecedent and consequent;
- Not be obviously false;
- Not use additional connectives (‘not’, ‘and’, ‘or’) in the antecedent or consequent;
- Vary in believability (where the most believable could be clearly true).

The researchers were also asked to state whether each of their conditionals was true or false, and to rank them from least to most believable. We intended that ranking would reinforce the requirement to vary believability.

From the resulting 40 conditionals, we removed seven that used phrasing too complex for a conditional inference task, and a further six that had true converses (this would confound analysis for a conditional inference task because it would make DA and AC inferences valid due to semantic content). We then included one of the true converses, added a conditional so that there would be more than one involving statistics, and made our collection up to 40 by adapting conditionals that have been studied or discussed in the mathematics education literature (Alcock, 2013; Alcock & Attridge, 2023; Dawkins & Norton, 2022; Durand-Guerrier, 2003; Houston, 2009; Hoyles & Küchemann, 2002; Selden & Selden, 2003). We standardised by phrasing each conditional with a comma followed by ‘then’ and by removing excess words like ‘must’ and ‘also’ in the consequent. In cases where it was natural to indicate quantification scope by naming an object type (e.g., ‘If a quadrilateral is cyclic, then it is convex’), we rephrased so that potential conditional inference items could be written to mirror everyday causal items (compare ‘If quadrilateral Q is cyclic, then it is convex’ with ‘If John studies hard, then he does well on the test’). Finally, the conditionals were formatted using LaTeX in a large font for side-by-side on-screen comparison.

Participants and Comparative Judgement Data Collection

We asked our eight researcher participants to complete the comparative judgement exercise using the online platform nomoremarking.com. They were presented with pairs of conditionals and asked to ‘choose which is more immediately believable’ by clicking ‘left’ or ‘right’. Each participant was asked to make 50 judgements, providing 400 judgements in total, so that on average each conditional was judged 20 times (each judgement involves two conditionals). We chose this number to balance the need to obtain reliable scores with avoiding judge fatigue (Verhavert et al., 2019). Some judges completed more than 50 judgements but, in line with our pre-registered analysis (https://aspredicted.org/ts39g.pdf), we used the first 50 in each case. The 400 pairwise judgements were fitted to the Bradley-Terry Model to generate a z-transformed parameter estimate for each conditional.

RESULTS

Reliability of Believability Scores

Consistently with guidance from Davies and Jones (2023), we report two measures of reliability: Scale Separation Reliability (SSR), and Inter-Rater Reliability (IRR). SRR is considered less robust but facilitates greater comparability with the literature; IRR is more robust but requires more data and is less often reported. SRR is often seen as analogous to Cronbach's Alpha and is interpreted using the same > 0.7 threshold. From our 400 judgements, we calculated SSR = 0.84. We calculated Inter-Rater Reliability by splitting the judges randomly into two groups, computing believability scores using the judgements from each group, then computing the Pearson correlation coefficient between the two lists; we repeated this process 100 times and took the median coefficient. IRR is often lower than SSR (Verhavert et al. 2019) and appropriate IRR thresholds remain an open discussion. However, for our data this caused no concern: we found IRR = 0.73. Together, these two measures mean that comparative judgement produced reliable scores. Despite the bivalent logic of mathematics, believability can be treated as a meaningfully shared construct that varies continuously.

Believability and Truth

Figure 1 and Table 1 show the believability scores listed from most to least believable, colour coded by the conditionals' truth values (false in grey). These results confirm our hypothesis that truth and believability would align imperfectly. True conditionals (N = 23) received higher scores (M = 1.02, SD = 1.67) than false conditionals (N = 17, M = −1.38, SD = 0.957); this difference was statistically significant, $t(38) = 5.308$, $p < .001$. However, there was considerable interleaving in the middle score range, and two true conditionals ranked within the bottom ten for believability.

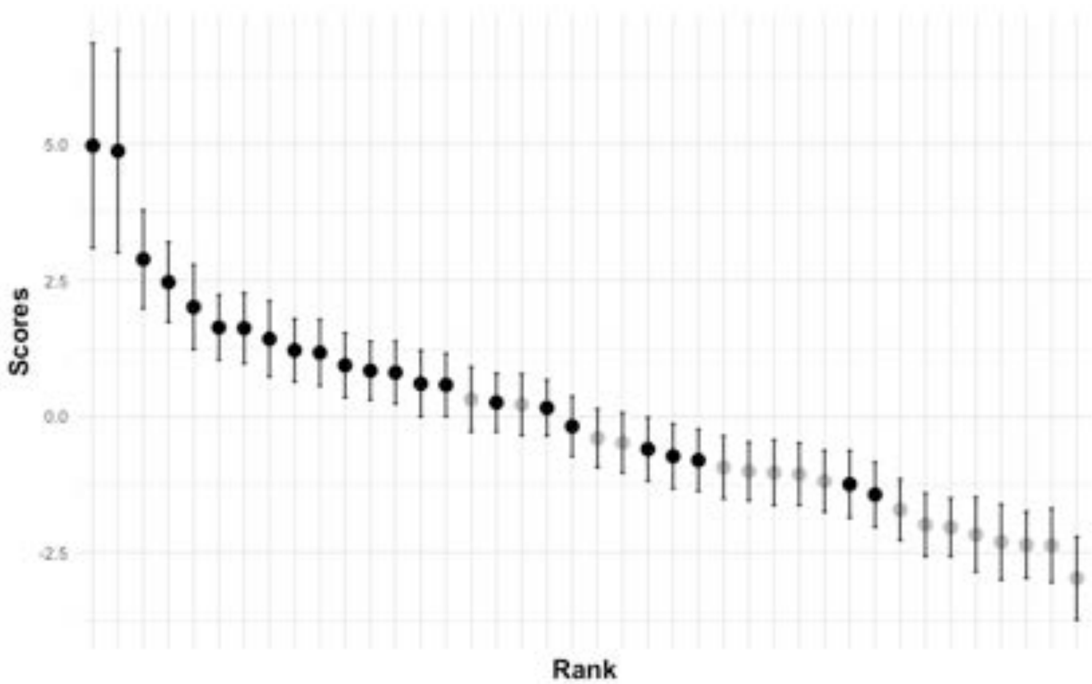

Figure 1: Believability scores from most to least believable; false conditionals in grey.

Conditional	Rank	Score
If $x < 2$, then $x < 5$.	1	4.97
If n is a multiple of 6, then n is a multiple of 3.	2	4.87
If line L is tangent to circle C, then L is perpendicular to a radius of C.	3	2.89
If n is a multiple of 4, then n^2 is a multiple of 4.	4	2.47
If C is a circle, then its width is the same when measured in any direction.	5	2.01
If polygon P is a square, then it is a rhombus.	6	1.64
If $a = b$, then $an = bn$.	7	1.62
If $x = \sqrt{y}$, then $x^2 = y$.	8	1.43
If $x = -4$, then $x^2 + x - 12 = 0$.	9	1.21
If X is a circle, then X is an ellipse.	10	1.17
If circle C and square S have the same perimeter, then C has bigger area than S.	11	0.94
If polygon P is a rhombus, then it has perpendicular diagonals.	12	0.84
If n is the product of two consecutive integers, then n is even.	13	0.81
If $x < 0$ then $x^3 < x^2$.	14	0.60
If $x^2 = y^2$, then $xy = yx$.	15	0.58
If fraction x has denominator 7, then it is equivalent to a non-terminating decimal.	16	0.32
If quadrilateral Q is cyclic, then it is convex.	17	0.26
If x is an integer, then $x^2 > x$.	18	0.22
If $x - 12{,}345 = 0.67$, then $x > -12{,}345.67$.	19	0.16
If polygon P is a rectangle, then every line through its centre cuts its area in half.	20	-0.18
If n is the sum of four consecutive numbers, then n is a multiple of 4.	21	-0.40
If line L is tangent to curve C, then L intersects C at only one point.	22	-0.48
If $x = 3$, then $2(x - 3) = 5x - 3(x + 2)$.	23	-0.60
If the product of two whole numbers is odd, then their sum is even.	24	-0.73
If rectangle R has area 10cm^2, then its perimeter is greater than 10cm.	25	-0.80
If $a > b$, then $a^2 > b^2$.	26	-0.94
If $x < 3$, then $1/x > 1/3$.	27	-1.01
If $a > b$, then $ac > bc$.	28	-1.03
If equation E is quadratic, then it has exactly two roots.	29	-1.06
If n is prime, then $n + 1$ is even.	30	-1.19
If quadrilateral Q has a reflex angle, then it will tesselate.	31	-1.24
If $\sin x > 0$, then $\cos x < 1$.	32	-1.44
If n is a multiple of 13, then it has an even number of factors.	33	-1.71
If x is positive, then $\tan x > \sin x$.	34	-1.99
If the side lengths of rectangle R are doubled, then its area is doubled.	35	-2.03
If function f is polynomial, then f has a real root.	36	-2.17
If the mean of dataset D is greater than 100, then the median is greater than 100.	37	-2.30
If the mean of dataset D is 7, then the median is 7.	38	-2.36
If $a = 42$, then $a \times b > 42$.	39	-2.37
If composite number c ends in a 3, then it is a multiple of 3.	40	-2.97

Table 1: Conditionals, ranks and believability scores; false conditionals in grey.

DISCUSSION

Our results show that believability of mathematical conditionals can be measured reliably using comparative judgement. This means that believability can be added to the list of important but hard-to-measure constructs for which comparative judgement has proved useful in mathematics education. The structure of our stimuli also provides a novel approach to assessing understanding of relationships between concepts. The extent to which people find a conditional believable tells us about the extent to which they consider its antecedent and consequent linked.

Our results also show that believability is imperfectly aligned with truth. As would be expected, true conditionals received significantly higher believability scores. But believability did not perfectly predict truth: in the middle range of scores, there was intermixing of true and false conditionals. This is in line with our theoretical suggestion that the relative inaccessibility of some examples in individuals' concept images or personal example spaces could make a conditional seem somewhat believable when it is false or somewhat unbelievable when it is true.

For our broader purpose, these results provide a good basis for designing a conditional inference task: the spread of true conditionals across believability scores means that we can select conditionals that are all true but that vary considerably in believability. However, we acknowledge two points requiring further investigation. First, these results could be specific to experts, so we will repeat the comparative judgement with undergraduates to ascertain whether they have similar views. Second, and more importantly from an educational perspective, it could be that people *can* judge relative believability when explicitly asked to do so, but that they are able to implement bivalent logic in mathematics so that believability does not affect their inferences. We will test this possibility by designing and testing conditional inference task (and will be ready to discuss the outcomes at the conference).

In the longer term, we aim to bridge a gap between research in mathematics education and in cognitive psychology. In undergraduate mathematics education, with its focus on proof, students must learn to distinguish valid from invalid inferences. We know that abstract conditional inference is predicted by mathematical study (Attridge et al., 2015) and predicts mathematical performance (Alcock & Attridge, 2023), but there has been no systematic study of inference from meaningful mathematical conditionals. In cognitive psychology, there is extensive study of inference from meaningful conditionals, but these use everyday causal content – again, there has been no study with mathematical content. Our work will open up the possibility of comparing reasoning in mathematics with reasoning in everyday settings. This will interest both mathematics educators who wish to leverage everyday reasoning to support mathematical development, and psychologists who wish to understand how conditional inference operates among experts in a field that uses bivalent logic.

NOTES

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Agential participation in resource-rich problem-solving activity

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In this paper, I analyse the responses of eight undergraduate students who were asked to solve a mathematical problem on their own time and with no restrictions in the used resources, and to reflect on their approaches. I consider this as a resource-rich mathematical problem-solving situation and I draw on the commognitive framework (Sfard, 2008) to capture the complexity of students' agentive participation, i.e. specific discursive practices within problem-solving situations. The analysis proposes six components – task setter; task; task solver; resources and their use; institutions; and, mathematical epistemologies – shaping what students' agential participation might be. I present accounts from students' work and discuss the analytical potential of the six components in studies of students' problem-solving practices with digital resources.

Keywords: Digital and other resources in university mathematics education, teachers' and students' practices at university level, agentive participation, problem-solving, commognition.

INTRODUCTION

Researchers and university mathematics teachers are more and more attentive on the availability of digital resources (e.g., search machines, Q&A and AI platforms, applications) and the impact such availability has on studying and teaching routines (e.g., Gueudet & Pepin, 2018; Lyakhova, 2023). So far, research findings have been advocating for benefits in using digital environments that afford opportunities for mathematical learning (ibid). However, recently, technological advances seem being integrated into study practices in ways that cannot be pre-determined (Biza, 2023). For example, students can solve a problem by searching online or by prompting an AI platform (ibid). Also, interaction with digital resources generates complex situations where those resources determine students' actions and teachers' expectations (Gueudet & Pepin, 2018). As a result, questions are raised whether and how seeking help from digital resources affects students' learning experiences and their ability to control those experiences – this is what I call students' *agential participation*, i.e., students' ability to have a transforming effect or impact on their learning experiences (inspired by McNay, 2015). In this work, I draw on the *commognitive framework* (Sfard, 2008) to analyse agential participation in eight undergraduate students' responses to a mathematical problem on their own time and with no restrictions in the used resources – this is what I call *resource-rich problem-solving activity* – and their reflection on those responses. In the next sections, I frame the conceptualisation of *resources* and *agential participation* in this work before presenting the context and the methods of the study. Findings are presented in a summary of students' mathematical responses and used resources as well as in accounts from the students' routines before discussing *agential actions* within those routines.

RESOURCES AND AGENTIAL PARTICIPATION

Digital and analogue resources

For this paper, in which resources are not pre-determined, the term 'resource' should stay quite open to include digital and analogue materials students can draw on for their problem-solving activity. Those resources might be human or non-human and designed (or not designed) for educational purposes (Gueudet & Pepin, 2018; Pepin et al., 2017). Although covering a comprehensive list of all the potential resources is beyond the scope and the needs of this paper, some indicative examples of what students may use are: Digital Curriculum Resources (DCR) organised "in electronic formats that articulate a scope and sequence of curricular content" (Pepin et al., 2017, p. 647), e.g. e-textbooks or repositories of lessons (e.g. Khan Academy); instructional technology (e.g. applications designed for learning purposes such as GeoGebra, Desmos); technology for programming and/or modelling (e.g. MATLAB); AI platforms; Theorem Provers (e.g. LEAN); Q&A platforms (e.g. Mathematics StackExchange); online search machines (e.g. Google); Social Media and Video platforms (e.g. Instagram, YouTube); online encyclopaedias (e.g. Wikipedia) or specialised knowledge and computational platforms (e.g. Wolfram Alpha). In the examples above, I would add analogue resources, such as paper-based textbooks, books and encyclopaedias, lecture notes, as well as communications with humans (e.g., peers, friends, teachers or family members).

Research has indicated that undergraduate students use a range of resources within and beyond their course materials (e.g. Anastasakis et al. 2018; Pepin & Kock, 2021). Often, the digital environment is pre-determined and its impact on students' engagement is studied in the context of the implementation and the affordances of this environment, regardless of whether those affordances are taken on board or not. However, when students draw on resources outside the course their actions are unpredictable. Pepin and Kock (2021) studied student use of resources in an open-ended environment of a challenge-based course. They observed that when the context becomes open with no pre-determined resources, the orchestration of resources becomes less linear (and less aligned to the course requirements). In another example, Biza (2023) asked undergraduate mathematics students to solve a problem without any restriction on the used resources. The use of digital resources in this study not only provided useful information, it, also, provided answers to the problem and facilitated hypothesis building or execution of time-consuming procedures. However, some of the students found the answer to the problem online and then, confirmed the answer with the conditions of the problem. Thus, instead of engaging with exploration routines, students engaged with searching online and then confirmatory routines.

Agential participation in resource-rich problem-solving situations

The theoretical perspective of this work is discursive and draws on the commognitive framework (Sfard, 2008). According to the commognitive framework (Sfard, 2008) the learning of mathematics is seen as the process of individualizing mathematical discourses established in a community and recognised by *used words*, *endorsed narrative*, *routines*, and *visual mediators*. In the context of this study learning is the process through which students gradually become capable of employing mathematical routines they encounter in their university studies *agentively* and *productively* (Nachlieli & Tabach, 2022). In this work, I endorse Lavie et al.'s (2019) contextualization of a routine within *task situations*, namely "any setting in which a person considers herself bound to act—to do something" (p. 159) and to a particular person who performs this routine. Very often, a task affords the routines that are expected by the problem setter. However, solvers may act in a way that does not align with setter's expectations. For example, a task in Linear Algebra might be designed to invite routines introduced in (and endorsed by) a Linear Algebra course. However, the students may engage with unexpected routines (e.g. by solving the task geometrically) or provide a response that is not endorsed in the context of the course (e.g. by providing a visual argument). So, what a routine is within a task situation might be seen differently by the task-setter and the task-solver. Such conflicts may question agentivity in application of problem-solving routines. Very often in educational setting, agentivity is seen within a "learning-teaching agreement" in which plays the role of the "ultimate substantiator" who "leads" the discourse (Sfard, 2008, p. 284). However, such "learning-teaching agreement" in the task-task setter-task solver triad might be challenged when other 'actors', such as resources, are coming into play. First, because digital resources provide opportunities that are not necessarily part of a taught mathematical discourses. Second, resources that are not part of the curriculum materials may endorse different narratives of the mathematical objects under consideration. For example, a mathematical definition that can be found online might not agree with the definition used in the lesson, not because one or the other definition is more accurate, but because definitions are narratives endorsed by certain communities in a certain context. So, when non-institutionalised digital resources are involved, the boundaries of the leading discourses are blurred – in practice, several, even conflicting discourses may lead the mathematical communication. In this work, I aim to capture what students' *agential participation* might be in the complexity of their engagement with resource-rich problem-solving situations by considering six components: the task setter; the task; the task solver; resources and their use; institutions in which the phenomenon takes place; and, mathematical epistemologies. I hypothesise that these six components, human and non-human, interact within the task situation, and, through the analysis of students' responses I examine the following questions: *What interactions are observed between task setter, task, task solver, used resources, institutions and mathematical epistemologies within a resource-rich problem-solving task situation*?

CONTEXT, PARTICIPANTS, THE PROBLEM AND METHODS

I discuss the work of eight students who attended a Mathematics Education course for finalist (Year 3) students of Bachelor of Science (BSc in Mathematics and BSc in Physics) programmes in a research-intensive university in the UK and is led by me since 2016. The aim of the course (entitled *The Learning and Teaching of Mathematics*) is to introduce students to the study of the teaching and learning of mathematics (see details in Biza, 2023; Biza & Nardi, 2023). Problem-Solving is one of the topics discussed in the sessions. The course is assessed through a *Portfolio of Learning Outcomes* that involves, amongst others, solving a mathematical problem (P1) and reflecting on problem-solving approaches (P2), see Figure 1. Here, I analyse responses from eight students from a BSc in Mathematics (Will, Terry, Simon, Sophie, Linda, Irfa) and a BSc in Physics (Hadid and George).

P1 In each one of (1) to (3) below, you are asked to identify a **NOT piecewise real function** that satisfies certain conditions. In each case, you will provide the function you have identified, and you will justify why this function satisfies the required conditions.

i. Find a function $f: R \to R$ if you know that the graph of f goes through the following points:

x	1	2	3	4
$f(x)$	2	4	6	8

ii. Find a function $g: R \to R$ if you know that:

(a) the graph of g goes through the following points:

x	1	2	3	4
$g(x)$	2	4	6	8

(b) g is a polynomial function with degree at least 5

(c) $\lim_{x \to +\infty} g(x) = -\infty$

iii. Find a function $h: R \to R$ if you know that:

(a) the graph of h goes through the following points:

x	1	2	3	4
$h(x)$	2	4	6	8

(b) the graph of h meets the x-axis infinite times.

Any mathematically correct and accurately justified response will receive full marks. Please present your solution to the problem in detail. In your investigation, you may consider using digital tools (e.g., computer or scientific calculators), software (e.g., Geogebra or other Dynamic Geometry Software, Desmos, Excel, MATLAB, etc.), or other digital resources. Also, you may seek help from others or from available online resources (e.g., online search, blogs, etc.). Please keep an honest and detailed record of the actions you have taken in order to solve the problem. In addition to your solution to the problem, you will attach your draft record of the actions you have followed. This record is not going to be marked but it will be essential to my understanding of your reflection on the steps you followed in the solution in P2. It does not need to be particularly polished or tidy; a scanned version of your handwriting suffices.

P1.1. My response to the problem

P1.2. Record of actions I have taken to solve the problem (*not to be marked)*

P2 Use Mathematics Education research literature to reflect on the actions you have taken to solve the problem in P1. Please note that an honest and detailed account of your problem-solving actions will give you more opportunities to provide a realistic, rich, specific and consistent reflection.

Figure 1: Problem Solving (P1) and Reflection (P2) Task

The Problem Solving (P1) and Reflection (P2) are seen in the analysis as one task (see Figure 1). In P1, students are asked to solve a mathematical problem (P1.1) and to provide their working on the problem (P1.2). In P2, students are asked to write their reflection on the solving process they have followed in the light of relevant research literature. The working on the problem (P1.2) is not marked but used as a reference for the contextualization of students' reflection in P2. The choice of the problem in P1 was strongly influenced by my experience with responses from previous cohorts of students (see the example of the divisibility task in Biza, 2023). For this reason, in the case I discuss here, I chose a problem that affords a variation of approaches and can be responded in various ways. I drew on the literature on *example generation* (e.g., Iannone et al. 2011) to identify an example generation problem that did not have only one 'right' response. The problem in Figure 1 was inspired by the work of Zazkis and Marmur (2021) with similar tasks with teacher. The mathematical content of the problem was related to school level Algebra (functions, function graphs, trigonometric functions) as I did not want students' (lack of) proficiency with advanced mathematics to be critical to (and potentially discourage) their explorations. As the task description states: "*[a]ny mathematically correct and accurately justified response will receive full* marks" (Figure 1). Responses to the problem were expected to be *analytic* examples of functions (*analytic example* for simplicity), namely variations of the following:

- P1.i: $f(x)=2x$
- P1.ii: $g(x)=-x(x-1)(x-2)(x-3)(x-4)A(x)+2x$, where $A(x)$ can be any polynomial function
- P1.iii: $h(x)= 2x\cos(2\pi x)+B(x)$ or $2x+\sin(2\pi x)C(x)$ or $(x-1)(x-2)(x-3)(x-4)\cos x +2x$

Students had the chance to seek help by asking peers, returning to textbooks, searching online or experimenting with applications. I wanted them to take the opportunity to reflect on their approaches, thus the description "*[i]n your investigation ... to solve the problem*" (Figure 1). Discussion around the role of digital resources in mathematical activities was part of the topics covered by the module. Thus, I did expect some influence of those resources on students' example generation and on their reflections.

FINDINGS

In this section, first, I offer a descriptive summary of students' proposed examples (Table 1) and used resources (Table 2). Then, I present accounts from students' exploration and substantiation routines.

A descriptive summary of students' proposed examples and used resources

All students responded that a function that satisfies the conditions in P1.i is $f(x)=2x$. There was a variation in students' responses to P1.ii and P1.iii though (Table 1). In each of these questions, four students proposed *analytic examples,* while three students proposed a function, an approximation of which meets the conditions with an estimation error (*approximation examples,* for simplicity).

One student (Hadid) proposed a function that does not meet the limit condition in P1.ii and proposed the function $axsin(bx)$ for large values of a and b without specifying those values in P1.iii.

A summary of the resources, digital or analogue, that were identified in students' responses to P1 and P2 (Table 2) indicate that all students used Desmos and almost all of them (except Terry) did some sort of online search.

	P1.ii	P1.iii
Analytic examples of functions (analytic examples)	Linda: $g(x) = -x^5 + 10x^4 - 35x^3 + 50x^2 - 22x$ Sophie: $g(x) = -\frac{1}{15}x^5 + \frac{13}{12}x^4 - \frac{13}{2}x^3 + \frac{215}{12}x^2 - \frac{613}{30}x + 10$ Simon: $g(x) = -x^5 + 10x^4 - 35x^3 + 50x^2 - 22x$ Irfa: $g(x) = -x^5 + 15x^4 - 85x^3 + 225x^2 - 272x + 120$	Linda: $h(x) = 2xcos(2\pi x)$ Terry: $h(x) = 2xsin\big(6\pi(x)\big) + 2x$ Simon: $h(x) = 2xsin(x^5 - 10x^4 + 35x^3 - 50x^2 + 24x) + 2x$ Irfa: $h(x) = -(\sin(x))(x-1)(x-2)(x-3)(x-4)(x-5) + 2x$
Examples of function an approximation of which meets the conditions with estimation error (approximation examples)	Terry: $g(x) = 2x - \frac{x^5}{10000000}$ Will:$g(x) = 8.881784e^{-16} + 2x + 3.058904e^{-15}x^2 - 9.077574e^{-16}x^3 + 1.257824e^{-16}x^4 - 6.616221e^{-18}x^5$ George: $g(x) = -0.001x^5 + 2x$	Sophie: $h(x) = 450\left(sin\left(\frac{x}{225}\right)\right)$ Will:$h(x) = 100sin(0.02002(x-1)) + 2$ George:$h(x) = 5.00242 - 35.3228sin(0.0566x + 2.99983)$
Other	Hadid: $g(x) = (x^2 - x)(x^2 - 2x)(x^2 - 3x)(x^2 - 4x) + 2x$	Hadid: $h(x) = axsin(bx)$, "for any large numbers *a* and *b*"

Table 1: Proposed examples of functions in P1.ii and P1.iii

Resource	Student
Desmos (Online mathematics application, instructional technology and DCR)	Terry, Linda, Sophie, Will, Simon, Irfa, Hadid, George
Online Search	Linda, Sophie, Will, Simon, Irfa, Hadid, George
Mathematics StackExchange (Q&A platform)	Linda
A-level maths website (DCR)	Linda
YouTube video	Sophie
Reduced Row Echelon Form (RREF) Calculator (Online Application) not named by the student, for example	Sophie
"MyCurveFit" Online curve fitting application (Online Application)	Will
Socratic Q&A (Q&A Blog Platform)	Simon
Implementing the Mathematical Practice Standards – Education Development Center (DCR)	Irfa
Symbolab (Online Application)	Irfa, George
BBC Bitesize (DCR)	Hadid
Calculator	Simon
Excel	George
Other "Playing around with webtools" (not specific)	George
Peer support (e.g. asking a friend)	Linda, Irfa
Paper and pencil graphs	Linda, George

Table 2: Digital (and analogue) resources identified in students' responses to P1 and P2

Accounts of student responses and reflections

Substantiation routines varied across student responses. Hadid, for example, chose the "form of a trigonometric function" for P1.iii and he proposed the function "$axsin(bx)$" with "a and $b \to \infty$", $h(x) \to$ the required co-ordinates. [next line] $= \infty xsin(\infty x)$" (Figure 2a). His argument was justified with the graphs in Figures 2b/2c; next to Figure 2c he wrote: "Screenshots of Desmos, on the left is an example of how I visualised manipulating a trigonometric function into crossing the required co-ordinates in part 3". Hadid substantiated the choice of the function on the grounds of a graph and his manipulation of this graph on Desmos, although the perceptual experimentation on Desmos is accompanied by mathematical narrative: "[f]or any large number, *a* and *b* will increase the frequency of the wave" and notations (Figure 2).

	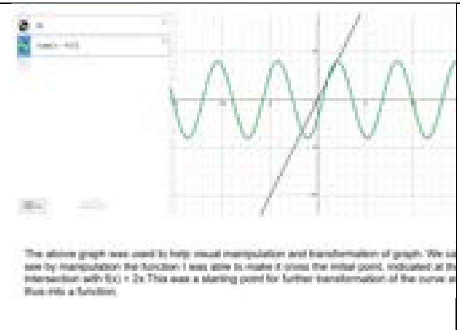	
Figure 2a: Response	***Figure 2b: First step***	***Figure 2c: Substantiation***

Figure 2: Hadid’s response to P1.iii

Simon, on the other hand, proposed an analytic example (Figure 3) while he had “a lot of playing round with Desmos”. He described a list of actions to: “verify answer with Desmos and by subbing numbers to the equation […] showing mathematically that this function works for the condition […] show with the graph and […] prove with numbers”. Even though Simon’s response to P2iii, was substantiated mostly with the graph on Desmos, it seems that he wanted to check himself what was provided by the “online tools”: “I only utilised online tools for operations I can do myself, so I could therefore check the workings shown by the calculator, preventing mistakes” (P2).

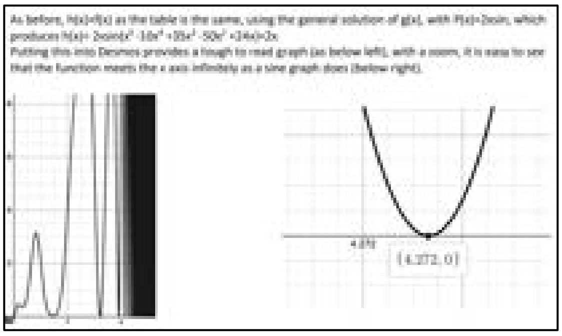

Figure 3: Simon’s response to P1.iii

Will and Irfa, both sought help online. Will used the “MyCurveFit” application to plot the points for P1.ii, and, then, to “specify the type of curve” of a 5^{th} order polynomial function that goes through the given points (Figure 4a). The function example he identified did not satisfy the limit to minus infinity condition, as he wrote in P2: “[…] I used my knowledge of limits to first see that the result I had gotten was incorrect as, in order for a limit of $x \rightarrow +\infty$ to cause a function $g(x) \rightarrow -\infty$, the term of the highest degree must have a negative coefficient which I did not get here”. To address this issue, he transformed the curve, still with the help of “MyCurveFit”, and ended up with an approximation example[1]. To confirm that the function is right, he plotted the graph on Desmos “in which you can zoom in to parts of a graph; so, doing this and plotting my points, I could see that the new equation I had gotten satisfied all the requirements”.

[1] g(1)= 2.00000095448; g(2)= 3.99999561036; g(3)= 5.99996881892; g(4)= 7.99988364555

Irfa, on the other hand, proposed the analytic example we can see in Figure 4b and Table 2. In her reflection in P2, she explains how she ended up with this approach. Firstly, she established the appropriate narratives about polynomials with a peer, then, she experimented with Desmos, but "this wasn't an efficient way to solve the problem", and she "googled 'forming a polynomial to fit a table'" [her quotations], which led to the appropriate online resource that helped her to solve the problem.

Both Will and Irfa searched online and sought help from online applications. However, Will prompted the online application he found *to solve the problem* for him, while Irfa prompted the resources to *help her to find out how she would solve the problem*.

	
Figure 4a: Will's work	**Figure 4b: Irfa's work**

Figure 4: Will's and Irfa's work on P1.ii

Terry, Sophie and Linda used Desmos to trial and improve several cases; an approach that would not have been possible (or would have been much more difficult) with paper and pencil explorations. Terry, for example, reports that, first, he plotted $f(x) = 2x$ and $g(x)=x^5$; then, $g(x)=2x\text{-}ax^5$; and, then he worked out appropriate scaling of a towards the approximation example he proposed eventually (see Table 1).

Finally, George kept a diary on his working on the problem, although he was not asked to do so. It seems that the openness in the approaches to the problem confused George. For P1.ii and P1.iii he reported several attempts in a period of 17 days (my estimation) in which he tried several methods and a significant online search. For example, he wrote in his "3rd look" to P1.iii: "Why am I not using $\cos^2$. Had these thoughts while reading Pólya's strategies … This did not work! … Playing around with webtools … Back to desmos => Maybe sines aren't the only $g(x)$". Later, most likely on the same day, he attempted P1.ii by creating a set of simultaneous equations. He progressed with the solution without success though when he wrote: "I'm done for today! I've made <u>SO</u> little progress and I am <u>TIRED"</u> [his emphasis]. The following day he writes that "he learnt that there are no gen[eral] solu[tion]s for quintics!", and he adds "[t]here are solvable cases … surely I'm not expected to do that …" and he wonders "[i]s the purpose of this question to test my reflective skills in giving up?". Eventually, he proposed an approximation function in Table 1.

DISCUSSION

In this paper, I analysed the responses of eight undergraduate students to a re*source-rich mathematical problem-solving task situation*. I drew on from Lavie et al.'s (2019) task-performer contextualization of routines and studies students' exploration and substantiation routines with attention to six components: *task setter, task, task solver, used resources, institutions and mathematical epistemologies*. Several students generated approximation (instead of analytic) examples of functions. Also, many of the students used online applications to find their examples or to confirm that the examples they found are correct. As a task *setter*, myself, I sensed the fluidity of what "mathematically correct and accurately justified response" might be. Students' trust of graphical representations conflicted with my analytic deductive discourse. On the other hand, the *task*, its description and the context it was used, is not agentless: it becomes an actor that affords expectations and *institutional* constraints loaded by the setter and their *epistemologies*, defined by the context, interpreted by the solver, and, finally, evaluated by the setter in the light of students' responses. *Resources* are not agentless either, they become actors as well that come to fill the gap of uncertainty by indicating a direction to the solver, a direction that may converge to or diverge from setter's position(s). It is not only the resources that act, but also how those resources are invited to act by the *solver*. Irfa searched online with the intention to seek assistance that will help her to find out how to solve the problem, while for Will, his prompts were oriented toward finding and delivering a response. So, going back to students' learning experiences and their ability to control those experiences, it seems that Irfa's approach was more agential in comparison to Will's. Overall, what the *solver* does cannot be seen outside what the *setter* and the *task* ask them to do and what the available *resources* can offer. Also, it cannot be seen outside the *institutional* discourses, especially outside dominant *mathematical epistemologies*. In conclusion, I argue that students' agential participation in resource-rich problem-solving situations can better be seen as a whole of the *sextet* of components with potential for research as well teaching, such as assessment and learning materials design.

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Learning to Engage Students as Partners in Critically-Oriented Reform of Tertiary Mathematics

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This paper describes the efforts of a Networked Improvement Community (NIC) to engage students as partners toward critically transforming cultural and institutional structures impacting introductory tertiary mathematics courses. In Spring 2023, stakeholders within a mathematics department, which included faculty and students, examined multiple forms of data to develop action plans addressing inequities and detrimental experiences within their math program. These plans were developed in partnership with students who shared their voice and unique perspectives. In this paper, we focus on the ways in which students engaged in this work, highlighting both how the NIC prioritized student voice, while also noting the persistence of power dynamics that influenced their engagement.

Keywords: Teachers' and students' practices at university level, curricular and institutional issues concerning the teaching of mathematics at university level, critical transformation, students as partners

MOTIVATION AND OVERVIEW

Mathematics operates as a gatekeeper for students by upholding structural inequities present in society (Ellis et al., 2016; Martin et al., 2010; Vithal et al., 2024). While, globally, there have been several calls and initiatives to address inequities in education, and mathematics in particular, researchers have critiqued the discourse and approaches surrounding equity and mathematics. Scholars have argued that improvement efforts have primarily focused on supporting or aligning with the interests of the dominant culture and have at times inadvertently contributed to deficit views of students (e.g., discourse about achievement gaps), rather than fundamentally changing cultural systems of oppression (Gutiérrez, 2011; Hughes, 2022). Instead of appealing to the dominant culture, transformative efforts should understand equity as a systemic issue and focus efforts on changing these cultural systems of oppression (Gutiérrez, 2011; Hughes, 2022). An example of a structural inequity in mathematics is the use of "tracking" students into different ability groups in mathematics courses which perpetuates or even exacerbates disparities for women and students of color, and can contribute to cycles of poverty for under-resourced communities (Ansalone, 2001). A critical approach to improving equity requires a recognition that change needs to happen at a systems-level, and that such change is meaningless unless it involves those being oppressed. Initiatives to support equity can be oppressive if they do not center those voices of those who are marginalized, a notion that is captured in the colloquial verbiage "nothing about us, without us" (Sharma, 2020, p.193).

In the context of mathematics, truly critical, transformative changes should involve the voices of those most impacted by change efforts - students. Students are often recruited to support the Scholarship of Teaching and Learning by acting as representatives of a particular course for a limited duration - e.g., a semester or term (Healey et al., 2014). The focus of this study is on one empirically-rich and unique single case-study of a group of students and mathematics faculty engaging in efforts to improve introductory mathematics courses at their institution. In contrast with previous studies, the students in this group had not taken an introductory mathematics course at this institution and have participated for the entire duration of the improvement efforts. This study emerged from a larger research study investigating how Networked Improvement Communities (NIC) could engage in critical participatory action research to address issues of equity in introductory tertiary mathematics courses in the United States (e.g., precalculus, calculus). NICs, guided by specific measurable goals, make improvements by engaging in multiple cycles of designing and testing solutions to problems (Penuel et al., 2020).

A growing body of research points to the transformative potential of engaging students as partners with faculty to humanize mathematics education (Cook-Sather et al., 2023). Successful partnerships follow a principle of reciprocity, in which students and faculty are both positioned as having expertise which can be leveraged to improve education (Mercer-Mapstone et al., 2017). While these partnerships have the potential to challenge existing structures that uphold inequities in math, there are barriers to positioning students as partners in this work. As such we address the following research questions: (a) How did the Networked Improvement Community include students as partners? and (b) What was the impact of including students as partners in efforts to critically transform introductory mathematics programs at the institution?

CONCEPTUALIZING STUDENTS AS PARTNERS

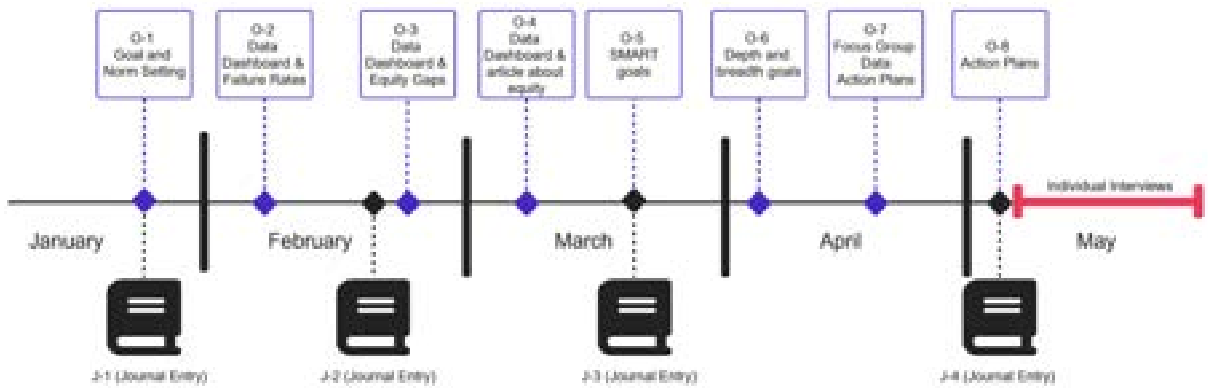
When working towards critical transformations via participatory action research, NICs engage in improvement cycles (observe, reflect, plan, act) that seek to empower stakeholders to improve their lives and communities through action. In Spring 2023, the NIC accomplished the observation and reflection phases while initiating the planning process. Throughout these phases, students were involved in every activity, actively participated in decision-making, and were responsible for specific parts of planning. Student voices were heard from within the NIC and through focus groups of students outside of the NIC who were influenced by the NIC's activities.

To better understand the roles students may take in such transformations, we consider Holen et al.'s (2021) framework of four different archetypes that capture varying partnerships between institutions and students. The four archetypes are: *Students as Apprentices, Students as Followers of Political Agendas, Students as Democratic Participants, and Students as Customers* (Holen et al., 2021). Partnerships in which students act as apprentices are characterized by relationships in which students and staff work towards the same goals, guided by the staff member. Usually these goals are related to academic-based research and self-development for both students and staff members. External pressure from government or other bodies can lead to students being cast as followers of a political agenda for institutions to receive financial support from external agencies. Students as democratic participants in a partnership represents students and staff members jointly working together to resolve obstacles involving compromise from both perspectives. The obstacles usually coalesce around the needs of students which in turn creates seats for students in stakeholder groups. Some student partnerships may form to satisfy or represent the needs of the "customers" at the university (in this case the students). As institutions compare their resources to other institutions' resources and the needs of the students, more opportunities for student feedback become available to lend a perspective lost within administration. Although this framework is intended to capture institutional partnership approaches, we argue that this framework can be adapted to study partnership approaches of NICs. In our context, the NIC's approach to including students as partners is influenced by the NIC's location within a mathematics department as well as within an external, federally funded NIC, both of which can exert external pressure influencing the character of these partnerships.

METHODS

This single case study (Yin, 2009) draws from data collected as part of the Achieving Critical Transformation in Undergraduate Programs of Mathematics project (ACT UP Math; NSF ECR# 2201486). ACT UP Math is examining the formation of NICs across three university mathematics departments addressing inequities in introductory math courses. Alpha University's NIC presented a unique case study because of their intentional recruitment of students. The NIC had eight members: five instructors (Angela, Caroline, Jeremy, Michelle, and Ruby) and three students (Chase, Chelsea, and Mallory). One student was a graduate student and two students were undergraduate students. The NIC leaders recruited students by posting flyers and sharing information about the project during classes and a department symposium. Final students were selected following an interview with the NIC leaders. The NIC met every other week for two hours from January-May 2023 and focused their meetings on using data to inform action plans that they could implement in the Fall 2023 semester. Alpha University's NIC created two action plans: 1) dismantling the placement system for introductory mathematics courses and 2) creating programming that connects students to the utility value of mathematics. The structure of the Spring 2023 semester is summarized in Figure 1.

Figure 1. Timeline of Spring 2023 NIC Activities and Data Collection



Data collected and analyzed include structured observational field notes of eight NIC meetings, semi-structured interviews of NIC members, and four reflexive journal entries completed by each NIC member. To analyze these data, the first three authors divided the observation notes, journal entries, and interview transcripts and open-coded the data for themes related to the experiences of students in the NIC. These three authors met to discuss the open codes and generated cross-cutting themes relevant to the NIC's inclusion of students. These themes were refined and confirmed by the remaining authors. In a second stage of analysis, we reviewed these themes through the lens of Holen et al.'s (2021) framework to describe the nature of the NIC's approach to student partnerships. Next, we share these themes with supporting evidence from the data. We use the notation "J-#", "I-#", and "O-#" to delineate between different iterations of referenced data (e.g., J-2 is the second journal entry).

FINDINGS

Our findings are organized in two sections. We first describe how the activities of the NIC engaged students as partners, using Holen et al.'s (2021) framework as a guide for describing variations in student involvement. We then synthesize reflections from NIC members to describe the impacts of students' involvement in the NIC.

Theme 1: Activities & Inclusion of Students in the NIC

Students were recruited to participate in the NIC with the intention that students would be the key stakeholders necessary to make critical transformation in Alpha University's mathematics department. Students were unintentionally and intentionally positioned in the NIC in two distinct ways as *apprentices* and *democratic participants.* Additionally, students were positioned as *customers* through focus groups conducted by the NIC. Although faculty NIC members valued student voice and expressed the necessity of having students in the NIC, there were moments where the partnership felt primarily instructor-led, encouraging less equitable partnerships.

External "Pressures" and the Incorporation of Students

Caroline, a NIC leader, shared that “it was really important to us [the NIC leaders] to have a significant student presence” in the NIC (I-1). Caroline’s intentions to include students in the NIC stemmed, in part, from their department’s goals to address one of the “weakest areas” of their change efforts to improve math courses, that is, the lack of student voice in “policy and conversations” (I-1). Throughout the semester, student voices were heard during full-group NIC discussions and smaller group conversations. As the three student NIC members became more comfortable sharing, faculty members elevated students’ voices in larger group settings. For instance, Jeremy, a faculty member, was paired with a student, Chase, during a NIC activity. During the activity, Jeremy provided space for Chase to share their thoughts on defining a high-value mathematics program, and publicly applauded Chase for their insights (O-5). This inclusion of students’ input remained central to the structure of the NIC meetings, as Caroline reflected:

> I think there is…a lot of intention put towards trying to make it equitable, and I hear the other faculty asking students explicitly, “what do you think about this?” And when we report out, we often ask the person with the least amount of power - in this particular department dynamic - to report out on behalf of the team. (I-1)

Efforts to Position Students as Democratic Participants

Early on in the semester, several faculty members reflected on how students were engaging in the NIC, noticing a tendency for students to be more reserved in their interactions than faculty. However, faculty seemed eager to mitigate imbalances in power felt by students during meetings. One member wrote that students “are a bit more shy to answer questions at times, but I think it is because they are used to [being] students. I can see that the people on the tenure track are very eager to have everyone participate” (J-2). Indeed, one of the NIC leaders wrote:

> I worry a lot about the power dynamic between faculty (of different ranks/positions) and students in the NIC. I hope we do a good job of making everyone feel as though their voice is valued, but I struggle with ‘leading’ in a way that doesn’t feel prescriptive or reinforce the existing power dynamics. (J-2)

Another faculty member echoed this worry, writing that they felt “self-conscious” when “partnered with a student” (J-2). For students, the intentions and actions of the NIC leaders to create a welcoming and inclusive space were clear, yet they still felt anxiety about interacting with faculty. Mallory, a student, described feeling “a little bit nervous” (I-1) to engage in meetings, but also recognized the efforts of the NIC leaders to mitigate power dynamics. In observations, the research team noted how faculty member Angela attempted to reduce the power differential between themselves and Mallory, by removing the formality of titles, “you can call me Angela in this space” (O-3). Faculty members continued to attenuate power dynamics by creating student-faculty pairings during initial NIC activities such as brainstorming goals (O-1) and discussing an article on equity in mathematics education (O-4).

While the NIC valued student perspectives and input, the influence that student NIC members had over the decision-making process and the type of partnership students engaged in varied. Leaders of the NIC strived to position student NIC members in the same ways as other members. This positioning was apparent to the student NIC members; for example, Chelsea noted that NIC members "all kind of have similar roles" and went on to describe how they view their own role within the NIC as being similar to others' roles: "We're just really providing our input and doing our best to come up with plans, and…I feel like I'm in that, too" (I-1). The recognition that student and faculty NIC members had similar roles supports the idea that students were successfully positioned as democratic participants throughout the semester.

The similarity in student and faculty NIC member roles continued into decision making. NIC members all participated in decision-making for the NIC by voting individually on major decisions. Since each person's vote carried the same weight, faculty described the system as "democratic" and "pretty equitable." While there was some risk that student votes would be overpowered by faculty, since there were simply more faculty in the NIC than students, students still felt that the voting system was fair. Chase believed that all NIC members held "very similar views as to what was important" (I-1) when it came to voting, suggesting that all members of the NIC had sufficient opportunity to make their voices heard during the voting process.

Students Instead Positioned as Apprentices

However, there was a tension between recognizing students as experts in their own experiences and viewing them as novices within the structures of educational systems. There were numerous moments where student NIC members would reference personal experiences that would spark conversation and be incorporated into the process of the group choosing topics of conversation. Yet, these conversations often continued amongst faculty with limited student interjection, minimizing the role that students played in initiating the discussion. To illustrate the tension, we offer the example of Chelsea, a student, bringing up a concern about having "such a large range of ability in the [math] class," where they noticed there is a divide in academic preparation that they personally experienced. Chelsea suggested being transparent with students, which prompted a discussion among three faculty members, who continued the conversation without additional student comment until the NIC moved on to another activity (O-5). Ultimately, the opportunity for Chelsea to be positioned as an expert and *equal partner* within the NIC was overshadowed by the subsequent discussion that neglected Chelsea's voice. Chelsea was instead positioned as an apprentice, following the lead and goals of the faculty members. Furthermore, although NIC leaders intended to include students as equal partners in the NIC, one NIC member asserted that "by virtue of the fact that faculty are planning and facilitating the NIC meetings, there is a power imbalance."

The power imbalance translated into some of the activities the NIC participated in, for example the creation of meeting norms. The NIC leaders utilized sticky notes as an anonymous way for NIC members to brainstorm non-negotiable norms for the group (O-1). NIC members were given space to reflect on the collaborative pile of sticky notes and were later asked to read out norms that resonated with them. NIC members were instructed to sort the sticky notes into general themes of meeting norms which prompted discussion and movement of the sticky notes. The majority of faculty NIC members actively engaged in moving sticky notes while one student, Mallory, attempted to move two sticky notes which were immediately relocated by a faculty NIC member. This action caused Mallory to physically disengage from the activity. The activity was intended to position student and faculty NIC members equally, recognizing that each individual would have space to share their opinion anonymously and work collaboratively to finalize a set of norms. However, the faculty (perhaps unintentionally) positioned Mallory as an *apprentice* or even an *onlooker* through this action. The inadvertent positioning of students as apprentices took place in multiple other instances as well. When exploring institutional dashboards from Alpha University, student NIC members were not privy to the dashboards themselves based on access criteria (O-3). Students' inability to access the data from their own accounts prompted the pairing of students with a faculty NIC member instead of individually exploring. Thus, outside systems prevented the NIC from successfully positioning students as *equal partners* and *democratic participants*, instead students were positioned as *apprentices* in this activity.

Positioning Students as Customers

The NIC also engaged students outside of the NIC through focus groups. Students in these focus groups were prompted to share their insights into introductory mathematics courses, including their experiences being placed in such courses. NIC members reflected on this experience in a journal entry referencing direct student experiences from the focus group: "Students would rather know which math topics they would use in their other classes rather than future career choices," "Students who are not able to register on time are automatically placed in the lowest level of math (even if they should have been placed higher)," and "even checking in [with students] can feel alienating if the student feels like they are the only one [who doesn't understand] (when we know they aren't!)" (J-4). However, some faculty members in the NIC expressed hesitancy regarding the value of this type of inclusion of students, given that the focus groups were small. In their interview, one NIC faculty member said, "I don't feel that confident yet about the data that's coming out of the focus groups because we get three students or something showing up." This suggests that some faculty only value contributions from students positioned as customers if those students are part of a larger group.

Theme 2: Impact of the Inclusion of Students in the NIC & What Assets Students Brought to Inform Critical Transformations

In this section, we call attention to how efforts to position students as democratic participants supported the critical change efforts within the NIC. We find that student members of the NIC bring their own critical perspectives to NIC activities, influencing the direction of the NIC; and, in turn, the NIC at-large works to provide a space where students can grow as change agents. The NIC also engaged a small number of students who are not NIC members, drawing on their firsthand experiences with introductory math courses to inform the NIC's action plans.

During NIC meetings, students used their own experiences as students to add depth and nuance to discussions of equity. Early in the Spring 2023 semester, the NIC explored data dashboards including course pass rates and equity gaps (down to a section level) and reflected on their noticings as a group. While everyone in the group gave careful attention to the data, student NIC members in particular pushed the NIC to consider anti-deficit perspectives while making sense of courses with high failure rates: Mallory drew on their personal experiences in the classroom to point out that not all students mesh well with all instructors' personalities and teaching styles, while Chase questioned whether instructors of courses with high failure rates and large equity gaps were communicating enough with students (O-2). Later in the semester, student contributions also shaped the NIC's goals and action plans. For example, Chase offered important insights to the NIC's goal to improve students' relationships with math by explaining how they use skills from their math classes in other academic pursuits, and have also taken valuable "life lessons" away from their experiences in mathematics (O-5). This influence from students is seen in part of the group's final action plan to use social media to feature how people in varying fields use mathematics in their lives.

Similarly, Mallory shared their experiences of mathematics placement at a previous institution, supporting the NIC in understanding the need for more accessible and transparent placement structures for students. In an interview, they described how their perspective on placement was taken up in NIC meetings: "My placement into [previous institution] was horrible. I didn't know how to go about redoing my placement and getting to a higher level quicker, which is something that we talked about and wanted to improve at [Alpha University] because a lot of people have the same issues." (I-1). By the end of the semester, the NIC had a goal of changing the placement system to be more equitable for students like Mallory.

The NIC's efforts to include students as partners by being attentive to the power dynamics between students and faculty were crucial in making students more comfortable expressing their thoughts and contributing to the efforts of the group. Indeed, sustained interactions with the NIC seem to be supporting students in feeling like equal partners. For example, early in the semester Chelsea felt hesitant to engage in the NIC, writing: "there are some current professors of mine in the group, some previous professors of mine in the group and so I want to make a good impression" (J-2). Chelsea was concerned with being "misunderstood," "say[ing] the wrong thing," or being seen as unhelpful (J-2). However, in later journal entries, Chelsea wrote that they felt more "confident now in speaking my mind" (J-3).

The NIC also solicited insights from students in introductory mathematics courses through a focus group. Three students attended and two were actively engaged for the allotted time, acknowledging their interest in connecting mathematics to their desired major, the difficulty registering for classes as an international student, and the differences between mathematics pedagogy across countries. In the concluding journal entry for the semester, multiple NIC members referenced student experiences shared during the focus group: “Students would rather know which math topics they would use in their other classes rather than future career choices,” “Students who are not able to register on time are automatically placed in the lowest level of math,” and “even checking in [with students] can feel alienating if the student feels like they are the only one [who doesn’t understand] (when we know they aren’t!)” (J-4). Two journal entries indicated how these student voices impacted NIC members’ teaching: “One thing that was mentioned, something I have not tried but am really looking forward to trying is giving students a way to ask questions anonymously…This way, no questions go unanswered due to students feeling afraid to ask or interrupt” and “This has made me think of what I could do differently in class and out [of class] to try and reach out and help a broader range of students.” The inclusion of student perspectives influenced both NIC activities and the actions of NIC members beyond their commitments to the NIC.

Faculty members in the NIC consistently affirmed the value of student contributions. Multiple faculty members shared how much they valued contributions from student NIC members: “I so appreciate the perspective of students in these conversations,” “...what has been so valuable for us in this group is the inclusion of students,” and “it’s informative to have the student voice and perspective in the group” (J-2, J-3).

CONCLUSION

Students within and outside of the NIC were, at various times in the semester, positioned as democratic participants, as apprentices, and as customers. In particular, the NIC endeavored to include student NIC members as democratic participants in NIC activities. However, outside power structures prevented students from fully viewing themselves as partners with faculty members even though faculty members and students held similar roles within the NIC. Student NIC members consistently viewed themselves as students, or apprentices, rather than colleagues due to external factors such as having NIC members as instructors and different institutional knowledge. Power dynamics imposed an important barrier in preventing students from viewing themselves as equals, despite continuous efforts by the NIC faculty to mitigate power structures. However, the NIC has made strides mitigating power dynamics over the semester and demonstrating the value of student voice. We are interested in how student voices will continue to be valued and how power dynamics will be addressed as the NIC implements its action plans.

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Role of example generation in implicit and explicit conjecturing tasks

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Conjecturing is crucial to mathematics and an activity in which it is believed mathematics learners of all ages should engage. It has been found that mathematicians productively generate examples when they are formulating conjectures. In this paper we explore whether this is also the case for undergraduate non-specialist mathematics students by means of an instrumental case study. We label conjecturing tasks as either explicit or implicit to distinguish between tasks which explicitly ask students to make a conjecture and those in which conjecturing is evoked implicitly, and we discuss the benefit of Comprehensive Example Generation (by which an example set is generated sequentially and systematically) in the context of such conjecturing tasks. The consequences of using a digital environment for such tasks are also discussed.

Keywords: conjecturing, example generation, task design, learning of calculus, students' practices at university level.

INTRODUCTION

Conjecturing and proving are fundamental practices of mathematicians. Indeed Bass (2015) describes most mathematical research as progressing from exploration and discovery, through conjecturing to formal proof. Yet conjecturing tends to be neglected in mathematics teaching and learning (Belnap & Parrott, 2013). This is despite the fact that many mathematics educators and researchers (e.g. Rasmussen et al., 2005) agree that learners of mathematics should be active participants in mathematical practices – including conjecturing, generalising and proving. In fact, ensuring learners' activities are synonymous with the practices of mathematicians is now a goal of national mathematics standards in many countries (Canadas et al., 2007).

Belnap and Parrott (2013) assert that the activity of conjecturing is crucial in mathematics but yet is accessible to learners due to its speculative nature and does not require the same rigour and precision as deductive work. Harel et al. (2022) agree that students should explore mathematical situations before being asked to construct proofs and contend that dynamic geometry environments have potential for *engaging* students in the process of conjecturing. Such environments allow students to focus on the core aspects of the mathematical phenomenon they are observing and draw them into a sense-making process. Advantages of using technology are also described by Breda and Dos Santos (2016) who explain how it allows information to be gathered and processed quickly and removes the burden of computation from the student, affording greater opportunities for experimentation and exploration.

Mathematicians use examples regularly and in several ways - for instance, to help them understand a statement or definition, refute a statement or generate an argument (Alcock, 2004). In particular, the contribution of example generation as an approach to a mathematician's fundamental activity of proof production has been acknowledged. Yet, despite the obvious role played by conjecture formulation in proving, Lynch et al. (2022) contend that little is known about the interplay between example exploration and conjecture formation; not alone for mathematicians but also for learners of mathematics. In fact, Furinghetti, Morselli and Antonini (2011) have cautioned that a focus on examples may "make students stick to the explorative stage and inhibit the need for generalization" (p.219) and that students often consider that checking examples constitutes a means of proof. The aim of this paper is two-fold. Firstly to distinguish between two types of conjecturing tasks for non-specialist undergraduate mathematics students: one in which students are explicitly asked to make a general conjecture, following their exploration of specific cases or examples; the other in which students are asked to generate examples of different phenomena but are expected to make a general conjecture as a result. We label the former as *explicit* conjecturing tasks and the latter as *implicit*. We present an example of each type of task (designed by the authors and used in first year Calculus modules) and the hypothetical learning process associated to each of them. Secondly, we put forward some evidence of the affordances for conjecturing provided by example generation tasks. In particular, we explore the research question: what is the role of example generation in facilitating the formulation of mathematical conjectures for undergraduate non-specialist mathematics students?

THEORETICAL FRAMEWORKS

Canadas et al. (2007) explain how different problems lead to different types of conjectures and propose a classification of conjecturing activity. They characterise conjectures as belonging to one of the following types: 1. Empirical induction from a finite number of cases; 2. Empirical induction from dynamic cases; 3. Analogy; 4. Abduction; 5. Perceptually based conjecturing. They identify the 'stages' of conjecturing associated with each type (e.g. observing cases, validating the conjecture) and explore how the context of a problem can encourage or discourage different types of conjecturing, cautioning that problem selection is important if specific types of conjectures are desired. We note that many of the problems described in Canadas et al. (2007) illustrating the different types of conjecturing discussed there are what we have termed 'implicit' conjecturing tasks.

The stages of conjecturing identified by Canadas et al. (2007) can be viewed as a 'hypothetical learning process' which together with learning goals and learning activities form a 'hypothetical learning trajectory' (HLT) for a student. Simon and Tzur (2004) recommend the use of HLTs in task design to ensure that sufficient thought is given to the development of student thinking through engagement with a task. This recommendation has been taken up by a number of researchers (e.g., Stylianides & Stylianides, 2009; Breen et al., 2019) to tie theory to practice and examine whether the intended goals of an instructional task or sequence of tasks have been achieved.

Lynch, Lockwood and Ellis (2022) focus on mathematicians' practice of generating examples when formulating new conjectures and introduce the term *Comprehensive Example Generation* (CEG) to describe the act of systematically and sequentially generating a data set. Two conditions must be satisfied for CEG; firstly, that the intention in generating the set is to reveal a structure or pattern, and secondly, that the systematic process used would reveal all examples of the phenomenon of interest if continued indefinitely. In interviews with thirteen mathematicians, Lynch et al. (2007) found that CEG had particular affordances for conjecturing activity.

SAMPLE TASKS

We describe two tasks here which encourage students to make conjectures following example generation, one explicit and one implicit. Both were designed by two of the authors as part of a previous research project. The aim was to develop tasks which would introduce undergraduate students to the (previously unfamiliar) habits of mind of mathematicians and provide them with opportunities to develop their mathematical thinking skills and understanding. The habits of mind on which we focussed included example generation, conjecturing and generalising (Breen and O'Shea, 2019). We use the types of conjecturing tasks identified by Canadas et al. (2007) to categorise the two tasks, and we outline the activity and learning we expect from students when engaging with each task.

The Subset Task (Explicit Conjecturing Task)

Students are asked to find examples of a phenomenon, in this case a set with exactly k subsets for different values of k, and then to make a conjecture about how many subsets a set with n elements has. By constructing examples it is hoped that the existence *and non-existence* of an example can give students opportunities to make conjectures.

A. Can a set with exactly 2 subsets be found? Explain.
B. Can a set with exactly 3 subsets be found? Explain.
C. Can a set with exactly 4 subsets be found? Explain.
D. Can a set with exactly 5 subsets be found? Explain.
E. Can a set with exactly 6 subsets be found? Explain.
F. Suppose a set S has n elements. Make a conjecture for the number of subsets that S has.

Figure 1: The Subset Task

<u>Hypothetical Learning Process (HLP)</u>

We expect students to

- determine that a set with one element has 2 subsets and then attempt to construct a set with 3 subsets by looking at a set with two elements,
- realise that a set with two elements has 4 subsets and conclude that it is not possible to have a set with 3 subsets,
- construct a set with three elements, determine it has 8 subsets and realise it is not possible to construct a set with 5 or 6 subsets,
- recognise the 'power of 2' structure in the sequence 2, 4, 8,
- conjecture that the number of subsets of a set of size n is 2^n,

- test their conjecture with $n=4$ to verify that a set with 4 elements has 16 subsets.

As the students generate this data set systematically and comprehensively, in theory all examples and non-examples would appear in time in line with CEG theory (Lynch et al., 2007), lending strength to the underlying structure observed.

Conjecture type: Empirical induction from a finite number of discrete cases

This task involves 'empirical induction from a finite number of discrete cases' and as such it is a 'Type 1' conjecturing task following Canadas et al (2007). It is expected that students would progress through all the stages of a Type 1 task as outlined there, namely: observing cases; organising cases; searching for and predicting patterns; formulating a conjecture; validating the conjecture; generalising the conjecture; justifying the generalisation. These stages, with the omission of the last, align well with the 'steps' outlined above in the Hypothetical Learning Process for the task.

The Asymptotes Task (Implicit Conjecturing Task)

Students were encouraged to use the dynamic geometry software Geogebra to look for examples of different phenomena. They were not explicitly asked to make conjectures.

Consider the graph of the rational function

$$f(x) = \frac{ax + b}{cx^2 + d}$$

Is it possible to choose values of *a, b, c, d* (between -5 and 5) in order to provide an example of a function of this type such that:

(i) The graph of *f(x)* has no vertical asymptotes;

(ii) The graph of *f(x)* has one vertical asymptote;

(iii) The graph of *f(x)* has more than one vertical asymptote;

(iv) The graph of *f(x)* has no horizontal asymptote.

Figure 2: The Asymptotes Task

The students were provided with a Geogebra applet with sliders; this allowed them to change the values of the coefficients *a, b, c, d,* and observe what effect the changes had on the graph of the rational function. The applet's initial state was to set all of the coefficients *a, b, c, d* to 1 (see Figure 3A). In this configuration, the function has no vertical asymptote and the *x*-axis is a horizontal asymptote of the graph. Note that the function will only have vertical asymptotes when *c* and *d* have different signs. It will have exactly one vertical asymptote in the case when the numerator is a factor of the denominator (for example if $a=b=c=1$ and $d=-1$ as in Figure 3B). If $c=0$ and *d* is non-zero then the function is linear and has neither vertical nor horizontal asymptotes.

Hypothetical Learning Process (HLP)

We would expect students to

- experiment with the values of all of the coefficients,
- realise that the existence of a vertical asymptote depends on the coefficients of the polynomial in the denominator and so begin by changing the values of c and d,
- look at the graphs of functions where $c>0$ and $c<0$, then similarly for d, and realise that the denominator of $f(x)$ only has zeros if c and d have different signs,
- observe that the x-axis is a horizontal asymptote for the function $f(x)$ except when $c=0$ and the function is linear.

Conjecturing type: Empirical induction from dynamic cases

This task is close to a Type 2 task as categorised by Canadas et al. (2007), as it affords 'empirical induction from dynamic cases'. The stages in such a task are described as: manipulating a situation dynamically through continuity of cases; observing an invariant property in the situation; formulating a conjecture that the property holds in other cases; validating the conjecture; generalising the conjecture; justifying the generalisation. However, it is noted that not all of the stages necessarily occur with every conjecture. The stages described correspond well to the 'steps' outlined above in the Hypothetical Learning Process for the task, although there are a number of conjectures which can be made in response to the different parts of the task.

CASE STUDY: IMPLICIT CONJECTURING TASK

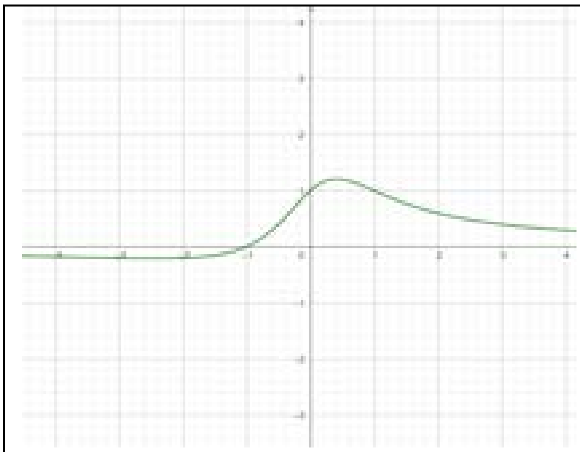
We present some evidence here that example generation tasks can provide opportunities for students to engage in conjecturing behaviours. We consider this to be an instrumental case study (Stake, 2000) where an instance of a phenomenon is explored in an effort to understand more about the general phenomenon. In our study, the case is the work of two students on an implicit conjecturing task.

Methodology

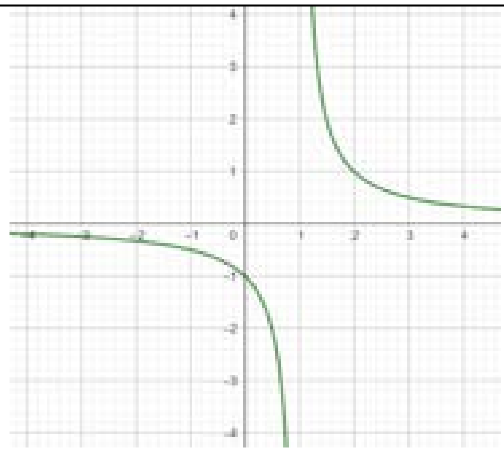
The second author carried out task-based interviews with four students from an introductory undergraduate calculus module where tasks like the Asymptotes task were assigned. The interviews each lasted for about an hour; during this time the students completed between 4 and 7 tasks and were encouraged to 'think aloud' throughout. Special software was used to record video, audio, the computer screen and any mouse movements. The transcription of the interviews included the audio recording along with a description of what was happening on screen. Two of the students, to whom we have given the pseudonyms Áine and Máire, worked on the Asymptotes task. Their transcripts have been analysed by two of the authors using a deductive approach to apply the theoretical frameworks and conceptualise the data. We sought episodes in the transcripts which provided evidence of the students engaging in CEG, reaching a particular point in the Hypothetical Learning Progression or working at a particular stage of the conjecturing process as envisaged by Canadas et al. (2007).

Student Data for the Asymptotes Task

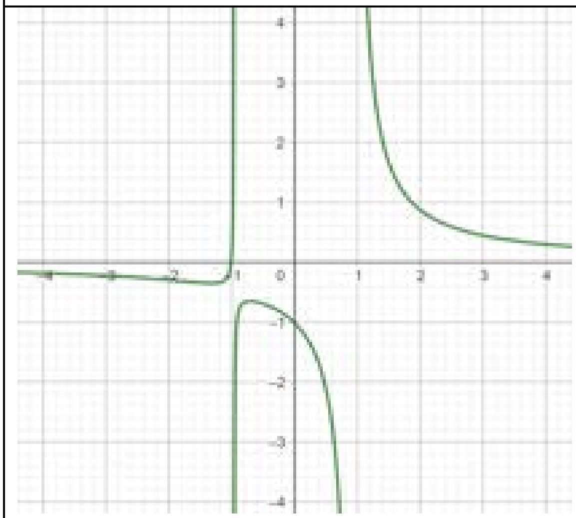
We consider the responses of Áine and Máire on the Asymptotes task. Prior to using the Geogebra applet, these students were given a paper version of the task. Áine gave a correct example (by setting c=0) to part (i) but was not able to provide examples for the other parts. For (i), Máire said that it was not possible to have an example of this type, for (ii) she said that x (not c) and d should both be zero, and she was unable to provide examples for parts (iii) and (iv). When using the Geogebra applet, both students started by moving one slider at a time while making sure that the other coefficients were set to 1. We will consider their work on this task individually.



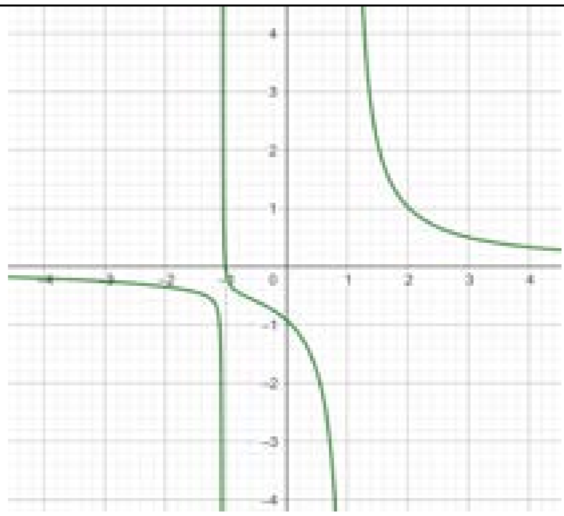
A: $a=b=c=d=1$



B: $a=b=c=1$ $d= -1$



C: $a=b=1, c=1.1, d= -1$



D: $a=b=c=1, d= -1.1$

Figure 3: Graph of $y=f(x)$ for various values of a, b, c, and d for the Asymptotes Task

Áine first selects the slider for d and changes the value to -1; she correctly observes aloud that the graph has a vertical asymptote (see Figure 3B). She changes the values of d to values ranging from -0.2 to -2.2 (the graph looks like the one in Figure 3C for values of d in (-1, 0) and like the graph in Figure 3D for values of d in (-5, -1)). Áine seems to notice the changes in the graph around d=-1, She then moves the value of d to values in (1,5) (and sees graphs similar to that in Figure 3A) and says '..higher values of d it looks closer to a curve'. At this point she sets d back to 1 and moves the slider for c. To begin with she looks at positive values of c, then she puts c=0 and notices that the graph is linear. She moves the slider for c back to 1 and then changes the value of *b* first and then *a*. She describes the resulting curves and notes that the *x*-axis is a horizontal asymptote of all of them. She then returns to the starting values of the coefficients and says ‘the vertical asymptotes occurred when I change *d* but not when I change other values…ok’. This could be taken as a conjecture. She then tries moving both *c* and *d* from their values of 1. She is able to give correct answers to parts (i)-(iv). She says there will be more than one vertical asymptote for ‘different values of *c* and *d* working together’. This seems like a conjecture that the relationship between *c* and *d* is crucial to the existence of vertical asymptotes, but she does not specify a relationship. At this point she seems to have revised her earlier conjecture that the existence of an asymptote depends only on *d*. So Áine has been able to use the task to experiment, to create examples, and to make and revise some conjectures.

Máire begins by changing the values of *a* from 1 to 5 and then down to -5, then returns *a* to 1 and changes *b* in the same way. After both sets of manipulations, she says that these functions have no vertical or horizontal asymptotes. Note that she is correct about the non-existence of vertical asymptotes here but all of these functions have a horizontal asymptote at $y=0$ (see for example Figure 3A). Máire sets *a* and *b* to 1, moves *c* up to 5, and says that there are no asymptotes. She then moves the value of *c* to -5 and says that there are again no asymptotes (which is incorrect as the graph looks like that in Figure 3C reflected in the *x*-axis). She sets *c* to be 0, notices that the graph is now linear and realises why. She changes the value of *c* to be 2 and moves *d* to negative values. This gives a curve similar to the one in Figure 3C. She notices that the *x*-axis is a horizontal asymptote and says that she thinks this function has a vertical asymptote also. Máire is able to find examples of a graph with no vertical asymptote and two vertical asymptotes by changing the values of *c* and *d* at the same time. She says ‘the more negative *d* gets the more vertical asymptotes we have’ which is not true but is a conjecture. Then she moves *d* from -5 to -0.8 and *c* from 1 to 5. She says that if we make *c* positive and *d* negative then we have a horizontal asymptote (this is true and possibly a conjecture but she does not mention the existence of a vertical asymptote here). She finds the example in Figure 3B, and moves *c* to 5 and *d* to -5 which again has one vertical asymptote. Note that the graph has two vertical asymptotes for most functions in this range of values for *c* and *d*. She says that ‘we have more than one vertical asymptote if $c<0$, $d<0$ and $c>d$’. This is a conjecture but it is not true. So Máire was able to use the Geogebra task to experiment, to find some examples, and to make conjectures. However, most of her conjectures are not correct.

DISCUSSION

It can be difficult in a course for non-specialist students to find opportunities to engage in authentic conjecturing activities. We have presented two types of example generation tasks here that can lead students to observe patterns and to naturally form their own conjectures. Lynch et al. (2022) have demonstrated how mathematicians use systematic example generation techniques to form conjectures, however little is known about students’ tendencies to use CEG. The Subset task was designed to elicit this behaviour. Although we do not present data on that task here, we have included it as an example of a task in which students are *explicitly* invited to engage in conjecturing and by which the process of CEG acts as a means of focussing students' attention on the inherent structure in a set of examples, thereby facilitating generalisation and the formulation of a conjecture. In the Subset task, students need not only to find examples, but also to realise that in certain cases this may be impossible – that is, they must combine the information from their examples and, crucially, non-examples in order to observe the expected pattern. In this way, there is greater agency and responsibility given to students in exploring the situation than might be if the task were to ask students simply to find the number of subsets for a set with (i) one, (ii) two, (iii) three elements. In addition, it may be that students realise that they can find a set with two subsets, and one with four subsets but none with three subsets and make a preliminary conjecture that the number of subsets must be even. Realising that there is no set with exactly six subsets would then cause them to refine this conjecture and arrive at a correct supposition for the number of subsets.

In the Asymptotes task, the students had difficulty in achieving CEG. It is possible that the issue was that there is a continuum of examples related to that task, and the students were able to move from one example to another very quickly. In some cases, a tiny change in one of the coefficients led to a significant change in the shape of the graph (compare the graphs in Figure 3B, C and D which arose from changes of 0.1 in the coefficients). Máire, in particular, changed the coefficient values quite quickly and this may have made it difficult for her to detect where important changes took place. Eventually the visualisation was valuable for her, and at the end of the task Máire said 'I didn't realise that the conditions had to be so specific until I actually looked at it …till I looked at it graphically really'. The task was difficult for the two students who attempted it; however, they were successful in generating examples and both conjectured that the relationship between c and d was important.

Technology helps students with these types of tasks because it allows them to look at a large number of possible examples quickly without the burden of calculation and to focus on the patterns emerging. In our study, both students had more success on the Asymptotes task when using the Geogebra applet than when they attempted the same task on paper. However, it may be that the speed at which students encounter examples when using such software is a problem and they may miss some important features. Harel et al. (2022) report that the facility and immediacy of generating large numbers of examples with technology may actually hinder the process of formulating and refining conjectures. As we saw with Máire and Áine, while the environment aided them in making a number of conjectures, their conjectures were often false, and they generally did not verify and subsequently refute or refine their conjectures. Lynch et al (2022) note that if examples are not generated using a systematic process, the example set might be non-comprehensive and may reveal misleading patterns. Furthermore, they caution that even when a student tests a diverse collection of examples by systematically varying one or more elements, all possible examples of a phenomenon may not be revealed. This appears to be true of Máire and Áine's work on this task.

The Asymptotes task can be categorised as an ‘empirical induction from dynamic cases’ (or Type 2 task) following Canadas et al. (2007). However, the students did not follow the stages outlined there in their approach to it. While they do manipulate the situation dynamically, they often do not observe an invariant property. They make conjectures in some cases, based on the manipulations that they have carried out, but do not attempt to validate or generalise their conjectures. It could be that the focus on giving examples impeded the act of generalisation as was found by Furinghetti, Morselli and Antonini (2011). Alternatively, it could be that the existence of *four* parameters which can be changed makes the task too complex for the students at this point in their learning and hampered their progress through the stages of conjecturing predicted by Canadas et al. (2007). The students in this study also completed simpler tasks using a Geogebra applet where only one parameter was involved. For those tasks, the students did identify the underlying patterns, and the use of the dynamic geometric environment seemed to be positive both from the perspective of engagement and the development of their thinking (Breen et al., 2019.)

While Áine and Máire did not closely follow the learning process hypothesised for the implicit conjecturing (Asymptotes) task, we believe there is a role for such implicit tasks in the curriculum. It may be that students need to become accustomed with CEG through multiple opportunities to complete tasks such as the Subset task in order for systematic and sequential generation of examples to develop as a ‘habit of mind’. However, we note that both students did make conjectures even though this was not explicitly asked for. This gives us confidence that a disposition of conjecturing is being developed. It may be that their learning could have been scaffolded more effectively by adding more structure to the Asymptotes task while retaining its implicit nature.

Implicit and explicit conjecturing tasks have characteristics that can help students develop their mathematical thinking skills. Explicit conjecturing tasks which involve CEG (such as the Subset task) have the potential to provide structure within which students can explore possibilities and make conjectures, while implicit conjecturing tasks which arise from example generation in a complex situation can encourage students to make conjectures naturally. While the two conjecturing tasks presented here were used in first-year undergraduate Calculus modules, we have designed and used similar tasks in other areas (e.g., Analysis, Number Theory) and with other undergraduate students and have found them to be equally useful in those contexts to engage students in conjecturing.

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What teaching practices should be used to introduce the limits of functions in the first year of university? A case study

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The introduction of the formal definition of the limit of a function during the first year of university is a source of many difficulties for students. In this exploratory research, we study the discourse of lecturers when they introduce this notion. To do so, we determine the 'relief' of the notion of limit, resulting from the combination of epistemological, curricular and cognitive studies. This relief enables us to envisage 'proximity opportunities', that is possible attempts to bring together students' prior knowledge and the notion of the limit. We demonstrate the value of using these tools by studying the discourse of one particular teacher, which enables us to identify the discursive proximities that he actually attempted.

Keywords: Teachers' and Students' practices at university level, Teaching and learning of analysis and calculus, Limit of a function, Discursive proximities, Lectures.

CONTEXT OF THE RESEARCH

We present here an ongoing study of university teachers' practices who teach the limits of functions. More specifically, we focus on the introduction of the formal definition of a limit, the first examples and the first results presented by the teachers. The choice of this topic is motivated in particular by the fact that limits of functions are taught in Calculus courses during the first year of university in many countries and that this content is often a source of difficulties for students (Oktaç & Vivier, 2016). Secondly, the limits of functions have been much less studied than the limits of sequences (see Chorlay (2019) for a state-of-the-art) and existing work does not place much emphasis on the study of teachers' discourse. Finally, this problematic, linked to the study of teachers' discourse, aims more globally at the study of the conduct of lectures and their impact on students' learning. In a previous research, we have shown that certain practices cause a discrepancy between teachers' objectives on the one hand and the way in which students receive the content delivered to them on the other: a main result attests that lectures are not necessarily a source of inactivity for students, and thus that this teaching space deserves to be studied (Bridoux et al., in press).

We begin by presenting our tools for analysing teachers' discourse and formulating the research issues that follow. We then show how we used these tools to study a lecture dedicated to the introduction of the limit of a function. Finally, we present our first results and a few prospects for further work.

THEORETICAL TOOLS AND ISSUES

Our research is based on Activity Theory, adapted to the didactics of mathematics (Bridoux et al., 2016). It leads us to study students' learning through the prism of their mathematical activities organised by the teacher through a coherent scenario. However, during the lectures, these activities are difficult to observe. We are therefore led to study the teacher's discourse specifically. We hypothesise that, in order to advance the students' knowledge, the teacher tries to use a discourse that is close to the students' work in order to introduce new knowledge, for example by building on acquired knowledge. Within our framework, this theoretical hypothesis is related to the ZPD[1] model of Vygotsky: it asserts that lectures can contribute to the appropriation of knowledge by students and, ultimately, to the conceptualization of this knowledge (Bridoux et al., 2016). The connections between the teacher's discourse and students that we seek to study are called 'discursive proximities' (Robert & Vandebrouck, 2014). Three types of proximities are distinguished. *Bottom-up proximities* lie between what students have already done and the introduction of a new object or property. In this case, the teacher's discourse therefore aims to move from the contextualised to the decontextualised by generalising the particular case. *Top-down proximities* are situated between what has been explained and examples or exercises. The teacher can then explain how the particular case fits into the general case, moving from the decontextualised to the contextualised. Finally, *horizontal proximities* do not lead to any change between contextualised and decontextualised. They consist of reformulations, explanations of the links between concepts, comments on the structure of the course, etc. Examples of proximities will be given in the next section for limits of functions.

To prepare for the study of teachers' discourse, the researcher must have an *a priori* reference, which we call '*relief on the concepts to be taught'* (Bridoux et al., 2016). The relief[2] of a notion to be taught is a cross-study combining epistemological, curricular and cognitive analyses. Relief thus makes it possible to study the specific features of the concepts to be taught, taking account of the curricula, while being aware of students' difficulties already identified by research. These analyses then make it possible to identify, *a priori*, *opportunities for proximities*, which will then be compared to the proximities actually attempted by the teacher to see whether or not these opportunities are taken during the course.

In this context, the "relief" helps the researcher to analyse the content taught by considering possible ways of introducing it and to study the distance between previous students' knowledge and the new notion. Thus the "relief" allows the researcher to describe the attempted conceptualization by taking into account students' difficulties.

[1] Zone of Proximal Development.
[2] The word 'relief' is a French word which is a metaphor for 'relief map'.

In the context of teaching the limits of functions and on the basis of the tools just presented, our research questions are formulated as follows: What are the relief elements that give rise to proximities in teachers' discourse during the course? What are the links between the content organisation choices made by teachers and the proximities attempted in their discourse?

ANALYSIS OF A LECTURE

In order to provide some answers to our questions, we present some curricular information about the limits of functions and then analyse the discourse of a teacher in a lecture by means of the tools described above.

In France, the limits of functions are intuitively introduced in *Première* (grade 11, students aged 16-17) on the basis of examples and without formalisation. They are taken up again in *Terminale* (grade 12). The objectives described in the syllabi are aimed at practising the operative aspects. At university, the definition frequently given is as follows: f has a finite limit l at a if $\forall \varepsilon > 0\ \exists \eta > 0\ \forall x \in D_f \left(|x - a| \leq \eta \Longrightarrow fx{-}l{\leq}\ \varepsilon\right.$. As students have already calculated the limits of functions at the end of their secondary education without using this definition, one of the challenges of university teaching is to make them feel the need for introducing this definition. It also requires knowledge with respect to logic, absolute values, real numbers and inequalities. However, this knowledge is not widely used in high school and is therefore probably not available to a large number of students. It is therefore difficult for the teacher to find an initial problem where the notion of limit would be the optimal tool for solving and where the students could construct the new notion independently. Thus, it is unlikely to find any bottom-up proximities in the teacher's discourse introducing this definition. To introduce the definition, teaching sequences developed by researchers are often based on the articulation between several semiotic representation registers (in the sense of Duval, 2006): natural language, graphs and algebraic (for example Bloch, 2003). This articulation seems to be an effective lever to give sense to the new notion which, in our view, would imply to find horizontal proximities within the teacher's discourse. However, our experience show that these sequences are rarely used in classic lectures. Instead, teachers often give a first intuitive formulation like "*f(x)* approaches *l* if *x* approaches *a*" and then formalize these words to build the definition. Sometimes they also use graphics.

It has been shown that students often develop a dynamic conception with respect to limits (Robert, 1983) in which the notion of limit is described as 'getting closer to', which can give rise to conceptions such as 'the limit is a number that the function cannot reach' (Mamona Downs, 2001). In contrast, a static conception in the sense of Robert (ibid.), in which the limit is associated with expressions such as 'as close as you want', allows students to give more meaning to the notion. The vocabulary used by the teacher can therefore have an impact on the students' conceptions. The teacher could also associate a graph with the definition (before or after its introduction). This link between different semiotic representation registers can lead the teacher to attempt horizontal proximities, for example via reformulations to interpret the inequalities present in the definition in terms of intervals or distances.

Once the definition has been introduced, the teacher often gives examples to show how to manipulate the formalism it contains. This type of task can lead to top-down proximities in the teacher's discourse, in particular to show the logical organisation required to manipulate the definition as an object or to make explicit the prior knowledge that students need, which could also be a source of horizontal proximities. But the first manipulative tasks, such as showing that $\lim_{x\to 2}(3x-5)=1$ or $\lim_{x\to a} x^2 = a^2$ by means of the formal definition, are already complex for many students. It is also not uncommon for the teacher to use the definition as a tool for proving results such as the uniqueness of the limit, calculation rules, etc., thus leading the teacher to attempt other top-down or horizontal proximities.

In this context, it is difficult for the teacher to introduce the definition of limit because of the distance between intuitive high school conceptions and the needed formalism at university. Furthermore, the skills needed to write the definition are not taught at high school level. That is why students are not able to build the definition by themselves or to solve first tasks where the notion is worked in its double dimension of object and tool. The proximities attempted by the teacher are thus crucial to show students feature of the notion of limit.

We now show how these elements of relief help us to study a teacher's discourse. We focus on a one-hour lecture given in the second semester to 200 first-year university students. To analyse the teacher's discourse, we compared what the teacher wrote on the blackboard with what he said orally. Tableau 1 in Bridoux et al. (2015, p. 48) gives an overview of the different phases organised by the teacher, showing their duration and content.

We have chosen to look specifically at the emergence of the formal definition and its first use as a tool to prove a property.

First, the teacher chooses to introduce the notion of limit intuitively by saying: '*How would you define an intuitive notion of limit?*' A student replies: '$f(x)$ *gets as close as you want to* l', then adds '*when* x *gets close to* x_0'. The teacher then shows a continuous function on the board and comments on the graph: '*We're trying to look at a diagram to explore this concept. So* x *is approaching* x_0 *the point* M *is approaching the point* M_0, $f(x)$ *is the ordinate of* M *is approaching* l *there, OK?*' Then he writes at the same time as saying: '$f(x)$ *is as close as we want to* l *if* x *is close enough to* x_0' (reformulation 1). The teacher thus reformulates the intuitive student's definition by combining the graphic and natural language registers and proposes a definition that can be associated with a static conception of the notion of limit, as we mentioned in the relief elements presented earlier.

The teacher continues: '*Here we're using sentences, what we'd like is to have a mathematical reformulation. Because these sentences leave a lot of room for ambiguity*'. He illustrates his point by giving the following example: '*We define a* *function of* $\mathbb{R}$ *in* $\mathbb{R}$ *by* $f(x) = 0$ *if* $x \in \mathbb{R} \setminus \mathbb{Q}^*, f(x) = \frac{1}{q}$ *if* $x = \frac{p}{q} \in \mathbb{Q}$ *where* $\frac{p}{q}$ *is irreducible*'. The teacher points out:

> '*Here, the intuitive notion becomes complicated, because you're going to have trouble tracing this curve. So we're not going to be able to use a geometric notion of the limit. So, to solve a certain number of problems, we need a more mathematical, more rigorous definition*'.

Here, the teacher uses an example to demonstrate the need for a definition that he would like to write in the algebraic register. However, the students did not have to work with this kind of function at high school. This example does not probably show the need of a formal definition for most students.

The teacher builds on reformulation 1 ~~from phase 1~~ to construct the formal definition of limit step by step, as shown in Table 1.

What is written on the board	What the teacher says (extracts)
$\lvert f(x) - l \rvert < \varepsilon$	*What is the distance from $f(x)$ to l ? Yes, it's the absolute value of $f(x) - l$ OK. So we want $f(x) - l$ to be as small as we want, $f(x)$ is going to be as close as we want to l. What does that mean? It means that the absolute value of $f(x) - l$ is less than epsilon, for epsilon to be as small as we want, we agree.*
$\forall \varepsilon > 0 \qquad \lvert f(x) - l \rvert < \varepsilon$	*We want epsilon to be as small as we want, which means that it will be true for any positive epsilon, so we'll have:*
$\forall \varepsilon > 0 \qquad \lvert x - x_0 \rvert < \alpha \qquad \lvert f(x) - l \rvert < \varepsilon$	*And thus we have that if x is close enough to x_0 that is, if the distance from x à x_0 is less than a certain alpha value.*
$\forall \varepsilon > 0 \qquad \lvert x - x_0 \rvert < \alpha \Longrightarrow \lvert f(x) - l \rvert < \varepsilon$	*So the if-then implication is that however small epsilon is, if $x - x_0$ is less than alpha, we must have $f(x) - l$ is less than epsilon, it's starting to look like the definition your friend gave.*

$\forall\varepsilon > 0, \exists\alpha > 0, \quad |x - x_0| < \alpha \Longrightarrow |f(x) - l| < \varepsilon$	*So alpha, how do we introduce it, because here we're introducing a notation, which means that alpha has to be in which set? It has to increase a distance so...* [student answer] *positive, that's it. Whatever epsilon is, as soon as x is close enough to x_0 i.e. if the distance from x to x_0 is less than alpha, so behind this is the notion that 'there exists alpha such that'. So our definition, if we want to write it in a rigorous way, looks something like this, OK?*
$\forall\varepsilon > 0, \exists\alpha > 0, \forall x \in Df, |x - x_0| < \alpha \Longrightarrow |f(x) - l| < \varepsilon$	*So there's still something missing, and that's x belongs to which set, for what x we have this involvement. It's for the x belonging to the definition set, so I must have...*

Table 1: Teacher's discourse during the emergence of formal definition

Here, the teacher builds up the definition step by step in the algebraic register, following the order of the quantifiers. With reformulation 1, he remains in the natural language register to reformulate the idea of closeness by introducing an inequality containing an absolute value and then in terms of distance. Our relief study enabled us to anticipate the presence of such horizontal proximities during the definition construction phase. The teacher's discourse does indeed contain several of these, in this case reformulations linked to the formalism and logical structure of the definition, but there is nevertheless a discrepancy between what is said orally and what the teacher writes in the algebraic register. Finally, the teacher writes the following definition: '*Let $x_0 \in \mathbb{R}$ and f be a function defined on a neighbourhood of x_0. f has a limit $l \in \mathbb{R}$ in x_0 $\left(\lim_{x\to x_0} f(x) = l\right)$ if and only if* $\forall\varepsilon > 0\ \exists\alpha > 0\ tel\ que\ \forall x \in D_f \left(|x - x_0| < \alpha \implies |f(x) - l| < \varepsilon\right)$.

While writing this definition, the teacher reads out the absolute values in terms of distances and then writes on the board: '*In other words, however small ε is, we can find a sufficiently small interval around x_0 over which the distance from $f(x)$ to l is less than ε*. The teacher continues orally: '*OK, the 'whatever' is 'as small as ε is'*, ''$\exists\alpha > 0$ *such that* $\forall x \in D_f$ $|x - x_0| < \alpha$*' this is an interval, so there is a sufficiently small interval around x over which the distance from $f(x)$ to l is less than ε*'.

During this phase, the teacher does not make links with the previous graphic. Horizontal proximities are thus not attempted in this regard.

We now analyse the first result proved by the teacher: 'If f *has a* limit l *a*t x_0 *and is defined* in x_0 *the*n $\lim_{x \to x_0} f(x) = f(x_0)$'. The use of the formal definition is compulsory for writing the demonstration. It is used here as a tool, as we had anticipated in the study of relief. To start the proof, the teacher draws a real number line and says: '*We are going to use reduction ad absurdum. l less than $f(x_0)$. What does the definition say?*' He writes: '*The definition of* $\lim_{x \to x_0} f(x) = l$ *is*

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in Df, |x - x_0| < \alpha \implies |f(x) - l| < \varepsilon$$

After recalling the negation of an implication $A \implies B$ the teacher writes on the board:

$$\exists \varepsilon > 0, \forall \alpha > 0, \exists x \in Df \text{ such that } |x - x_0| < \alpha \text{ and } |f(x) - l| \geq \varepsilon$$

Then:

> '*So how can we prove an existence? What are the possibilities? There are cases where you can't produce any. The simplest way is to give an example: we show that for a certain number, it works. That's what we're going to do here, we're going to show that there's an epsilon for which we have the property*'

The teacher makes a methodological comment here, which we interpret as an attempt to relate it to the students' knowledge of logic.

The teacher then takes over the proof:

> '*I'm going to take epsilon equal to half the distance. What is this distance? $f(x_0) - l$! This distance is not zero because we said that these are two different numbers. It's the absolute value of $f(x_0) - l$, here it is $f(x_0) - l$ because I considered that l is smaller but it can be larger [...] I take half of this absolute value, so epsilon is this distance*.'

After choosing ε, the teacher writes on the board $\forall \alpha > 0\ |x - x_0| < 0$ and says: '*So what could we say to show that for any alpha, there is a x in the definition set such that we have the following property? What do you think the x that will cause the problem?*' A student gives the right answer but the teacher continues to comment: '*So what does it mean that* $\forall \alpha > 0\ |x - x_0| < 0$, what *numbers verify this? The distance from x à x_0 is as small as I want it to be, that's what it means [...] If we want to find an x that works, it's bound to be [...], x_0, yes?*' He then points to the logical sentence whose negation he has considered: '*So what happens with the two propositions for? x_0 ? The distance from x_0 to x_0 is zero, so less than any alpha, it works*'. He writes on the board $|x_0 - x_0| = 0 < \alpha$ then $|f(x_0) - l| < \varepsilon = \frac{|f(x_0) - l|}{2}$ and comments: '*So can we have the second property? Epsilon, we defined it as being... So we still have this property, so we do have the two properties on the right*'. Here, the teacher's discourse contains reformulations based on the notion of distance which can be more intuitive for students and linked to the graphic register. However, this notion is lost in a very complex reasoning, to which students are not used at this level of teaching.

Our interpretation of this episode is that horizontal proximity opportunities are not attempted by the teacher during this phase.

Finally, the teacher says '*Our demonstration is as follows*' and then writes on the board:

We pose $\varepsilon = \frac{|f(x_0)-l|}{2}$*. For all* $\alpha > 0$*. We have* $|x_0 - x_0| = 0 < \alpha$ *and* $|f(x_0) - l| = 2\varepsilon > \varepsilon$*. Conclusion:* $l \neq f(x_0)$ *is contradictory to* $\lim_{x\to x_0} f(x) = l$*.*

After that, he says: '*OK, so why does it work, where does* f *defined in* x_0 *came in, it's here* [He shows $|x_0 - x_0| = 0 < \alpha$] *in the choice of* x *belonging to the definition set equal to* x_0*, it's only possible if* x_0 *belongs to the definition set*'. The written trace on the blackboard does not make explicit what has been presented orally by the teacher on the structure of the demonstration and contains no trace of the different reformulations given orally by the teacher. In our opinion, there is therefore a lack of bottom-up proximities here.

DISCUSSION

Our relief study showed that the articulation of different semiotic registers associated with horizontal proximities in the teacher's discourse helps students to give sense to the new notion. Top-down proximities allow the teacher to show how the definition is manipulated like an object or a tool. Horizontal proximities also lead students to understand how the formalism is used in the first tasks (examples of proofs of properties).

First of all, this relief study helped us to identify horizontal proximities in the construction phase of the definition formulated in the algebraic register. These proximities take the form of reformulations supported by work in different registers of writing (words, graphs, symbols) and revolve around the notions of absolute value, distance and interval. This choice is perhaps linked to the fact that the teacher imagines that this is old knowledge that has been stabilised among the students, whereas the links between these concepts are very little explained at secondary school.

We also hypothesised that the first examples could lead to attempts of top-down proximities. However, the teacher does not give an example to illustrate the definition, as is often done in a textbook (see for example Ramis and Warusfel, 2022). Instead, he chooses to create a gap associated with the use of the natural language register, which makes it impossible to deal with the example. The teacher will therefore not mobilise the definition as an object, thus causing the absence of top-down proximities in his discourse. What's more, the example chosen requires knowledge about numbers that is probably not readily available to a majority of students.

Finally, we have seen that the use of the definition as a demonstration tool is based on knowledge of logic which the teacher tries to take into account in his discourse, but we think that these attempts are once again only accessible to a small number of students.

As we had anticipated in the relief, we did not identify any bottom-up proximities in the teacher's discourse. In our opinion, this absence is linked to the choice of introducing the formal definition by attempting to (re)formulate the inequalities in terms of distance and with the idea of closeness. Another choice of introduction is to use sequences, which are often studied before the limits of functions, to construct an initial definition using the limits of sequences (see, for example, Ramis & Warusfel, 2022). The question then arises as to how to go about constructing the $\varepsilon - \alpha$ definition. Didactic engineering has also been developed to link the notion of the limit of a function to other knowledge (e.g. in topology, Branchetti et al., 2020), or to get students to interact more (in the form of a debate, for example, Lecorre, 2016) or to articulate different registers while leaving students more autonomy to construct the notion (Bloch, 2003).

Thus, the teacher's organisational choices mainly lead to attempts at horizontal proximity, but in our opinion these connections will have little impact on the students' conceptualization and thus on their learning, given the unavailability of the concepts on which these proximities are based.

This work, which remains exploratory at this stage, nevertheless shows how the tools presented make it possible to apprehend the teacher's discourse and to formulate hypotheses about the way in which students may receive this discourse. The aim now is to extend this work by studying more lectures and to compare the different proximities contained in the teachers' discourse in relation to their choice of introduction.

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Exploring Conceptions of Mathematical Creativity: Calculus Instructors' Views

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There are calls to incorporate creativity in tertiary mathematics courses. Given the centrality of teachers' beliefs on their instructional practices, we explore the conceptions of Calculus instructors on mathematical creativity. Specifically, we report on their views of mathematical creativity in three different realms: as research mathematicians, as mathematics teachers, and their views on students' mathematical creativity. In each realm, we found their conceptions of mathematical creativity aligned with the literature on making connections, taking risks, and various teaching actions that are reported to foster mathematical creativity.

Keywords: mathematical creativity, instructors' views, Calculus.

INTRODUCTION

Numerous research studies, policy, and curriculum-standard documents call for a focus on mathematical creativity in mathematics courses and programs (e.g., Borwein et al., 2014; Levenson, 2013; Silver, 1997). The inclusion of mathematical creativity in the classroom necessitates an understanding of this construct, particularly from an instructor's perspective. In fact, a review of the literature (Thompson, 1992) indicated that "teachers' beliefs about mathematics and its teaching play a significant role in shaping the teachers' characteristic patterns on instructional behavior" (p. 131).

Historically, exploring mathematical creativity dates to early twentieth century (as cited in Borwein et al., 2014 and Sriraman, 2009). This exploration continued its momentum through Hadamard's (1945) surveying (via mail) of mathematicians' creative processes. More recently, Sriraman (2009) and Borwein et al. (2014) also shared mathematicians' perspectives on creativity. Additionally, there is a recent effort to explore students' views of mathematical creativity (Cilli-Turner et al., 2023; Satyam et al, 2022). These explorations have led to a plethora of definitions or conceptions of mathematical creativity, with Mann (2006) reporting that there are over 100 different definitions of creativity that consider this construct from different perspectives. However, we found little exploration of instructors' views of mathematical creativity, which is striking considering their potential to impact classroom practices.

In this paper, we explore the views or conceptions of instructors participating in professional development on supporting students' mathematical creativity in their Differential Calculus (Calculus 1) courses. While they were provided with an operational definition of mathematical creativity at the beginning of their participation, we found their end-of-semester conceptions of mathematical creativity could be categorized into several broad themes. In this paper, we address the research question: *What are the conceptions of mathematical creativity held by Calculus 1 instructors?*

Specifically, we will report on their conceptions as research mathematicians, as instructors, and their views on students' mathematical creativity.

THEORETICAL PERSPECTIVE

We employ a *process view* of mathematical creativity, which underlines the mechanism of creativity while a person is engaged in a mathematical activity (Balka, 1974; Torrance, 1966). In our work, we adopt a process-oriented relativistic perspective to define mathematical creativity as a process of offering new solutions or insights that are unexpected for the individual, with respect to their mathematics background or the problems they have seen before (Liljedahl & Sriraman, 2006; Savic et al., 2017).

This theoretical framing guides our research on mathematical creativity through which we explore instructors' views and conceptions of this construct. Thompson (1992) defines teachers' conceptions as "general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like" (p. 130). Teacher conceptions can influence practice, in how teachers frame a task, lesson plan, and their teaching actions (Buehl & Beck, 2014; Song & Looi, 2012). Therefore, it is essential that we understand how instructors define and view mathematical creativity if we want them to promote their students' mathematical creativity through their teaching.

LITERATURE REVIEW

Much of the research on defining mathematical creativity is largely studies based on the perspectives of mathematicians engaged in mathematical research. The first documented exploration is Wallas (1926)'s model describing mathematical creativity through a four-stage creative process: preparation (gathering information and problem solving), incubation (taking a break), illumination (AHA moment of new idea), and verification (making sure the idea is a solution). Hadamard (1945) built on Wallas's model to explore mathematical creativity by surveying prominent mathematicians at the time. Around the same time, Guilford (1950) hypothesized characteristics that are common to creative people and developed three components of creativity: fluency (number of ideas), flexibility (changing ideas or approaches), and novelty (unique or original ways). Expanding on these components, Torrance (1974) added a fourth component of Guilford's definition of creativity, elaboration (describing or elaborating on those ideas), and developed tests to assess a person's creativity. More recently, Sriraman (2005) added to the literature by interviewing five research mathematicians about their creativity, finding views that aligned with Wallas's four stages.

While literature on mathematical creativity from research mathematicians' lens has seen a significant growth in the last few decades, research on instructors' views on their own teaching and their students' creativity remains rare. Moore-Russo and Demler (2018) found that when asking in-service mathematics teachers about their beliefs on mathematical creativity, all participants eschewed the "genius" view of creativity, believing that all students were capable of creativity in mathematics. Additionally, all of the teachers in this study supported the process view of creativity.

Tang et al. (2015) also explored mathematicians' views of mathematical creativity and found their views related to specific mathematical actions such as *making connections* and *taking risks* (Savic et al., 2017). Karakok et al. (2020) investigated the creative actions involved in problem solving culminating in the development of a reflection tool, CPR on Problem Solving. Specifically, making connections identifies "processes involved in connecting a given problem with definitions, formulas, theorems, representations, and examples from the current or prior courses and connecting the attempted problem solutions or approaches to each other" (p. 987). Taking risks focuses on "processes of actively attempting a solution, demonstrating flexibility in using multiple solution paths, using a tool or a trick, posing questions about reasoning within solutions, and evaluating solution attempts or solutions" (p. 988).

To foster students' processes of making connections and taking risks, certain instructional actions are needed. In Satyam et al., (2022), students reported actions they felt honed their creativity in their Calculus 1 courses. There were four categories of teaching actions: *Task-Related, Teaching-Centered, Inquiry Teaching, and Holistic Teaching.* Task-Related was defined as "any action that mentions properties of a mathematical content task that were (re-)designed, evaluated, or assessed by the instructor" (p. 540). Teaching-Centered was defined as "any action that was mostly focused on the instructor, whether it be verifying correctness or connecting topics" (p. 540). Inquiry Teaching was defined as "any action that can be linked to inquiry-oriented (or -based) instruction" (p. 541), as defined by Shultz and Herbst (2020) and Kuster et al. (2018). Holistic Teaching was defined as "any teaching actions that do not require a response from students yet psychologically build an environment for fostering creativity" (Satyam et al., 2022, p. 541).

METHODS

The data reported here comes from a larger project studying impacts of fostering and valuing creativity in Calculus 1 (Karakok et al., 2020; Satyam et al., 2022; Tang et al., 2022; El Turkey et al., 2024). There were three cohorts of 12 instructor-participants (3 in Fall 2019, 6 in Spring 2020, and 5 in Fall 2021) at various universities in the United States. Calculus 1 generally covers single-variable functions, their limits, derivatives, applications, and basic integrals. Instructors participated in 10-12 professional development sessions during the semester and implemented six creativity-fostering tasks (four of which they developed themselves during the professional development) in their calculus course. For reasons of scope and length, this report examines only the conceptions of five mathematicians who taught Calculus 1 from our Fall 2021 cohort as reported in end of the semester interviews. We selected this cohort because their end-of-semester interviews included a list of questions that specifically addressed our research question.

We take a social phenomenological perspective (Denscombe, 2017) through qualitative inquiry to capture instructors' conceptions of mathematical creativity from the three aforementioned lenses. Two authors coded the data using an iterative process. First, we split the data into three categories and used an in vivo coding process

(Saldaña, 2009) to code interviewee's views on creativity in their own research (R), how they described their own fostering of creativity in their teaching actions (T), and what they observed in their students' creative moments (S). Each author coded separately and then met to concur each code where we relied on an "intensive group discussion and simple group 'consensus' as an agreement goal" (Harry et al., 2005; as cited by Saldaña, 2009, p. 28).

After aggregating the codes into themes, we relied on our process-oriented theoretical framing to code the data using the CPR on Problem Solving (Karakok et al., 2020) categories, *Making Connections* and *Taking Risks*, for the (R) and (S) data. For the (T) data, we used the creativity-fostering teaching actions (Satyam et al., 2022) as a categorizing mechanism. Thus, the (T) data was categorized using the themes: *Task-Related, Teaching-Centered, Inquiry Teaching, and Holistic Teaching* (Satyam et al., 2022). We also report any utterances that did not belong into those categories.

RESULTS

Views on Mathematicians' Creativity in their Research

We found that four of the five instructors cited making connections as their main approach to being creative in their own mathematical research. We italicize their quotes for readability. For example, Claude Louverture[1] (White/Turkish, they/them) stated:

> *The connective tissue is logic and we're connecting, you know, axioms or assumptions to conclusions somehow. The way that I see the creativity happening is that I envision myself stepping across a river on stones and in the process, I'm trying to get to this this conclusion, starting from the opposite bank which is my assumptions, and I'm trying to make these intuitive creative leaps.*

Other phrases used to describe making connections as mathematical creativity included "*synthesizing different fields together*" (Bartholomew Jackson, Black, he/him) and generating examples to prompt investigation of definitions and theorems. Three out of five instructors talked about taking risks as mathematical creativity. This included "*explor[ing] math without having an end result in mind*" (Carmen Williams, Hispanic/Mexican, she/her) and having "*no idea it is going to help… there's just something that tells me that this could be helpful, and I really didn't know if it was helpful or not*" (Dr. Watson, White, he/him). We also report on two aspects that did not fall in the categories of making connections or taking risks. Two participants (BMil, Dr. Watson) stated that interacting and collaborating with others were part of math creativity, as well as affective aspects. For example, BMil (Asian/Chinese, she/her) stated that, "*it's still more about working with other people and valuing their ideas and seeing where it goes*." Dr. Watson also praised mentoring students as a catalyst for mathematical creativity for him. He went on to associate affect with mathematical creativity, stating that "*part of success in math is embracing the uncomfortable feelings that come with being stuck on a problem*."

Views on Teaching to Foster Creativity

Satyam et al. (2022) found four major instructional categories that students reported fostered their creativity: task-related, teacher-centered, inquiry teaching (or active-learning), and holistic teaching. Many responses from instructors about teaching to foster creativity aligned with those reported by students.

Four instructors mentioned a teacher-centered action (provided guidance or shared big-picture thoughts about Calculus) and associated that to fostering creativity: "*I tell my students that Calculus is a close study of infinity where we're trying to use this abstract concept that we're not built to fully understand in useful ways to apply to actual real-world concepts*" (Claude). BMil stated that "writing out everything on the board and showing different ways of doing a problem" while also utilizing random functions to demonstrate creativity: "*I'm being creative because I'm coming up with whatever pops into my head is the function you're going to get*." BMil and Bartholomew Jackson stated that they both show different approaches on the board so the students can see more than one approach and where it leads. Finally, Bartholomew reported being "*willing to go off-script and work out a solution that people suggest instead of just answering it verbally - actually writing it out on a board and showing it to everybody.*" Bartholomew's quote is an example of soliciting students' ideas to perform a teacher-centered action. Other active-learning actions included asking for more questions in the class, allowing students to pick topics, asking for examples, and ideas to solve problems, all of which came from BMil. Dr. Watson, BMil, and Claude all encouraged collaborations in class: "*all the assignments they're allowed to talk with other students and sort of ... extend their zones of proximal development by talking to each other*" (Claude). Dr. Watson stated: "*then I see students comparing their answer with another student. 'Is that the same answer… Am I right, are you wrong?' I think that's when their creative wheels start turning a little bit*." Dr. Watson also had an active-learning whole-class success plan: "*I'm going to redefine success on this problem, as we will all succeed if we collectively can solve the problem*."

The actions above were prompted from a task or problem posed to the students. The statements that instructors gave about tasks fostering students' creativity were categorized as task-related actions. These included assigning creativity-fostering tasks, which was mentioned by all instructors: "*Some of it was developing some of my tasks, so I think I felt like I was being creative and also thinking about, 'how can I make this task so broad to just get like lots of different ideas from students right?'*" (BMil). Bartholomew also stated that group-work tasks worked to foster creativity, as well as purposefully having open-ended tasks like BMil: "*group work assignments, allowing them to be open-ended and to make their own choices… giving them a blank graph with several conditions and then they need to draw a function that satisfies that*." Instructors also utilized these tasks on their assessments: "*I put it on several of the homeworks, it was on exam two, and then it also showed up on the final*" (BMil). In terms of assessments, Claude chose a standards-based grading system (Cilli-Turner et al., 2020), and stated that assessment played a large role in fostering creativity: "*I think* *that allowing revisions on almost all my assignments hopefully helped - there'd be less pressure on the first try to get the right answer. Which to me, that limits creativity if they're aiming for a specific right way of doing things*." Dr. Watson admitted to his class that, "*maybe I should be more creative in the way I assess you guys' learning so that I can infuse my assessment with the things I'm saying I believe about creativity*."

Dr. Watson's example was his way of being holistic with his teaching choices – valuing creativity not only in his assignments, but also in his beliefs about assessment. We also found that he was holistic in how he supported his students verbally in the classroom. For example, he would "*make a big deal when anybody asks a what if question because I think that's a, that's like a key to being creative*." He also "*talked more this semester, I think, than before about some of the emotional aspects of doing mathematics*." Claude was also thoughtful about the classroom environment, stating that "*to me a lot of creating an environment that fosters creativity is creating space for mistake making and for unexpected detours*." BMil went deeper, explaining that fostering an environment meant "*maybe a student gave me an answer that wasn't going to lead to the final answer, or maybe it was just a wrong thing to try, like it didn't connect with what we were doing. But I would still write on the board and then ask, either that student, why did you suggest this? what led you to think about trying this next thing?*" She was adamant that she would have not done this in previous semesters of her teaching. Finally, soliciting different approaches to a problem was a common theme in the instructors' responses. Bartholomew also acknowledged a change in his teaching, stating that "*continuing to affirm different approaches, so that they feel empowered to try different things… encouraging them to think outside the box and be flexible and to draw on things that they've learned in the past… it's kind of been there, but I just haven't always been as intentional about it, especially in my language*."

There were two aspects of the instructors' views of teaching to foster mathematical creativity that are not fully explained by one of the four teaching-action categories. One is that four instructors talked about improvising in their classroom. Usually this took place in a whole-class discussion (active learning) of tasks when a student would demonstrate a surprising approach. BMil, prior to this focus on creativity in her Calculus course, would have "*shied away*" from incorrect approaches. In this course, she wrote down the approach on the board (teacher-centered) and asked the student why they suggested it (active-learning). Bartholomew did the same action of going "*off script*," often writing out the student approach on the board instead of verbally discussing why the approach did not work. These actions in class made an environment (holistic) that, according to BMil, is "*not just shutting down an idea because I knew it wouldn't work*." The other aspect that we found instructors stating about teaching is that they had philosophical approaches to fostering creativity that were driving many of the four categories of teaching actions. Dr. Watson stated that,

> *"I would love my students to be able to take some things away from my classes that are more than just I can do these math problems and I can think logically. I'd love them to take*

away some virtues: risk taking, making connections, seeing what's fundamentally true and what is only peripherally true, evaluating solution paths."

BMil had a goal that mathematics was not "*owned by one person*," nor wanted to be "*gatekeeper*" in their understanding of mathematics.

Views on Students' Mathematical Creativity

Parallel to instructors' views on their own creativity, three instructors' views on students' mathematical creativity included making connections. Claude stated that, "*I think of creativity as being a lot about connection and connecting ideas to other ideas, connecting things that [students are] learning to things that they already knew, connecting math to other subjects that they care about*." Dr. Watson cited both making connections and taking risks in his viewpoint of students' mathematical creativity, along with evaluating students' solution paths. In fact, taking risks was a common theme with all five instructors. Carmen associated creativity with exploration, "*trying things that felt maybe unsafe*." BMil also stated that, "*my students did take risks in suggesting different things that they didn't know were going to work out or not*."

Other instructors could not separate the action of taking a risk from the affective state that allows for that action. Claude associated being "*passionate*" with students being "*risky*," while BMil stated that being "*comfortable*" allows students to approach "*a problem in more than one way*." Confidence and comfort in the classroom were associated with mathematical creativity by all five instructors. It was so prevalent that creativity and confidence were interchangeable; with instructors often stating that a change in creativity came from a change in confidence. For example, Bartholomew Jackson said "*I think students got more comfortable with those [tasks], there were students who got more comfortable with those as the semester went on. So, I guess that's an increase in creativity*." There was other affect associated with creativity including enthusiasm (Dr. Watson) and perseverance (BMil).

DISCUSSION

We found that our instructor participants, when talking about creativity in their research, often adhered to the two categories "making connections" and "taking risks," aligning with a process-oriented view of their own creativity. Only two participants mentioned ideas of mathematical creativity outside of those categories; collaboration, either situated within working with other mathematicians or in mentoring, and affective moments, which include "embracing the uncomfortable feelings" involved in exploration. On the other hand, all five instructors demonstrated knowledge of their students' affective states and noticed the need for addressing affect in class. They saw students having an affective state of confidence, comfort, or passion as a near requirement for creative mathematics. These results align with students reported affective outcomes when discussing creativity (Tang et al., 2022).

Through the different categories of instructors' conceptions in our coding of their interviews, we observed similarities with students' views of mathematical creativity as reported in Cilli-Turner et al. (2023). For example, Carmen highlighted the importance of students' willingness to explore which aligns with the Actions and Attitudes theme describing creativity as an action or attitude of a person deemed creative (Cilli-Turner et al. (2023). The Application theme (creativity as applying mathematics to another field or discipline) was evident in Bartholomew's conceptions of mathematical creativity as a mathematician working in mathematical biology. All instructors designed and implemented tasks that elicit different approaches from students. In that, their conceptions encapsulated the Different Ways theme (Cilli-Turner et al., 2023). By being explicit and intentional about creativity, the instructors' conceptions aligned with the Originality and Against Authority themes (creativity as going against an established authority) from Cilli-Turner et al. (2023).

Of note is that even though this study took place within the context of a Calculus course, the majority of the participants did not situate their responses around a particular topic or concept in Calculus. Instead, most instructors spoke of teaching to foster creativity as actions that could be more broadly applied in the classroom, regardless of the mathematical topic. Thus, it seems that instructors' conceptions of creativity in their teaching are not specific to a single course and align more with conceptions of mathematical creativity in problem solving.

One implication is that teaching actions to foster creativity include many more encompassing actions, which may not be observed by students. The improvisation that instructors go through with in-the-moment decisions can have a large impact on students' creativity (Beghetto & Kaufman, 2011) and aligns with literature that in-the-moment decisions can stem from a teacher's beliefs (Song & Looi, 2012). By instructors having creativity embedded as a dimension in their conceptions of teaching, we believe that can translate to more in-the-moment decisions that center students' ideas, to support students feeling more creative and confident in mathematics.

Limitations

As mentioned previously, both making connections and taking risks were categories from the CPR on Problem Solving, which was introduced to the participants in the first professional development session. Introducing this reflection tool had the potential to impact instructor responses to the interview questions.

NOTES

1. All names reported are self-chosen pseudonyms. Self-reported pronouns are also given after each pseudonym.

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(Un)doing gender in university mathematics – theoretical challenges and practical recommendations

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After many years of research and interventions around mathematics and gender, mathematics as an academic domain still faces a participation gap. While recent scholars underlined the importance of action against this, practical implications, and recommendations about teaching mathematics in a gender-equality-promoting way, are rare. This paper aims to provide such practical implications based on a review of current literature and the theoretical approach of (un)doing gender that is further discussed.

Keywords: Preparation and training of university mathematics teachers, Teachers' and students' practices at university level, Gender and Diversity in mathematics

INTRODUCTION: EVERYONE CAN LOVE MATHEMATICS, BUT?

After years of historical development most countries around the world provide the same formal possibility to engage in mathematics, for all genders and recent years have shown evidence, that a broad range of different persons study mathematics successfully. However, while this formal opportunity is given, and positive examples of diverse participation do exist, most research around gender and diversity in mathematics identifies that actual opportunities to succeed in mathematics are not gender equal.

Non-male genders are still underrepresented in mathematics, as the popular observations of the leaky pipeline in mathematics describe. The higher the academic level, the more are non-male genders in mathematics underrepresented. While in some European countries, up to 50% of women (including prospective teachers) start to study mathematics, in Germany for example only 18% of professors in mathematics (partly including mathematics education professors) are non-male (Göller et al., 2021).

Differentiation-oriented research further investigated this gender participation gap for many years. Differences in affective variables, such as interest, self-concept, and anxiety have been identified to explain some variance in participation. Furthermore, stereotyped beliefs (Kaiser et al., 2012) as well as stereotype threats were heavily discussed (Maloney et al., 2013). Based on those findings, as well as under political pressure of equality demands, many academic institutions implemented interventions to foster gender equality. Bias training, targeted recruitment and gender equality action plans have been implemented during the last years (Grzelec, 2022), as well as mentoring and drop-put prevention programs specifically for women. However, when being evaluated, those interventions faced two main challenges. Specifically, situated and temporary projects often seemed to fail in reaching long-term outcomes. In meta-analysis almost, none of the current interventions around Europe truly managed to increase the participation of non-male genders in mathematics (Grzelec, 2022). Secondly, many of these interventions have been criticized for their often differentiating perspectives, which may be reproducing gender issues and promoting an image of deficit women who need to be trained and mentored to participate equally in mathematics (Leder, 2019).

Other theoretical approaches thus aimed at shifting the perspective from differentiating towards formatting powers of discourse. Mostly grounded in sociocultural or sociological approaches, scholars investigated available positions and narratives for women in mathematics (Mendick, 2005; Solomon, 2012) and unravelled underlying social structures of gender being connected to power, positioning, and participation in mathematics. To some extent, these new insights gained from such theoretical perspectives may explain that differentiation-oriented interventions that do not manage to address the relevant underlying social structures and that are not connected to the relevant discourses and power in the field can barely have any long-term effect. However, their theoretical approaches are mainly descriptive and aim to investigate the complexity of social structures, that are in most cases even more complex to change.

Given this somewhat restrain from concrete practical implications on the one hand and the other hand also implementations that have been criticized theoretically and evaluated as ineffective, this leaves abstinence of practical implications. This counteracts the willingness and necessity to react in terms of gender equality, specifically in the day-to-day practices where discourses are being negotiated as well, such as they happen regularly in teaching mathematics at university.

In the following, I thus aim to develop practical implications for gender-equal teaching practices in university mathematics. I therefore explore further the contrary theoretical perspectives of differentiation and discourse and analyse their individual (dis)advantages to promote gender equality.

A THEORETICAL PERSPECTIVE OF (UN) DOING GENDER

To generally frame and situate mathematics teaching practices within gender equality I refer to the idea of doing and undoing gender. This has been widely used in many contexts and has recently also been successfully applied in teaching contexts (Deutsch, 2007; Goris-Hunter et al., 2018).

Doing and undoing gender (equality)

Doing gender may be understood as an action or practice that (re)produces any kind of gendered description or gendered addressing. One of the most common examples Butler (2004) provides is the baby that is born and initially gendered by the doctors who describe it as either “It’s a boy” or “It’s a girl”. Doing gender in mathematics would thus describe actions, that directly address a differentiation of gender, e. g. if a professor says, “the women outperformed the men in the exam”. Furthermore, I understand actions, that are based on certain assumptions about gender, as doing gender, too, for example, if a professor is providing different applications in their mathematics lecture, that are known to be interesting for different genders. Similarly, a professor pointing towards the various accomplishments of women in mathematics is doing gender, underlining, that what they are presenting, was achieved by female mathematicians. The introduced differentiation perspective of describing and mostly equaling out differences between genders can be framed as doing gender, as well.

In contrast, undoing gender describes actions or practices that restrain from (re)producing gendered descriptions or addressing. This would be a doctor congratulating for a healthy baby, for example. In mathematics, a professor may describe that several students were challenged by exercise X, so the exam showed different results. Equally describing how different persons accomplished different things in mathematics is what I would frame as undoing gender.

In terms of gender-equal teaching practices, doing gender is often understood as unfavourable for gender-equality, because like the differentiation perspective, it may reproduce stereotyped beliefs and categorized assumptions. Undoing gender is seen as favourable to reach gender equality, because if gender does not matter anymore, gender equality may be reached. However, if undoing gender is for example performed by individuals that experience gender discrimination, it may not be favourable in terms of reaching gender equality. It is known that many successful women in mathematics are undoing their gender within mathematics or that their gender is being undone by others (Solomon, 2012). This describes that in order to fit into the mathematics world, they try to make their own gender invisible. Similarly, many successful women in the history of mathematics have been described in terms, that undo their gender – e. g. she is more a machine than a woman, or this woman is truly a man (Gildehaus & Oswald, under review). While at least on the individual level often successful, these practices of undoing gender in reaction to a gender-discriminating field are obviously not changing the field towards gender equality. It leaves structural restrictions against everyone not willing to undoing gender as not equal to participate.

The other way around doing gender does not necessarily need to be unfavourable for doing gender equality. As stated earlier, there is empirical evidence that gender can have an impact for example on interest in mathematics. The lecture which aims to provide interesting applications of mathematics for all genders, taking into account possible differences, clearly practices doing gender but may have a positive impact towards non-male genders' interest in mathematics. At least for a short period, doing gender (and in line with that differential perspectives) may thus be favourable for doing gender equality. From a long-term perspective though, this may not be the case (in line with the interventions being evaluated as rather ineffective), as stated earlier. Grounded in the discursive perspective is, that gender practices of doing and undoing gender in teaching mathematics, do not exist detached from power relations or in a historical vacuum. They are connected to their discourses, relations and power distributions and the current situation is mostly still doing gender in mathematics in unfavourable ways. A differentiation-oriented approach to doing gender thus always faces the challenge of shifting a structural and complex problem onto individuals, while origins of gendered participation mainly seem to be based on strong societal discourses and distributions of power within society, not on individual choice-making (Mendick, 2005).

Yet, the idea of most socio-cultural approaches would be misunderstood, if we frame participants of a social world as somewhat empty figures between different social structures and discourses. While the formatting power of gender discourses can be strong, individuals still enact agency, for example in negotiating such discourses in their day-to-day practices and actions (Holland et al., 1998). Doing and undoing gender are practices that are being performed in almost every daily action in the field of university mathematics. Every actor and participant in the field contributes to gender equality and inequality and may be profiting from at least some practical recommendations on how to reflect these day-to-day actions.

Positioning of mathematics lecturers within this perspective

This perspective also points towards a specific position of researchers and mathematics lecturers. If mathematics lecturers or tutors are being asked, if they care about gender issues in their teaching, about half of them describe doing so at least to some extent. Still, some lecturers distance themselves from considering gender issues and the most popular opinion to neglect them, is to state, "No necessity, I just treat everyone the same" (Gildehaus & Oswald, under review). Thus, while these lecturers do not necessarily deny the problem or situation of gender issues in mathematics itself, they still distance themselves from being part of the problem. Furthermore, mathematics is perceived as an objective discipline where content cannot be gendered. It is important to state though, that, while the mathematics that is being taught may be objective, the way it is taught most likely is not. This does not only follow from the sociocultural approaches, but already from basic ideas on mathematics learning, more people may agree with. Given that learning mathematics is an interactive process, where knowledge is constructed and developed, learning takes place individually, but in an interactive way. From this perspective, teaching mathematics and treating students cannot be objective or the same. The way students are being addressed are being reacted to, are being heard as well as the way students themselves recognize being addressed, being reacted to and being heard, differs. The same words from the lecturer may have different impacts for different people. They can be gendered if non-male genders take them in and understand them differently and these words perform doing gender if one is not getting aware of them. If they are connected to unfavourable interpretations, they are not only doing gender but doing gender inequality.

We may thus follow, that at least some kind of awareness and reflection about one's own doing and undoing gender practices in teaching mathematics is essential for every mathematics lecturer. Furthermore, it is not only being aware of one's own gendered teaching and acting but being responsible for this. As stated earlier, gender issues are not individualized problems that could be solved by those, that they affect. Neither could powerful gender discourses be changed by mathematics lecturers alone. However, lecturers are powerful actors within the social field of mathematics, who in many situations can decide about what to represent, how to assess performance, how to request and promote participation and how to address different students.

Aim and approach

These theoretical perspectives should have clarified that undoing as well as doing gender demands that we address the tension between the pursuit of equality and the respect of difference. It is important to state, that there are no simple solutions to promote gender equality by just changing teaching mathematics teaching practices a bit. Practical recommendations need to consider the grounding discourses and power distributions that surround and structure the field of mathematics. In the following, I thus aim to balance the given tensions, by incorporating both perspectives into possible practical recommendations – doing and undoing gender practices, differentiating as well as discursive approaches. I therefore discuss the following question:

> What practical recommendations can be derived from current literature on gender and mathematics?

To answer this question, I mainly rely on the current literature reviews of gender in mathematics education, basically the one from Leyva (2017) and the most current from Becker and Hall (2024) as well as the given overview from Leder (2019). Based on these reviews I identified the field of participation (in line with Leder, 2019 and according to what Becker & Hall, 2024 named "lived stories" and "attitudes and beliefs") as well as performance (in line with Leder, 2019 and Becker & Hall, 2024 and according to what Leyva, 2017, identifies as achievement) to be of great relevance within the field. I then tried to derive practical recommendations based on the presented insights in the reviews as well as the original papers. I hereby also stumbled upon further possible recommendations which I could not frame within my approach. These are presented in section 3.

PRACTICAL RECOMMENDATIONS

1. *What we know about gendered mathematical participation and how to react*

Gendered participation is mainly observable in terms of affect (e.g. interest, value, self-concept) as well as in terms of general visibility and acting, such as learning strategy use. We usually face low gender differences in mathematics in affective variables, when it comes to university since the study choice itself is already gendered (Yazilitas et al., 2013). However, during the first semester, women still report lower self-concept than their male students (Sax et al., 2015). A lower self-concept is often observed hand in hand with higher insecurities and more fragile identities (Solomon et al., 2011). This can be specifically important if it comes to performance or drop-out intentions. It is known that specifically personal relations to tutors, role models and safe spaces of exchange and learning, as I discuss in the following, can promote women's identity work and self-concept in mathematics.

Women often report lower intrinsic value for mathematics at university than men (Johns, 2020). Qualitative insights suggest that women may value different aspects of mathematics compared to men, e. g. responsible applications as in Solomon (2012), but there is very little knowledge about these possible different values. This might be of relevance though, in terms, of what is being addressed in lectures, how mathematical study programs are being promoted and what contents are being taught. Addressing gender-specific interests in one's lecture would be doing gender, but it could provide undoing gender inequality to some extent.

Further known differences in participation can be summarized under visibility. Rodd and Bartholomew (2006) investigated the "invisibility" of young successful women in mathematics as some kind of self-protection (and possibly also undoing gender). Concretely speaking, this means that these women often participated less visible in their studies, e. g. they were less involved in class discussion, they were less likely to ask questions and they were also not valued for what they achieved (e. g. other students referred to a student as the best men of the year, while the best exams were actually performed by a woman). This problem of invisibility is also often discussed in historical perspectives on mathematics, which I address later (3.). In part, it works hand in hand with the findings that narratives of "being a genius", as well as just "being good" in mathematics are less available for women than for men (Mendick, 2005). For lecturers, it may be of great importance to reflect on whom they recognize in their seminars and teaching. Specifically, when it comes to decisions of who could eventually be asked to become a tutor, and who is being recognized as good, it is relevant to be aware of less visible candidates, that can be identified from the written homework rather than the lecture.

Recent research also investigated, that learning strategy use is gendered throughout different mathematical courses. In major, engineering and teaching programs, women report using more organization and less elaboration strategies (Gildehaus & Liebendörfer, 2021). Since learning strategy use can be connected to performance (Liebendörfer et al., 2022) and useful learning strategies are rarely addressed in lectures, we may keep in mind that a general goal transparency and recommendations about learning and learning strategy use in courses may help to provide a framework for all students and thus be undoing gender, in terms of reducing possible gaps, without categorizing.

2. *What we know about gendered mathematical performance and how to react*

Throughout the last years, most studies and meta-studies indicated that mathematical performance is not gendered anymore (Lindberg et al., 2010, Leder, 2019). We need to be aware though, that these greater studies sometimes focus on different mathematics courses, e.g. engineering as well as pure and applied mathematics. If we are taking a closer look at specific domains (Klieme et al., 2010) or assessment formats, gendered performances can still be found. It is known that competitive written or oral exams, where performance is requested under time pressure, can be unfavourable for some women. For example, women may be outperforming men in their weekly homework exercises, but men outperformed women in the written exam at the end (Göller et al., 2021). Furthermore, we often do find differences in oral exams, where anticipated insecurity is often unfavourable for grading. Here it seems important to reflect on what is to be graded and if a mathematical answer that is explained insecurely, but correct, actually is a correct answer. Moreover, alternative, mainly less competitive, and more cooperative ways of assessment could be discussed. Specifically, in mathematics, it is known that cooperation is of high relevance for later publication outcomes (Hu et al., 2014), which could underline that alternative forms of assessment would not only promote gender equality.

3. *What else do we know, that can enlighten our action*

Aside from these insights of gendered participation and performance, we can find many further insights that can be helpful for (un)doing gender and promoting gender equality, I briefly summarize some in the following:

Educate tutors: Given the somewhat more fragile and sometimes insecure participation of some students, we do know, that personal relationships, for example with tutors can have a very important influence on women's identity development in mathematics (Solomon et al., 2011). Thus, it seems of importance to invest in tutors, that often have the nearest relationships and impacts with students. There even are tutor education programs available, that include gender issues (Scharlach, 2022).

Provide learning support: Similar to the role of tutors we also do know learning support centres can provide an important place of safe exchange among students, specifically women can benefit. While support centres promote the learning of all students, women may profit proportionally higher (Ní Fhloinn et al., 2016).

Reflect on your representations: I discussed earlier the problems of invisibility. Given a historical underrepresentation of women's mathematical work, it can be challenging to provide somewhat equal references about male and female mathematicians. However, there are lots of open-access materials available in different European countries, that provide overviews on women's contributions to mathematics. Blunck (2008) for example created a complete mathematical lecture based on the contributions of women over the years, that can be assessed online, including exercises. Doing gender, in terms of being aware of an equal representation, may promote gender equality here.

Be a true mentor: There are many mentoring programs for young women in mathematics that shall for example strengthen their mathematical identity development. However, we may reflect again on the structures and power distributions in a social field. Mentoring would be most efficient if it would actually provide access to what is valued within the field (Colley, 2003) e. g. to informal information, visibility and great reference. If possible, mentors are taking the chance to actually provide and share their access within the field, this could be of great advantage for mentees.

Do not be afraid of disadvantaging men: Some scholars and lectures are concerned, that taking up practices towards gender equality, specifically in terms of undoing non-male discrimination, may actually backfire and result in male discrimination. However, recent empirical research showed no disadvantages for men, whenever interventions to promote women's engagement, were held. For example, Laursen et al. (2014) investigated different learning environments, that turned out to increase women's participation but had no effects for men. Similarly, many interventions, e. g. against performance anxieties (Zhang et al., 2013) showed smaller or no effects for men, but greater effects for women.

DISCUSSION

The aim of this paper was to provide practical implications to promote gender equality in mathematics at universities. Based on a perspective of doing and undoing gender I discussed current findings on gender and mathematics with respect to practical implications for teaching mathematics at university. Herby, I argued that mathematics lecturers should be aware of and responsible towards (un)doing gender in their teaching. I further discussed some recent findings in gender in mathematics from different perspectives: differentiation-oriented and discourse-oriented. Mathematics participation and performance can be gendered and practices of undoing and doing gender can be relevant to handle this.

However, as stated in the contradictions of the theoretical framework already, these recommendations may provide a first step and idea of how to act as a mathematics lecturer, but there is by no means any guarantee, that they would work out as suggested. As Grzelec (2022) stated in her research on practices in organizations, every action implied as promoting gender equality can counteract current practices (that we may not have been aware of) of doing gender inequality. This again promotes the necessity of structural, institutional and formal change, where lecturers as powerful actors can be a start, for the day-to-day practices, but other measurements need to follow. From the nature of this paper, it occurs, that these suggestions being made, are on the one hand based on current literature about gender in mathematics, but on the other hand, based on my own reading. Thus, one may have considered different papers, as well as one could have gotten different implications from the papers. In any case, I provide a starting point, on how gendered day-to-day practices in university mathematics can be identified, reflected, and questioned. It seems an open desiderate to discuss further implications that align with these first insights.

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How Emergency Remote Teaching Inspired a Blended Learning, Flipped Classroom Intervention in a University Mathematics Course

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Emergency remote teaching (ERT) during the pandemic presented university mathematics lecturers with numerous challenges which despite the lack of preparation, they were able to rise to. By employing a commognitive lens to lecturer interviews, we explored how the mathematical discourse in the lecturers' pedagogical practices shifted from before the pandemic to ERT. Here, we explore how these challenges facilitated shifts in lecturer and student routines including changes in pedagogical approach and in digital resources employed. We also explore how these changes have provided an opportunity to reflect on what digitally rich(er), inquiry-based learning environments are possible post-pandemic and to consider transforming the learning experience of students enrolled on a university mathematics course.

Keywords: emergency remote teaching (ERT), blended learning, flipped classroom, theory of commognition, communities of practice theory.

INTRODUCTION

At the time of publication of our first paper (Hooper & Nardi, 2021) we had begun emerging from the COVID-19 pandemic. Our paper drew on the first author's MA dissertation which explored how university mathematics lecturers had risen to the challenges of emergency remote teaching (ERT) (Engelbrecht et al., 2023). It was inspired by his first-hand experiences as an undergraduate student who had experienced chalk and talk (Cevikbas & Kaiser, 2023) lectures throughout his BSc in mathematics. From interviews with practicing lecturers during the pandemic – and subsequent commognitive analyses (Sfard, 2008) of these interviews, we aimed to identify discursive shifts in the lecturers' pedagogical practices during ERT. Two themes emerged: the presence of a "Faceless Audience" which captures how students attended lectures with no camera or microphone on; and "Coping without the chalk and blackboard" which highlights digital and other resources employed to manage during the online delivery of ERT (Hooper & Nardi, 2021). These themes provided the inspiration for what then became the first author's doctoral project which focusses on implementing innovative, digitally rich teaching pedagogies in a university mathematics course. In this paper, we set about addressing two aims: The first is to briefly report results of lecturers and students pedagogical shifts during ERT (Hooper & Nardi, 2021). Secondly, we want to elaborate how these findings have influenced an intervention in a first-year university mathematics course. We start by outlining the influences from university mathematics education (UME) literature on digitally rich, blended learning (BL) and flipped classroom (FC) innovations. Then, we introduce the Theory of Commognition (utilised in the MA analysis) followed by presenting the findings from our analysis of lecturer interviews. We then address how these utterances turned into actions for the doctoral study. We finish by exploring how we plan to network ToC and Communities of Practice for the doctoral project.

INNOVATION IN UME PEDAGOGY

During the rapid transfer to ERT, it became apparent that the chalk & talk pedagogy, which had until then been the pedagogical norm in lectures, was obsolete (Engelbrecht et al., 2023). This left a vacuum in place of the chalk and blackboard, with innovative teaching approaches and digital resources filling the void. In this section, we explore literature on the use of BL, FC, and innovative digital resources. Both helped university mathematics lecturers tackle pandemic teaching challenges.

Blended learning (BL) and Flipped Classrooms (fc)

Normally, BL classrooms combine online and face to face (f2f) components with the inclusion of synchronous and asynchronous tasks (Cevikbas & Kaiser, 2023). During lockdowns, the f2f component was digitised, but due to the flexible nature of this approach, the lecturers were able to create a unique blend of pre-recorded videos and live sessions, thus preserving the synchronous and asynchronous tasks. One of these blends is the FC. Cevikbas and Kaiser (2023) describe this as an environment where students view the lecture content outside of the classroom in preparation for activities in class. The literature indicates benefits in the classroom environment such as opportunities for active learning, improved communication, and increased engagement (Cevikbas & Kaiser, 2023). However, students are often concerned about the increased workload associated with these FCs (Cevikbas & Kaiser, 2023). Despite this, the FC provided an alternative teaching approach during ERT as it allowed flexibility and the possibility of introducing engaging activities which were important for students’ online engagement. The advancement in technology helped facilitate FC implementation: we now briefly outline these.

Innovative digital resources to facilitate FC online learning.

The ability for lecturers to rapidly shift their sessions online was facilitated by advances in technology such as Blackboard Collaborate (BC). BC is a learning management software whereby course information is organised, resources managed and during the pandemic lectures were live streamed. Tools like BC provided lecturers with the ability to organise asynchronous tasks via course zones and synchronous sessions were live streamed. Additionally, the live stream had useful features for students such as chat functions, traffic light emoticons and polling tools. For example, multiple choice questions gave students opportunities to engage with content outside of summative assessment which, (Feudel & Unger, 2022) comments has benefits for students’ revision, consolidation, and error correction in mathematics. Transitioning to an online platform and the use of tablets facilitated the incorporation of digital resources into live sessions. For example, different applications were easier to include in online sessions as lecturers could easily switch between tabs (e.g., From BC to Matlab) (Hooper & Nardi, 2021). Since the pandemic, digital innovations are continuing with tools such as LEAN (Thoma & Iannone, 2021), an interactive theorem prover that students engage with to explore and develop proof routines. Applications such as LEAN, Desmos and MATLAB are increasingly more accessible in f2f sessions as students are equipped with laptops, smartphones, and tablets. Hence, there is greater scope to include these digital resources in f2f learning environments to support the construction of digitally rich and engaging mathematics classrooms.

THEORY OF COMMOGNITION

The Theory of Commognition (ToC, Sfard, 2008) is a discursive approach in which learning (cognition) is seen through various acts of communication: with others, non-verbal (e.g., written word) and with oneself. The ToC emphasises the growth of discourses described by Sfard (2008) as *special types of communication* made unique by the community in which communication takes place. The discourse of the mathematics community can be described in terms of four characteristics: word use, visual mediators, endorsed narrative, routines.

Word use: This concerns the use of key words whether they are specifically mathematical words (e.g., linear independence) or do they also have meaning in colloquial discourse (e.g., Set.

Visual Mediators: Any visual objects that student or teacher may act upon to help facilitate the communication of mathematics to oneself or others in the classroom community (Sfard, 2008). The visual objects can vary in form such as graphs, diagrams, mathematical symbols, and physical objects (e.g., a calculator) that can be acted upon.

Endorsed Narratives: A series of utterances that describe the mathematical objects, the processes, and the relationships between them. The acceptance (or rejection) of these utterances is scrutinised against established rules defined by the community (Sfard, 2008).

Routines: We define routines as repetitive, regularly employed patterns and well- defined practices of the mathematics discourse (Sfard, 2008). Routines can be evidenced across the other three mathematical characteristics of discourse.

Interviews with mathematics lecturers (Hooper & Nardi, 2021) were analysed using these characteristics to trace shifts pedagogical discourse during ERT. Since "*commognitive accounts of teaching and learning tend to be fluid – non-binary, non-deficit- small-scale, snapshot dissections of communication*" (Nardi et al., 2021, p. 1) this felt like the natural choice to understand from the lecturer accounts how the discourse shifted from f2f to ERT. Next, we will delve into the two key themes that emerged from this commognitive analysis.

DO OLD HABITS DIE HARD?

In (Hooper & Nardi, 2021), we reported from a small-scale study focussing on understanding how university mathematics lecturers rose to the challenge presented by the immediate shift to ERT. Towards that study, Hooper conducted six semi-structured interviews (each lasting an hour) with practicing lecturers (before and during) the pandemic. This was an important requirement to identify shifts in the pedagogical choices of the lecturers in the different learning environments. The interviews covered multiple topics from pre-COVID-19 norms, digital resources, student engagement, challenges, and COVID-19 "keepers". The aim was to trigger reflections of the pedagogical routines during and before COVID-19. It also offered an opportunity to reflect on what resources and pedagogies lecturers might preserve for future teaching (known as COVID-19 “keepers”). The ToC was selected as the theoretical lens to analyse interview transcripts, identifying utterances that indicated shifts in lecturer’s pedagogical discourse. A coding system was implemented whereby rewatching the interviews led to highlighting instances where the designated characteristic appeared. In total eight codes were used to analyse the data where both mathematical and colloquial characteristics were represented (Hooper & Nardi, 2021). Two broad themes emerged from the analysis that encapsulated the experience of the participating lecturers: *the faceless audience* and *turning the dusty to digital.* The first explores challenges that the lecturer faced when students’ cameras and microphones were switched off. Within this, we see how digital resources were utilised by the lecturer to overcome this barrier and generate student engagement. The second theme focused on how lecturers utilised the digital resources to conduct lectures without the physical chalk and blackboard. We now explore these two themes along with other noteworthy data from the interviews.

THE FACELESS AUDIENCE

During the period of ERT, the learning and teaching environment drastically changed. In-person attendance of students and lecturers in a theatre had to be translated online. Although, it was as if an error occurred during this transfer period as lecturers reported student’s cameras and microphones remaining switched off. During the interviews, several lecturers, such as Zeta, reported that.

“What is really difficult is that you don’t get to see the students faces…No one has had their cameras on almost all year.”

The challenge presented by this was highlighted by Lecturer Alpha’s comment regarding f2f sessions.

“You can see if there are blank faces and whether they are getting it or not.”

From a commognitive standpoint, this indicates the challenges lecturers were having with the gestural endorsements of narratives during ERT. The absence of visible faces on camera posed a challenge for lecturers in gauging engagement with mathematical content, as they traditionally relied on students’ gestures. With this ritual no longer viable, lecturers adapted by incorporating three digital resources: Multiple choice questions, traffic light emoticons, and the built-in chat function. These tools, each in slightly different ways, facilitated a shift in the lecturers’ narrative endorsement routines, from relying on gestural cues to question posing techniques. Multiple-choice questions were utilized by lecturers to provide a quantitative gauge of students understanding of prerequisite mathematical content and content covered in f2f sessions. In this exert from lecturer Gamma we see an example of this new routine

“…this would give some indication of how far they had got and whether they got the answer right…If 80-90% got it right you can move on, but you know if only 50% get it right it’s time to go back and revisit.”

Similarly, traffic emoticons acted as digital replacements for f2f gestural cues. The lecturer, after completing a section of material such as a proof, would ask students to indicate their feelings towards the mathematics covered. Students could respond in three ways (happy, neutral, or sad) indicating to the lecturer whether to continue or revisit the content. These two resources demonstrate shifts in the narrative endorsement routines. Also, the physical selection of an answer by students suggests a shift (albeit a small one) in student engagement routines from the in-person sessions where gestures in the moment are typically drawn on by the lecturers (for example, nodding heads or confused facial expressions). Using multiple-choice quizzes creates opportunities for students' active participation in sessions rather than being passive observers. This shift in student engagement poses the question: Can digital resources help facilitate other forms of question posing? This question provides inspiration for the doctoral study as we seek to incorporate digital resources into a classroom intervention to foster an active learning environment.

Turning to the chat function (which operated like an online message board), this was a space for students to post comments and questions during live session. Here, we identified a shift in the student's question posing routines – that is they engaged in these routines compared to f2f sessions. This shift is highlighted by lecturer Epsilon

"The students really liked to use the chat and I erm felt more students were less intimidated to ask questions via the chat…Sometimes other students would respond, and a little discussion would happen".

Evidently, we see digital resources have helped facilitate students' inquiry routines (via the chat function) during online sessions. Students felt less intimidated to ask questions and respond to their peers helping foster discussions. Therefore, the chat function provided opportunities for active learning via question posing and peer to peer discussions. This demonstrates a potential covid-keeper as we utilise digital discussion boards to promote student engagement. In Hooper's emerging doctoral project, a digital resource we will utilise is Padlet, a digital discussion board software which allows for anonymous posting of text and multimedia. Padlet's inclusion is inspired by the reported benefits of the chat function during ERT to promote student engagement and peer to peer discussion in sessions. The aim is for this software to provide a space to promote student discussion, debate and question posing to shift from a chalk and talk environment to a student-centred active learning environment (Biehler et al., 2022; Cevikbas & Kaiser, 2023).

To summarise, the *faceless audience* presented lecturers during ERT with challenges regarding student engagement and endorsement routines. Incorporating new digital resources (e.g., Chat function) helped address challenges and facilitate a shift in the lecturer and student question posing and engagement routines. The reported benefits of these resources have provided inspiration for Hooper's doctoral project to introduce similar resources to help prompt a shift to an active learning environment.

TURNING THE DUSTY DIGITAL

During chalk and talk sessions, the writing on the blackboard was the prominent form of visual mediation which in the shift to ERT was no longer sustainable. To meet this challenge, lecturers turned to digital resources to mitigate the lack of chalk and blackboard. Advances in technology such as tablets and digital ink were incorporated to aid the communication of mathematics during online sessions. They played a role supporting visual mediation in two ways: firstly, in the physical writing of mathematics onto digital planes in time, secondly, by making access to other digital resources. This is highlighted by lecturer Delta

"I could open up another window and show them an animation of some curved geometry…in some ways this was better than normal lectures".

Commognatively, this demonstrates a shift in visual mediator use from pre-ERT to ERT as online lectures were facilitated by advances in technology. Lecturers could incorporate a variety of visual mediators like digital diagrams produced on MATLAB. This diversification of visual mediator tools helps shift the classroom to a dynamic (as opposed to static chalk drawings), active learning environment. Lecturer Alpha provides an example of visual mediators being implemented online

"You get a solution, and it will be an algebraic expression which may be a function of x, y & t and actually seeing an x,y map changing in time you know brings the solution to life".

Hence, digital resources can support the inclusion of dynamic visual mediators in f2f sessions to create a more dynamic and engaging student experience. In Hoopers doctoral project we will look to incorporate digital resources (e.g., MATLAB, & Desmos) to support lecturers content delivery and students' inquiry activities. By giving students agency over these tools, it is hoped they engage with the inquiry and active learning tasks.

OTHER POTENT RESULTS

At first glance of our (Hooper & Nardi, 2021) paper, there were utterances that didn't appear to be significant - or directly related to two emergent themes. But, upon reflection, these comments have contributed towards inspiring elements of Hoopers doctoral research. One lecturer commented on occasional (but infrequent) sharing of student's screens to display their attempt at a mathematical question. Whilst not identified as evidence of shifting routines, it clearly demonstrates the potential for student responses to mathematical questions to become focal points of discussion and mathematical knowledge building. This is infrequent posting of solutions shows a potential shift in pedagogical routines in the classroom as students respond to questions agentively (taking the initiative to present their own solutions) as opposed to passively being presented a model solution. This response sharing has potential to facilitate students' participation in the mathematics community's shared practices and accelerate discursive shifts (e.g., in word use, visual mediators or in participation in substantiation routines). By presenting solutions crafted by themselves (or with peers) they use words, notations, and language of their own to communicate mathematics. Whilst informal, the lecturer can utilise these contributions as discussion points to build formal mathematics. In the doctoral project we plan to use digital resources such as Padlet to help facilitate shifts in students answer sharing routines which will support the shift to an active learning environment.

Additionally, presented with an opportunity for change, some lecturers described overhauling their approach to lecturing during ERT. Two lecturers revealed how used pre-record content, set asynchronous tasks and utilised live sessions for solution presentation. This demonstrates a clear shift in the lecturers' pedagogical routines away from chalk and talk towards FC/BL pedagogy. They reported students were generally positive towards this approach, students liked viewing content at their own pace and using the recordings as revision tools. For the lecturers, it gave them more opportunities to explore examples and delve deeper into topics – an affordance not presented in f2f sessions previously. These self-reporting support similar benefits reported by (Cevikbas & Kaiser, 2023). This has inspired Hoopers doctoral project by introducing elements of the FC, the intervention course will have more free time for students to engage with inquiry and active learning tasks. By engaging with these tasks, we hope to see a shift in the environment to one built on discussion, inquiry, debate, and endorsement of mathematical content.

FROM UTTERANCES TO ACTIONS

As has been reported in preceding chapters, we have reported lecturers' reflection on implementing innovative pedagogies and digital resources to respond to ERT. However, these innovations were no result of a sudden enlightenment for reform towards student-centred active learning environment but a consequence of a medical emergency. In this section we will discuss conjectures to transform interview utterances into intervention actions. Firstly, both lecturer and students can shift their pedagogical routines – as evidenced by the varying forms of FC being implemented during ERT facilitated by digital resources. Therefore, we'd like to incorporate a FC approach in our course intervention to foster an environment which promotes exploration, discussion, and inquiry-based tasks (Laursen & Rasmussen, 2019). Inquiry- based learning approaches promote tasks which challenge students' mathematical knowledge, invoke collaboration, and give students agency over the learning process. This approach presents the lecturer with new challenges as their role in lessons changes, no longer engaging in mathematical monologues but shifting to actively steer students through sequences of pedagogically informed tasks. Our intervention will utilise asynchronous tasks (e.g., watching pre-recorded videos, attempting question sheets) to free up time in f2f sessions for student exploration.

Secondly, digital resources have an important role to play in facilitating shifts in pedagogical and mathematical routines of both student and lecturer. We saw how mathematical and colloquial digital resources aided lecturers in the shift to ERT, we appeal to these resources again to help facilitate active learning environment in our course intervention. One such colloquial digital resource is Padlet. Incorporating this online discussion board software will provide a safe space (due to its anonymity features) for students to share responses without the need for ownership of the response. These responses will still provide opportunities for classroom discussion as solutions are debated and refined with the aim of students eventually endorsing a particular narrative on the given mathematical topic. By utilizing student answers, we are shifting the roles of lecturer and student as the centre of classroom activity shifts from monologues lectures to active learning tasks led by student responses. Additionally, students' contributions have a role in knowledge creation: instead of polished presentations by the lecturer, students can see conflicts in their submissions and adjust accordingly after discussions, with endorsement following. Due to the multimedia capabilities of Padlet, we can implement the use of dynamic mathematical resources such as Desmos and MATLAB to facilitate inquiry-based tasks. Students will have opportunities to explore results and examples using software and share their observations using Padlet. The aim is to provide agency over the learning process so that they have a central role in mathematical discovery in their classroom as opposed to being passive observers.

These utterances were identified by a commognitive analysis of interview scripts and provided fruitful inspiration for the doctoral project. However, when planning ideas for the course intervention, it dawned upon us that we needed an additional theory to help us with the strategic pedagogical planning we needed to create the course intervention. In the next section we introduce the Communities of Practice Theory.

COMMUNITIES OF PRACTICE

A Community of Practice (CoP, Wenger, 1998) is established overtime as group interacts and pursues a set of shared goals. To achieve these goals, members of the group must engage in the practices of the community (which can be explicit or tacit). In a mathematics department we may observe several CoPs operating simultaneously with members switching between them (research Cops, Teaching CoPs, etc.) However, these communities are defined by their own shared repertoire which make them unique from one another. At first, members may engage with the community peripherally which overtime shifts to more legitimate participation by newcomers as they gain more confidence; also referred to as the process of *legitimate peripheral participation*. For example, a student enrolled on a Fluid Dynamics course will engage with a distinct set of activities and goals to achieve legitimate participation than the Real Analysis course they study. At the start of these courses, the lecturer will define the CoP in terms of the meta and object level learning goals expected of students as they engage in practice. There students enter the community as newcomers, since, as per (Wenger, 1998), they have not engaged with the community, its shared repertoire, activities, and goals before. The lecturers described here as *old- timers*, function as guides who are fully experienced, integrated members of the CoP. Students will transition from newcomer to old-timer by engaging in legitimate peripheral participation whereby newcomers will interact with a learning trajectory to gain experience and confidence – and become fully active members of the mathematics course. The trajectory involves moving through various levels of the community which can be broken into smaller sequences of tasks for students to engage in as they pass through the course and become legitimate participants in the mathematics community.

NETWORKING TOC AND COP TO INFORM AN INTERVENTION

Historically, ToC has been used to analyse observation of (university or school) classroom interactions (Nardi et al., 2021). From the above results we can see the potential for lecturers to shift their pedagogical discourse towards attending to student needs. However, whilst deciding on the theoretical framing of this course intervention their appeared to be a lack of directionality. By directionality, we mean that to understand if the intervention has worked, we need a theory that allows us to define where the course is starting, where we would like students to end and how we plan on getting them to that end point. It is this element of directionality that we sense a CoP lens will help us with. Having selected a course on Sets, Numbers and Proof (SNP) the intervention will explore how a FC pedagogical innovation facilitated by digital resources can foster a rich active learning environment. To design this course intervention, we plan to utilise the strengths of both ToC and CoP, slotting them together like two pieces of a jigsaw puzzle. Using CoP concepts, we can build a structural image of our SNP course by defining the elements of mutual engagement, joint enterprise and shared repertoire (Wenger, 1998). These three concepts will provide us with a unique set of values that set SNP apart from other courses the students are enrolled upon. At first, newcomers (in this case the newly enrolled students) will engage with the SNP community peripherally which overtime shifts to more legitimate participation as they grow in confidence; also referred to as the process of legitimate peripheral participation. To gain this confidence to engage in the community we want to incorporate activities and tasks that prompt shifts in the student's mathematical discourse. This is where we will utilise the four characteristics of mathematical discourse - when planning the activities, we will look for opportunities to shift the mathematical discourse e.g., can we spark a discussion leveraging students' informal utterances to build a formal definition. Or perhaps, when presenting the concepts of injectivity, subjectivity and bijectivity to support communication of the definitions we utilise visual mediators. To organise these activities and tasks in a systematic way we once again appeal to CoP, particularly the concept of learning trajectories. The learning goals of the course can be further broken down into smaller unit by unit goals which will inspire a learning trajectory that the students will engage with throughout the semester. The learning trajectory will be guided by the meta and object level learning goals that define the community, and will be constructed utilizing active, inquiry-orientated tasks that encourage shifts in the student's mathematical discourse. To analyse the fine-grained elements of interactions as students engage with the intervention, we shall turn to the ToC. The four characteristics of mathematical discourse will be used to identify shifts in student discourse as they engage with the various activities and digital resources on the intervention. By noticing shifts in the student's discourse this will evidence to us whether the learning trajectory goals are being accomplished by the students (or not). Additionally, we can compare these shifts along the trajectory to determine whether the process of legitimate peripheral participation has occurred among students and if so, how close to the centre of the community have they come during the semester. Looking at both the shifts in discourse and their legitimate participation we can begin to understand how effective the course intervention has been on students' (and their lecturers') learning and teaching experience.

CONCLUSION

In this paper we have addressed two interrelated issues. Firstly, we have reported results from Hooper's MA dissertation (Hooper & Nardi, 2021) which highlighted the challenges for university mathematics teaching during ERT. Through a commognitive lens, we identified shifts in student and lecturer mathematical discourses. It is the presence of these shifts that inspired Hooper's ongoing doctoral study and its focus on designing a first-year course intervention. Secondly, we have seen how active, inquiry-based learning, facilitated by digital resources, forms the basis of this intervention whose theoretical underpinnings network CoP theory and ToC. At present, a co-design team is constructing an intervention on the SNP course drawing on these theoretical elements. Additionally, the team is drawing on the work reported in (Hooper & Nardi, 2021) and other relevant research to inform the design and implementation of the course intervention. The aim is for the course intervention to be ready to implement in the Autumn semester 2024.

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Insights into transitions to, across and from university mathematics that arose through consideration of mathematical identity

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In this paper, we present selected findings from the recently completed PhD study of the main author which is entitled Mathematical Identity of Science and Engineering students (MISE). The sample for the study included 32 students of science (including science education) and engineering from Dublin City University. A longitudinal research design facilitated the collection of data at the start, middle, and end of participants' programmes. Thus, the findings from this study of mathematical identity are expected to be of interest to those members of the INDRUM audience who are concerned with the lived experiences of students as they progress to, across and from university mathematics. For the purposes of this paper, participants from the science education and engineering programmes will be the focus of the discussion.

Keywords: Mathematical Identity, Transition to across and from university mathematics, Teaching and learning of mathematics in other disciplines, Pre-service teachers, Engineering.

INTRODUCTION

The aim of the MISE study was to explore the relationship of science (including science education) and engineering students with mathematics and to investigate how this relationship changed as they made the transition to university education (Howard, in press). Thus, the study addressed issues related to participants' navigation of Klein's first discontinuity (Klein, 1908/1939), meaning the transition from post-primary school to university, from first year their first year up until their final year. In this paper, we address the following research question: How does the relationship of these students with mathematics change over time?

Data was collected at three stages of participants' undergraduate studies, including in their fourth year as they prepared for the transition to their post-university careers. For those participants who entered the teaching profession, this transition is known as Klein's second discontinuity, which is more widely researched than the transition of engineering students to their workplace (Hausberger and Strømskag, 2022). We expect this study to be of interest to the INDRUM audience because the data reflect the lived experiences of students as they made the transition to, across and from university mathematics, and gives insight into the experiences that shaped the transition.

THEORETICAL BACKGROUND

Although a definition of mathematical identity is offered in this section, the authors agree with those of a recent review of the area (Radovic et al., 2018), who concluded that knowledge of how the definition was conceptualised and operationalised is required to fully understand the theoretical stance of a mathematical identity study (p.

32). To this end, mathematical identity is described first in the context of broad conceptualisations of the phenomenon that are common in the literature and subsequently, in the context of some theoretical dimensions of identity proposed by Radovic et al. (2018). Finally, the approach taken in MISE is presented.

In the most general terms, mathematical identity can be thought of as a student's relationship with mathematics, which evolves over time and guides the manner in which the student interacts with the subject. Grootenboer & Marshman (2016) explained the relationship as follows: "When students are learning mathematics they are simultaneously developing mathematical identities, and, their mathematical identities are enabling and constraining the way they are learning mathematics" (p. 116). Mathematical identity has been the subject of increased attention by researchers in recent years (Darragh, 2016, Figure 1). This has been attributed, in part, to the potential for identity to weave interrelated dimensions together, including "beliefs, values, attitudes, emotions, dispositions, cognition, abilities, skills and life histories" (Grootenboer & Marshman, 2016, pp. 27-28). Investigating the mathematical identity of pre-service teachers has proved popular because of the influence of teachers' identities on their teaching practice (Goldin et al., 2016, p. 14). For other students who study mathematics, identity can highlight issues that contribute to marginalisation, and thus influence whether students' continue, or not, to study and engage with mathematics (Solomon, 2007).

In their review of mathematics learner identity, Radovic et al. (2018) stressed that rather than choosing between a social or subjective emphasis, studies in identity often address the interaction or tension between the two (p. 27). The MISE study is best described as a narrative study which emphasises subjective aspects of identity (students' self-concept and their interpretation of their own experiences), but also allows the social aspects of identity to emerge through participants' narratives (how participants manoeuvre themselves within the possible identities which are made available to them) (Radovic et al., 2018, p.29; Skott, 2019, p. 470). Mathematical identity was defined as the "multi-faceted relationship that an individual has with mathematics, including knowledge, experiences and perceptions of oneself and others" (Eaton & OReilly, 2013, p. 280). This definition allowed the pursuit of a narrative approach to identity because students' subjective self-concept as well as the social world around them were explicitly included in the definition. Consistent with a qualitative paradigm, rather than seeking to generalise the results of the study, one should consider whether the "thick description" provided by the author establishes trustworthiness and transferability of the results (Guba & Lincoln, 1985, pp. 124-5).

Connections between mathematical identity and transition

Although various definitions of identity are offered in the mathematics education literature, identity is usually positioned as a fluid and constantly renegotiated phenomenon, a so-called Meadean viewpoint (Darragh, 2016; Goldin et al., 2016). It has been argued that thinking of transitions in terms of identity may encourage students to see the challenges they have encountered as "troubles overcome in their rite of passage" and "as an affirmation of who they are now" (Hernandez-Martinez et al., 2011, p. 128).

Narrative studies have often paid attention to points of change or transition which have appeared under various names: turning points, leading activities, critical events, core events/episodes. The use of such transition points for analysis is aided by participants telling a full sequential story of their experiences with mathematics. The popularity of autobiographical methods of data collection, often through interviews (Goldin et al., 2016), can be explained by recent research (past 20 years) into analytic approaches relying on points of change/transition, and more generally on the desire to infuse a temporal property into definitions of mathematical identity (past events are seen from the perspective of the present) (Kaasila, 2007, p. 212).

The longitudinal research design of MISE allowed the investigation of changes in mathematical identity over time and the experiences that affected students' transition to, across and from university mathematics. In other words, mathematical identity can be seen as a lens through which we can investigate the lived experience of students as they navigate such transitions and put mathematics into action through teaching or industry placement (in science education and engineering programmes respectively).

METHODS

This longitudinal study comprised of three stages of data collection and analysis, involving thematic analysis of questionnaire and focus group data followed by a narrative analysis of interview data. The data was collected in students' first year (semester II), third year (semester I), and fourth year (semester II). A summary of the research design is presented in Table 1.

Stage of data collection	Collection method	Sample size	Date collected	Type of analysis
Stage 1	Questionnaire (Q)	32	March 2018	Thematic analysis
Stage 2	Focus groups (FG)	6	October 2019	Thematic analysis
Stage 3	Interviews (I)	6	March 2021	Narrative analysis

Table 1: Stages of data collection and analysis in MISE.

The questionnaire for the study was adapted from that developed by Eaton et al. (2013) and consisted of three open-ended questions: a broad opening question, a follow-up question which included some prompts, and a final evaluative question (more details in Howard, 2023, p. 260-262). The second stage of data collection consisted of two focus groups each with the aims to elaborate on the themes developed from the first stage, clarify some researcher interpretations, and examine changes in mathematical identity over time. Participants' questionnaire and focus group responses were analysed through thematic analysis using inductive codes, which were derived from the data, and deductive codes which were developed from the literature and a pilot study (see Howard et al., 2019 for further details of this stage of data analysis).

The third stage of data collection consisted of narrative interviews in which participants were invited to recount their experiences of university mathematics year-by-year, consider and visualise the key events which influenced the journey, and compare their past and present mathematical identities with reference to their questionnaire responses from first year (more details in Howard, 2023, p. 72-75). Narrative analysis of the data mainly drew from an approach entitled "storying stories" (McCormack, 2004) to identify the stories that participants told about their experiences with mathematics and generate a personal narrative (or interpretive story) for each interview participant. The storying stories framework, along with the adaptations implemented in MISE, have been presented elsewhere by Howard et al. (2021). Thematic analysis of the first two stages of data collection permitted the development of themes which hold across the participants' data, while narrative analysis of interview data allowed attention to the individual identities of a selection of participants through their personal narratives.

FINDINGS

In a previous conference publication, the authors of this paper have presented MISE participants' resistance to teaching which promotes instrumental understanding of mathematics (Skemp, 1976, p .20), and their stated desire to "understand" mathematics (Howard et al., 2022). In the remainder of this paper, we will discuss three findings related to participants' transition to, across and from university mathematics respectively. Quotations from students of science education are denoted SE01, SE02, etc. and those of engineering students are denoted EN01, EN02 etc. The suffixes Q, FG, and I indicate from which stage of data collection the quotation is drawn.

University mathematics requires hard work for everyone

Several MISE participants described struggling with mathematics for the first time at Senior Cycle in post-primary school (years 4-6) or in university, relying on the idea of "natural ability" to explain their, sometimes effortless, success in mathematics at earlier stages of schooling. SE01 described having a "mental block" against the "quite abstract" linear algebra module he encountered in second year at university: "[I] told myself that I couldn't understand it … it was kind of the first time that that happened … even in school" (SE01_I). Another participant referred to the transition from post-primary school to university (third level): "When I entered third level, I expected I would be prepared from the leaving cert. However, I was very very wrong" (SE03_Q). In accordance with other research from the UK (Hernandez-Martinez et al., 2011), MISE participants who described mathematics as requiring minimal effort referred only to primary school or Junior Cycle in post-primary school (years 1-3):

> "I didn't study for the Junior Cert in Maths because it seemed common knowledge" (EN03_Q).
>
> "[W]hen I entered senior cycle my relationship with maths completely changed. I went from being bored by how easy maths was to really struggling with it" (SE03_Q).

It appeared, therefore, that participants were acutely aware of the increased difficulty associated with Senior Cycle and university mathematics. In first year at university, many MISE participants indicated that they supported the belief that university mathematics requires hard work from everyone, regardless of any natural ability that may have given them an advantage in post-primary school. Furthermore, as the study progressed, participants' endorsement of this belief appeared to strengthen. In the focus groups, SE05 and SE04 agreed that outsiders often dismiss their hard work, believing that they must be effortlessly good at mathematics:

> "People will say 'oh you're always good at maths' sort of thing. We've all heard that probably, but they don't say it about history or geography ... I just think you need to put in as much work to maths as you do for all the other subjects" (SE05_FG).

Solomon (2007) identified a dominant belief in the undergraduate community, that those who are good at mathematics are effortlessly successful, and that "you can either do it or you can't" (p. 89), while Boaler (2016) reported that ""students who are successful through hard work often think that they are imposters" (p. 148). MISE participants rejected this viewpoint, and there was widespread agreement that while they might have been able to rely on natural ability before now, in university everyone needs to work hard to be successful in mathematics.

Collaboration

In the first stage of data collection, there were no indication that MISE participants collaborated with each other when studying university mathematics. Although it was not expected a priori that collaboration would be a main feature of participants' mathematical identity, it was decided to investigate how this evolved as participants moved further along the transition to university mathematics. In subsequent stages of data collection, the development of collaborative groups over time was revealed for both science education and engineering participants (Howard et al., 2022). Engineering participants described how they were routinely required to work as part of a team on group assignments and projects, and they established friend groups to work on assignments together in the library:

> "Definitely with assignments. … There's a particular few people in my friends group I'd say, that, we'd always kind of compare answers, just to see if we were getting them right. And if we got, had the same answers we knew we were right" (EN01_I).

EN02 agreed that "A lot of people formed friends so they could figure stuff out in the library" particularly for the engineering mathematics modules in first and second year: "Maths in particular was actually one that kind of brought the year together, I think" (EN02_I). EN02 elaborated that she sees the ability to collaborate in this manner as an essential quality of an engineer:

> "Since first year, everyone has kind of said engineering is one of those degrees that you need to be able to talk to people about to actually pass and do well in. Because no one person has all the information. ... Then one maths question you end up under-

understanding a lot more than just one person standing up and kind of teaching you that way" (EN02_I).

Science education participants similarly described working together by the end of their programmes, but collaboration among the group seemed to develop more slowly compared to engineering participants. In the focus groups, SE01 and SE05 explained that they collaborated only a little with their classmates, particularly for one module which required them to answer mathematics questions through an automated online system. However, in the interviews in their fourth year, science education participants described that they "did the assignments together" (SE02_I) as well as practice exam papers, and even "just general, kind of keeping up with content of modules" (SE04_I). SE03 further explained that they would coordinate via text to study and attend, as a group, the Mathematics Learning Centre (MLC) in the university library.

Placement and work experience

For many students, placement and work experience offer them their first insight into the practical, day-to-day details of their future career opportunities, as was the case for two engineering and two science education interview participants in MISE (EN01, EN02 and SE01, SE02 respectively). These experiences informed students' expectations about Klein's second discontinuity: transition from university mathematics to the workplace. In the case of pre-service teachers, like those in the science education programme, this is also referred to as a "double discontinuity" as their workplace is a post-primary school (Klein, 1908/1939).

SE01 described a transformation of his perspective from studying a science programme and doing "maths for the sake of maths" with "no end in sight," to a focussing on education and thinking "this is going to help me [in the classroom]" (SE01_I). He presented teaching placement in third year as a catalyst which "changed my thinking about the degree on a whole, as well as attitudes towards what I was doing" (SE01_I). By focussing on putting knowledge into practice through teaching placement, he realised that "some of the stuff you could nearly pick from the third year modules and actually bring it into a fifth year class" (SE01_I). SE02 similarly appreciated teaching placement because he could work with teachers who were experienced in the practicalities of post-primary teaching, and appeared to believe that this provided an important platform for his development at that stage of his studies:

> "The woman I'm working with she's a year head, and she has what, 40 years of experience, and it really shows in the sense that, she's old time right, she's been there for a long time, she's been there through way before Project Maths, then phasing in Project Maths. … and her experience clearly shows" (SE02_I).

Perhaps influenced by teaching practice, SE02 expressed relief at completing his final pure mathematics module in second year (he did not specialise in mathematics subsequently) and appreciated later modules which focussed on pedagogical content knowledge, adding that it was "not even theoretical it's very practical. That you can literally bring this into a class and use it if you need to" (SE02_I). He provided a second year module (Teaching and assessing Junior Certificate mathematics) as an example: "They build upon the initial methods, and they get to the final method, which works amazing and you don't need to think about how you did it at the beginning … that module is about building that back up" (SE02_I).

SE01 and SE02 disagreed with regard to the connections between their university programmes and their real-world practice during teaching placement. S01 appreciated the "deeper understanding" of Leaving Certificate mathematics that his early university modules provided and described that they were "given tools to verify or prove theorems" rather than practicing specific methods to answer particular questions, as was the case in his Leaving Certificate class. His descriptions of this deeper understanding resonated with Skemp's characterisation of relational understanding as "knowing what to do and why" (Skemp, 1976, p. 20; cf. Howard et al., 2022). On the contrary, S02 did not choose mathematics as a specialty in third year, and appeared to position himself as a good teacher relative to his classmates precisely because he had a lesser knowledge of, and had remained untainted by, advanced mathematics. Instead, he believed that it was easier for him to "get back to basics" with the post-primary school mathematics curriculum:

> "[I]t almost backfired, having us do maths in college, because we have gone so far, and I can only imagine how much like people who did maths as a subject [speciality] ... I wonder if they go back to teach will they struggle with it more than I would. Because I haven't gone that far" (SE02_I).

Although both EN01 and EN02's work placements were cancelled in third year due to the COVID-19 pandemic, they both appreciated the real-world experience they had encountered in summer internships. EN02 appreciated being given the responsibility and agency to design her own products and solutions to problems, using what she had seen in lectures:

> "I think internships are a real good one for being like, oh ok, this that we're seeing in college, it's being applied. … I was just presented with a problem, and they were like 'fix it.' What are you going to do to fix it? … it was eye-opening to see that the stuff that you're learning is actually, it's in practice as well" (EN02_I).

Both engineering interview participants saw their summer internships as formative for judging the role of mathematics in their future careers, but, interestingly, neither felt that they used mathematics in their real-world practice, at least not directly. EN01 said "it is maths but it's not what I would think of maths" and EN02 concurred that "[it's] not just maths alone, definitely not." EN02 clarified her comments with an example to explain why she felt that an engineer's role in the workplace is more akin to what she called problem-solving:

> "There was a machine that wasn't working properly, and you had to understand like the actual mechanics of it to see what was happening, and it was that kind of problem solving. But I suppose like, the base layer of all of that stuff is maths" (EN02_I).

Goold (2015) showed that engineering students in another Irish institution perceived a gap between their academic learning and their workplace practice, which was attributed in part to the dominance of students' "objective" mathematical learning in class compared with the "subjective" mathematics competencies that are necessary in the workplace (p. 552). Indeed, we found that MISE participants' mathematical identities were built around absolutist views of mathematics, wherein mathematics is seen as pre-existing, immutable, and objectively true (Ernest, 1988), yet mostly believed that its purpose is realised through application to real-life situations.

CONCLUSION

This paper presented selected findings from the main author's PhD study into the mathematical identity of science (including science education) and engineering students in Dublin City University (Howard, 2023), and in particular, the research question "How does the relationship of these students with mathematics change over time?" In this paper, selected findings for this question were presented in three parts, which addressed the transition to, across and from university mathematics respectively.

Firstly, we found that participants believed that while they might have been able to rely on natural ability before now, in university everyone needs to work hard to be successful in mathematics. In part, this finding corroborates research from the UK which reported that while students felt that they could achieve good grades at GCSE level with little effort, A-level mathematics required more hard work and understanding (Hernandez-Martinez et al., 2011). For MISE participants, a natural ability allows one to be successful in mathematics with less effort, but Leaving Certificate and University mathematics do not exhibit the same vulnerability as earlier stages of schooling. Secondly, although in first year participants appeared to see mathematics as an individual endeavour, collaborative groups developed over time. This development seemed to occur more quickly for engineering students, who banded together to work on mathematics modules, group projects, and assignments.

Thirdly, science education participants appreciated the value of teaching placement for several reasons. For SE01, placement inspired a change in identity from a scientist-in-the-making to a teacher-in-the-making. He appreciated the "horizon knowledge" provided to him by his mathematics modules and the direct relevance to teaching practice of his education modules in later years. SE02 expressed a similar appreciation of modules which directly helped his classroom practice but explained that he considered it easier for him to "get back to basics" with Leaving Certificate mathematics precisely because he had not studied as much advanced mathematics as those who specialised in the subject. Absolutist views of mathematics were observed among MISE participants generally, and EN01 and EN02 appeared to present evidence that their real-world practice was closer to what they called "problem-solving" (applying or implementing mathematics) than to their absolutist conception of mathematics. This finding resonates with the pronounced divide between classroom mathematics and its real-world applications which was reported by another study in Ireland (Goold, 2015). Our analysis suggests that the nature of this divide may be explained by students drawing a line between absolutist mathematics and their own conception of problem-solving in real-world engineering work. Such results are important because it is argued in the literature that structural engineers need to decide "how, when, and when not to use mathematics" (Gainsburg, 2007, p. 500), but that the in-school perception of mathematics as immutable might constrain the development of such a disposition among students.

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How do Q-A logs support inquiry-based teaching? A case of a mathematical modelling course

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In the situation of inquiry-based teaching, teachers are supposed to play the role of facilitators and should intervene appropriately and at the right time. To do this, they should be able to monitor the status of students' inquiries in real time to some extent. To this end, we expect the Q-A log to be a useful tool for teachers, where the Q-A log is a simpler variation of the Q-A map from which the tree structure has been removed, leaving only a list of Q-As. A case study of a mathematical modelling course at a Japanese university showed that Q-A logs supported the teacher's intervention in inquiry activities. In particular, the teacher's way of using Q-A logs accelerated students' inquiry processes by promoting the groups to generate subsequent questions and by diffusing questions generated by one group to the other groups.

Keywords: Teachers' and students' practices at university level, assessment practices in university mathematics education, modelling task, teacher's feedback, in vivo analysis.

INTRODUCTION

Recently, educational needs for so-called inquiry-based teaching and learning have been increasing from primary school to university around the world. This can be described as the emergence of a 'new paradigm' happened in our didactic worldview (cf. Chevallard, 2015). However, our resources for realising such a teaching process still maintain properties optimised for the old paradigm basically: curricular programme, school equipment, time management for teaching, group composition of a class, assessment tools, the professional capacity of teachers, and so on. Here, we can recognise the *transitional tension between the forthcoming paradigm and the prevailing infrastructure*.

In this paper, we introduce our teaching device, which we call the Q-A log, for supporting teachers who want to conduct inquiry-based courses at a university. It is a minimalist variant of *questions and answers map* (Q-A map; cf. Winsløw et al., 2013) formulated in the ATD, the anthropological theory of the didactic (cf. Chevallard, 2019). The Q-A map was originally proposed as a technical tool for designing inquiry-based didactic processes beforehand, *a priori analysis*, and/or for describing them afterwards, *a posteriori analysis*. On the other hand, as we will explain later, the Q-A log can be useful in its 'real-time' use during the inquiry, *in vivo analysis* [1]. Indeed, for many teachers, the in vivo analysis of students' activities in their inquiries is problematic because the dominant teaching format is still based on the traditional system of a community of many students and a team of a few teachers conditioned by the old school paradigm. Such compositions make in vivo analysis difficult due to the essential diversity of inquiry paths, but we have no choice other than being based on them under the circumstances. In our view, the Q-A log has the potential to defuse this difficulty of inquiry-based teaching in the transitional period.

QUESTIONS AND ANSWERS LOGS

Within the framework of the ATD, an inquiry is a human activity of a generic type, that is, to study a generating question Q_0 to bring an acceptable, final answer $A^♥$ to it into being. Moreover, its process is accelerated by *dialectics of inquiry* of different kinds: e.g., the dialectic of media and milieus (cf. Chevallard, 2020). In particular, the *dialectic of questions and answers* is often recognised as principal. In fact, among the popular tools in the ATD is the Q-A map, which is tightly connected with this dialectic. We, too, consider the dialectic of questions and answers as a chief dialectic in inquiry, so that we decided to focus on a variation of Q-A maps, which we call *Q-A log*—any list of *Q* and *A*—as a descriptive tool *not for researchers but for students* (Kawazoe & Otaki, 2023a). On the one hand, following the hypothesis of Barquero and Bosch (2022), we assume that some ATD tools, especially the Q-A map, are useful for students to reflect on their own mathematical inquiry. However, such tools are basically not easily available for them because of their complexity. For instance, Q-A maps usually have tree structures, thereby making explicit not only the flow of didactic time but also the knowledge organisation in the inquiry. Such functionality is complicated for students, even if it is crucial for researchers. Then, we deliberately remove the tree structure from the Q-A map.

The Q-A log is simply a list of *Q* and *A* without the tree structure, which makes it easier for students to record and reflect on their work (Kawazoe & Otaki, 2023a). In addition, we conjecture that it can be a useful tool for teachers in teaching and assessing inquiry activities during the inquiry. In the process of students' inquiry, teachers should play the role of facilitators or supervisors who intervene appropriately at the right time. To do this, they need to be able to monitor the status of students' inquiries in as real time as possible. We expect that Q-A logs will enable teachers to carry out *in vivo analysis*, which will enable them to give feedback without much delay, because the records made in the form of Q-A logs will allow teachers to grasp the status of students' inquiries in a less time-consuming way. Based on this idea, this study examines the functionality of the Q-A log in an inquiry-based course that focuses on mathematical modelling.

METHODOLOGICAL SETTING

Basic method

This study uses a mathematical modelling course at a Japanese university as an inquiry-based course. Implementing Q-A logs into the course, we collected Q-A logs and data on the teacher's interventions. Details of the course, the modelling task on which we focus, and the data on the teacher's intervention are explained below. The collected Q-A logs were compared with the data on the teacher's intervention. Then, we identified which students' questions were addressed by the intervention and analysed what impact the intervention had on the students' inquiry activities by identifying the questions and answers from the Q-A logs that were affected by the intervention. With these results, we discuss the role and potential of Q-A logs in inquiry-based teaching.

Design of the course, students' activities, and teacher's intervention

The mathematical modelling course to be reported was designed by the first author as inquiry-based. It is a 2-credit, 1-semester course that meets for 90 min per week for 15 weeks. The course is not compulsory and is open to all students. Students are invited to participate in three mathematical modelling tasks during the semester. The viewpoints of the mathematical modelling cycle and the Q-A log, which are hypothetically relevant for students, are introduced to students before modelling activities (Kawazoe & Otaki, 2023a). They work in groups on each task. They record their modelling activities in the form of Q-A logs and submit group reports. Students are also asked to submit their individual reflection comments each week. Two different types of teacher's intervention are designed to facilitate students' modelling activity: one is to give comments directly to a group during the modelling activity in the classroom; the other is to give feedback to the whole class at the beginning of each week's lesson. In the second type of intervention, the teacher's feedback is based on the Q-A logs and the students' individual reflection comments submitted in the previous week. To condition adidactic situations, the teacher plays the role of facilitator, and the teacher's intervention is carefully controlled and minimised so as not to over-teach the students.

The modelling task: Halving the population of deer and wild boar

In this study, we focus on the first modelling task of the above course, which was conducted in the autumn semester of the academic year 2023. The three same modelling tasks were used in academic years 2022 and 2023. For the other two tasks, see (Kawazoe & Otaki, 2023b). Here is the first modelling task, which we call as '*the halving task*' in the following.

> **Halving the population of deer and wild boar**
>
> Wild animals cause serious damage in many parts of Japan. Agricultural damage reaches around 20 billion yen annually. As of FY2011[2], the number of deer and wild boar in Japan was estimated at 3,250,000 and 880,000, respectively. If nothing is done, the number of deer will increase at a rate of approximately 20% per year. The Ministry of the Environment and the Ministry of Agriculture, Forestry and Fisheries have set a goal of halving the populations of deer and wild boar by FY2023. Local governments are stepping up their catching activities. [An English summary of "Damage by wildlife: How to reconnect with people" (2015) by the authors. [3]]
>
> (1) At the time the article (Damage by wildlife: How to reconnect with people, 2015) was written, what specific measures could be taken to achieve the goal within the deadline? Make a plan that includes numerical targets.
>
> (2) Examine the information available at the present time and discuss the future prospects in light of the plan that you made in (1).

This task lasted for four weeks, including a 90-minute reflection. Students are divided into groups of three to four. Students can use any digital devices (laptops, smartphones, scientific calculators, etc.), the internet connection, and the computer algebra system *Mathematica* in the classroom.

Useful pieces of information for performing this modelling task can be found in the newspaper article: the number of deer and wild boar in FY2011 was estimated at 3,250,000 and 880,000, respectively; the government aims to halve the population of these animals by FY2023 through catching; the number of deer increases at a rate of about 20% per year in the natural environment without any intervention. For each of the two animals, details of past population estimates and the numbers caught each year can be found in government documents on the Internet. The rate of natural increase in wild boar is not described in the newspaper article, but it is also found in government documents. This task can be approached by creating a mathematical model that represents the change in the population of deer or wild boar from year to year. There are various possible mathematical models for this task. Some of them are obtained as numerical sequence models, putting a_n the number of deer or wild boar in the nth year from the starting year: a numerical sequence model with the recurrence relation $a_{n+1} = pa_n - q$ or $a_{n+1} = p(a_n - q)$ with the natural rate of increase p and the annual catch q as parameters; a geometric progression model $a_{n+1} = p(1 - r)a_n$ with the same p and the proportion of the annual catch r. Other possible models include linear approximation models of population trends based on historical data. More complex models can, of course, be constructed (cf. Croft et al., 2020).

Data collection

The data of students' modelling activity and of teacher's intervention for *the halving task* were collected in the course described above, which was conducted in the autumn semester of the 2023 academic year at a Japanese university. The course was taught by the first author without any teaching assistant for teaching and preparing the course. The participants of the course were first- and second-year students whose majors were engineering, agriculture, computer science, psychology, and so on. Eighty participants were registered in the class, 50 of whom were engineering students. Almost all of them were first-year students, and only five were second-year students. The participants were divided into groups of four, and 20 groups were formed in advance. However, one of the 80 participants had never attended the class, and hence one of the 20 groups finally consisted of three students. For most of them, this was their first time experiencing mathematical modelling in the course.

During the four weeks of modelling activity for *the halving task*, the Q-A logs of 20 groups were collected in digital format (PDF or Word) using the Moodle learning management system. Students were asked to put the two questions (1) and (2) in the task as Q_0 and $Q_{0\text{-}2}$ in their Q-A logs; hence, their logs start from Q_0. For the subsequent questions that arose in the modelling activities, the students were asked to number them in order of time, writing them down as Q_1, Q_2, ...; if questions occur in parallel at the same time, write $Q_{2\text{-}1}$, $Q_{2\text{-}2}$, etc.; for an answer A to a question Q, write A with the same subscript as Q (e.g., A_1 for Q_1); if there is more than one answer to a question Q, then add sub-numbers (e.g., $A_{1\text{-}1}$, $A_{1\text{-}2}$, ... for Q_1). The students' individual reflection comments and the groups' reports were also collected via Moodle. As the data of the teacher's intervention, we collected the slides that were used to give feedback to the whole class each week. The majority of the teacher's feedback to the students was provided through these slides. We use these data as teacher's intervention data and omit other minor interventions in our analysis.

RESULTS AND DISCUSSION

Preliminary analysis of the student's modelling activities

Before moving on to an analysis of the teacher's intervention, we examine the students' modelling activities as observed in the Q-A logs and the groups' reports. Regarding the Q-A logs, following the same manner in (Kawazoe & Otaki, 2023a), we performed deductive coding with each pair (Q; A) of Q and A as the target unit of coding, using the list of the seven steps of Blum's modelling schema (Blum & Leiß, 2007) as the code list: 1) understanding, 2) simplifying/structuring, 3) mathematising, 4) working mathematically, 5) interpreting, 6) validating, and 7) exposing. (Q; A) is possibly in the form (Q; ∅) in the case without any answer or (Q; A_1, A_2, …) in the case with multiple answers. More than one step is assigned to (Q; A) if it involves several steps. There were 291 pairs of (Q; A) and 476 steps corresponding to the pairs (Q; A). Table 1 summarises the frequency (%) of each step identified in the Q-A logs of all groups. Table 1 shows that the process of the students' modelling was most activated in Step 2, the *simplifying/structuring* step. The distribution of the frequencies in Table 1 was very similar to the result of our previous study (Kawazoe & Otaki, 2023a), where we investigated students' modelling activities using Q-A logs for the same modelling task in the autumn semester course of the 2022 academic year. We found that Step 2 was the most frequent step in every group.

Step in Blum's schema	1	2	3	4	5	6	7
Frequency of the step	1.5	49.4	17.2	13.2	9.9	6.9	1.9

Table 1: Frequency (%) of each step in Blum's schema corresponding to pairs (*Q*; *A*)

Regarding the mathematical models developed by the students, the three types of number sequence models ($a_{n+1} = pa_n - q$, $a_{n+1} = p(a_n - q)$, and $a_{n+1} = p(1 - r)a_n$) mentioned in the previous section were observed in the group reports. Some groups modified their first models by replacing the annual catch q with a linear function on n or by a constant multiple of q. Most of the groups used 1.2 as the value of p for deer, but one group used their own estimate. Linear models ($a_n = c–dn$) and a model using a differential equation were also observed.

The teacher's intervention

During the four weeks of modelling activities on *the halving task*, the teacher gave instructions and feedback to the whole class at the beginning of each weekly lesson.

The teacher's slides used in these four weeks showed that the teacher's feedback, based on the Q-A logs, was only given in the second week. Feedback on students' individual reflection comments and general comments on the construction of mathematical models were also provided in the second week. In the first week, only the instructions for the task and the initial questions Q_0 and $Q_{0\text{-}2}$ were given. In the third week, only additional instructions on the task and comments on how to write a group report were given. The lesson in the fourth week was dedicated to reflection; therefore, the slide used this week had no interaction with the students' modelling activities.

In the second week's slide, 13 questions, some of which were presented with answers, and 2 answers were selected by the teacher from the Q-A logs produced in the first week. Excluding the questions and answers picked up to explain the method of producing Q-A logs, 10 questions and 1 answer were presented in the following four categories: (a) questions about the rate of natural increase of wild boars (3 questions); (b) questions about the order of breeding and hunting (4 questions); (c) questions about the population of deer and wild boars as of 2015 (2 questions and 1 answer); and (d) a question about whether parent deer or fawns are killed (1 question). All of these questions and answers relate to important factors that should be considered when mathematising and correspond to Step 2 (simplifying/structuring) of Blum's schema. That is, the teacher's feedback on Q-A logs was concentrated on this modelling step. This is consistent with the fact that the students' questions were concentrated on Step 2.

Interaction between students and teacher via Q-A logs

Let us now look at the dialectical interplay between the students' Q-A logging and the teacher's feedback to it pertinent to each of the four *Q-A categories* (a)-(d) above. In Figures 1-4, we describe the interaction for each of (a)-(d) with three rows: the top row shows the students' *Q*s and *A*s that were mentioned in the teacher's slide used in the second week; the middle row shows the teacher's feedback on the *Q*s and *A*s in the top row; and the bottom row shows the students' *Q*s that occurred after the teacher's feedback. The 20 groups of students are described as G_1, G_2, …, G_{20} below. In the following, the students' Q-A logs and the teacher's reaction to them are shown in English, which is our translation from Japanese.

Figure 1 shows the interaction starting with the Q-A category (a): questions about the rate of natural increase in wild boar. The teacher presented the three questions in the top row of Figure 1 to the class and gave feedback about how to find the data of the increase rate and how to estimate it. The bottom row shows all the questions on the rate of natural increase in wild boars that appeared in the students' Q-A logs after the teacher's feedback. Groups G_3 and G_{17} did not question the rate of natural increase of wild boar in the first week, but Q_6 and Q_7 in Figure 1 appeared in their Q-A logs in the second week. Group G_2 questioned it in the first week, as shown in the top row with a grey box, and in the second week, they questioned their estimate obtained in the first week in response to the teacher's feedback.

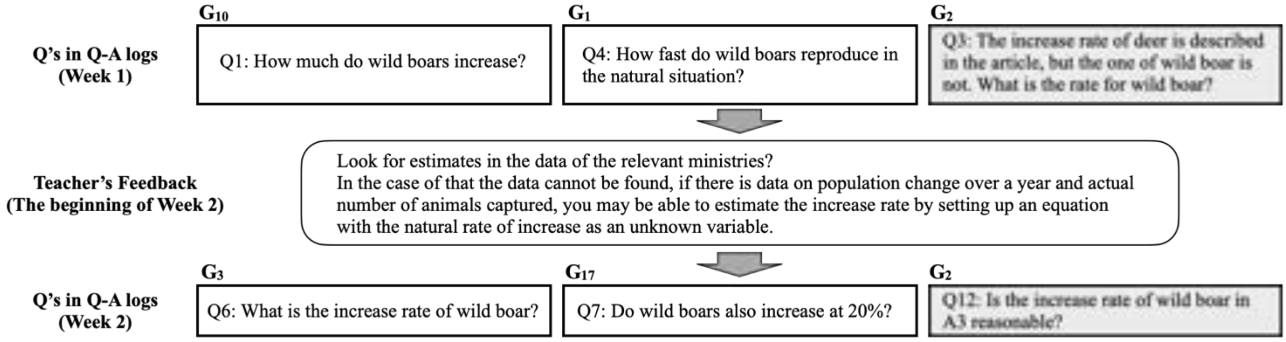

Figure 1: Interaction caused by the teacher's feedback on the Q-A category (a)

Figure 2 shows the interaction starting with the Q-A category (b): questions about the order of breeding and hunting. The teacher presented the four questions in the top row of Figure 2 to the class and gave feedback. As one question in the top row mentioned the breeding season for deer but did not mention the hunting season, the teacher asked about the hunting season in his feedback. Also, the teacher suggested students to look at the relation between the breeding season and the hunting season. This feedback promoted students' modelling activity. Ten groups had not looked at a hunting/breeding season and their relation in the first week, but in the second week, questions about hunting or breeding season appeared in the Q-A logs of these 10 groups. Group G_{20} questioned the breeding season for deer in the first week, as shown in the top row with a grey box, but the group did not question the hunting season. In the second week, a question about the hunting season appeared in the Q-A log of this group (a grey box in the bottom row in Figure 2).

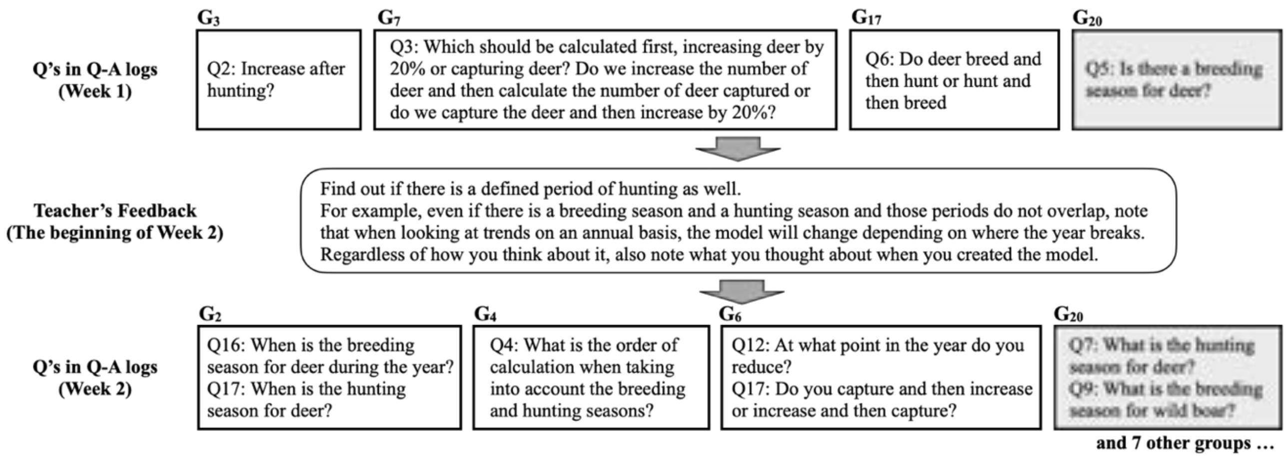

Figure 2: Interaction caused by the teacher's feedback on the Q-A category (b)

Figure 3 shows the interaction starting with the Q-A category (c): questions about the population of deer and wild boar as of 2015. The teacher presented the three questions in the top row of Figure 3 to the class and gave feedback. The teacher gave the information that the Ministry publishes the estimated number of deer and wild boar every year, but with a delay of two years. The teacher also reminded the class that hunting has been going on since before 2015. The teacher's feedback prompted the group G_8 to reconsider how to estimate the population in 2015. Group G_2 (shown in grey boxes) had already questioned the population in 2015 in the first week, and the group raised a further question in response to the teacher's feedback.

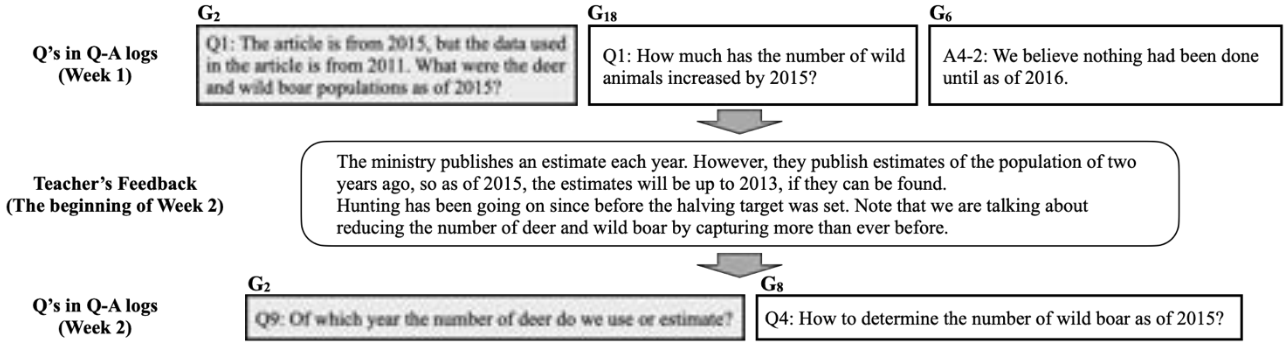

Figure 3: Interaction caused by the teacher's feedback on the Q-A category (c)

Figure 4 shows the interaction starting with the Q-A category (d): a question about whether parent deer or fawns are killed. The teacher picked up Q_3 of G_{17} and shared it with the class. The teacher's feedback was very simple. He just asked all the groups to discuss whether they would consider it in detail or not. Groups G_2 and G_{19} had not raised such a question in the first week, but in the second week, these two groups considered it in response to the teacher's feedback.

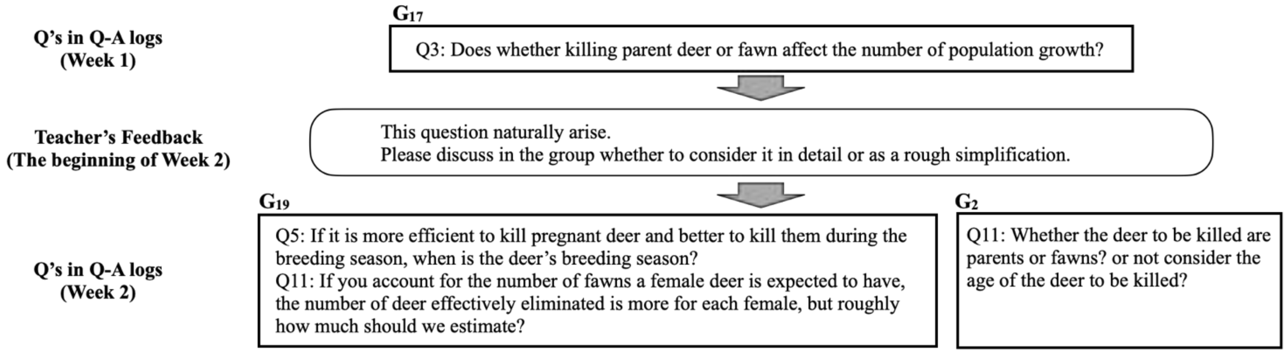

Figure 4: Interaction caused by the teacher's feedback on the Q-A category (d)

Let us summarise the teacher-student interaction mediated by the Q-A logs observed in Figures 1-4. The teacher's feedback, in the form of picking up Qs from the Q-A log and encouraging further activities for the whole class, was observed to have two effects on the students' modelling activities: first, the group that raised the *Q*s picked up by the teacher generated their subsequent questions in response to the teacher's feedback (G_2 in Figures 1 and 3; G_{20} in Figure 2); second, the group that did not have the idea of the *Q*s presented in the teacher's feedback incorporated these questions into their questions (G_3, G_{17} in Figure 1, etc.). For the latter case, as shown in Figures 2 and 4, G_2 incorporated Q_{11}, Q_{16}, and Q_{17} from the questions of other groups. These three questions contributed to proceed their mathematical modelling process; Q_{16} and Q_{17} led to the interpretation and validation of the model in Q_{18}, whose answer A_{18} led to A_0 ($= A^{\heartsuit}$) to the initial question Q_0, while Q_{11} contributed to the setting of assumptions when producing the answer A_0. Thus, it can be said that the teacher's feedback helps facilitate the modelling activities of group G_2.

CONCLUSION

In our previous study (Kawazoe & Otaki, 2023a), we observed that the Q-A log supported students' mathematical modelling as a resource for managing their own activities. In contrast, the present study clarified that the Q-A logs supported teacher's interventions in the mathematical modelling course. In fact, it was observed that the teacher's feedback with Q-A logs indeed 'fed' the students' inquiry activities in the following two ways: first, by promoting certain groups to generate subsequent questions, and second, by disseminating questions of one group to the other groups. The first author, who was the teacher of the reported course, feels that the Q-A log allowed him to easily monitor students' inquiry processes. Although our study examined the teaching practice of only one teacher, the findings further strengthen the authors' belief that the Q-A log can be an effective didactic tool for the teaching of mathematical modelling, and even more generally, for inquiry-based teaching.

We have referred to *in vivo analysis* in the Introduction. Indeed, the analysis and intervention by the teacher's using of the Q-A log presented in this study seems not to be 'real time' in the strict sense, that is, taking care of the students *all the time*, at first glance. But that is actually an analysis *during* the modelling process, which takes four weeks. Quick analysis of the students' inquiry activities, which the Q-A log allows the teacher to do with, enabled appropriate feedback to the students before their work in the next class. From the perspective of such a long time-scale, the Q-A log functioned as a tool for 'in vivo' analysis well with no doubt. When we need more prompt analysis, digital tools can be helpful. One way to get it even closer to real time would be for teachers to share Q-A logs with students and monitor them in class, for example, by using online sharing tools. However, this is an issue for the future, including technical aspects.

NOTES

1. See also (Barquero & Bosch, 2015) for a priori analysis, in vivo analysis, and a posteriori analysis.

2. FY2011 refers to Fiscal Year 2011. Japan's fiscal year starts in April and ends in March of the following year.

3. There is also an English article on the same news (Wild animal population control, 2014).

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Transitioning to proof via writing scripts on the rules of a new discourse

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Through the lens of the commognitive framework, proof-based mathematics emerges as a distinct discourse, the transition to which requires special rules for endorsement of claims. We investigate newcomers' learning of these rules when being taught them explicitly. Our data come from academically inclined high-school students who took a special undergraduate course. The course assignment included typical proof-requiring problems and a scriptwriting task, asking students to compose a fictitious dialogue about a proof-related mistake. The analysis showed that while students' solutions to typical problems required rule implementation, the dialogues involved rule formulation and substantiation. In many cases, the students discussed the classroom rules, extending, elaborating, and specifying the teacher's formulations.

Keywords: Learning of specific topics in mathematics, novel approaches to teaching.

RATIONALE

Two leitmotivs emerge from our reading of mathematics education research on proof. One is concerned with how fundamental proofs are to mathematics and how paramount it is to engage students in proving throughout their mathematics education (e.g., Stylianides et al., 2017). This leitmotiv is reflected in some curricula (e.g., NCTM, 2000), which cannot be taken for granted considering the marginalization of the activity in the curricula of other countries (e.g., Hanna & Jahnke, 1996). The other leitmotiv pertains to how challenging proof is for newcomers. This finding comes from a broad variety of students, including school high-achievers and future mathematics majors (e.g., Stylianides & Stylianides, 2022). The two leitmotivs feed into long-standing research on the complexity of transitioning to proof-based mathematics. In this study, we explore a didactical innovation to support students in this transition.

Philosophers, mathematicians, and mathematics educators have been considering proof from different perspectives. We adhere to the social perspective, viewing proof as a human endeavor that is cognizant of the discipline and the community that practices it (e.g., Stylianides et al., 2017). In the words of Balacheff (2008, p. 502),

> The issue of truth and validity cannot be settled in the same way in everyday life, in law, in politics, in philosophy, in medicine, in physics or in mathematics. One does not mobilize the same rules and criteria for decision-making in every context in which one is involved.

The notion of *rule* is common in the area of proof and it is central to our study. In the third section, we ground the notion in the commognitive framework. In the meanwhile, it is sufficient to associate the rules of proof (*RoPs* hereafter) with canons proofs are expected to abide by, such as logical inference and generally accepted conventions of proof-writing (e.g., Durand-Guerrier et al., 2012; Selden & Selden, 2015). We focus on the rules written proofs are expected to follow while acknowledging that the process of proof-seeking does not necessarily follow the same rules.

Kitcher (1984) notes that conventional rules of a mathematical practice may not be visible to mathematicians:

> Ideas about how one does mathematics may simply be included in early training without any formal acknowledgment […]. It is usually at times of a great change that metamathematical views are focused clearly, in response to critical questioning. The metamathematics of a practice is most evident when the practice is under siege (p. 163).

Mathematics education research echoes this view by arguing that some aspects of proof often remain tacit in instruction (e.g., Stylianides & Stylianides, 2022). Accordingly, the lion's share of proof research can be reframed as students grappling with RoPs that have not been presented to them as explicitly as they could be (e.g., Dreyfus, 1999). This grappling has been extensively studied in the context of the transition from secondary to tertiary mathematics – a time of great change, where many familiar mathematical practices are under siege (e.g., Gueudet, 2008).

Some scholars argue that the shift to proof-based mathematics requires a fundamental change in the rules underpinning key mathematical activities, such as justifying the validity of mathematical claims (e.g., Stylianides & Stylianides, 2022). Kjeldsen and Blomhøj (2012) maintain that developing appropriate rules for these activities is indispensable for mathematics learning. Sfard (2008) posits that modifying the rules that govern students' mathematical discourses is an educational goal. Then, we propose that a direct engagement with RoPs may be of didactical value for newcomers to proof-based mathematics. We investigate this proposal in this study.

This study aims *to characterize RoPs that students develop after having been explicitly introduced to them as part of the transition to proof-based mathematics*. Specifically, we focus on how newcomers to proof formulate, explain, justify, and implement RoPs.

LITERATURE BACKGROUND

Students' challenges with rules of proof

A "solid finding" in mathematics education is that "many students rely on validation by means of one or several examples to support general statements, [and] this phenomenon is persistent in the sense that many students continue to do so even after explicit instruction" (ECEMS, 2011, p. 50). Another occasionally reported issue pertains to logical circularity. Pinto and Cooper (2019) shared a case of a student, who relied on a corollary of the theorem he was attempting to prove. Selden and Selden (1987) offer a comprehensive list of students' issues with proof, including making invalid inferences and beginning with the conclusion to arrive at a true statement.

Research often frames the abovementioned issues as errors and misconceptions since they have to do with logic and proof validity (e.g., Weber, 2002). This framing is consistent with the cognitive perspective that dominates proof research (Stylianides et al., 2017). From the social viewpoint (Stylianides et al., 2017), these issues constitute deviations from the rules of logic practiced in professional mathematics communities.

Abiding by these rules requires implementation of additional rules that are less universal. Stylianides (2007) argues that in a mathematics classroom a proof uses accepted statements, employs forms of reasoning and expression that are known to or within the conceptual reach of the students. Hence, classroom proofs vary depending on what each of them renders "accepted," "known," and "within students' reach." RoPs are needed to organize mathematical statements, forms of reasoning and expression into fully-fledged proofs. These RoPs can be conceived as community-specific sociomathematical norms that determine "what counts as an acceptable mathematical explanation and justification" (Yackel & Cobb, 1996, p. 461). Dreyfus (1999) reflects on undergraduates' grappling with RoPs of this sort. Specifically, he discusses students' explanations and proofs that do not "go back" enough and are not "deep" enough to be considered fully-fledged proofs.

Studying proof learning through scriptwriting

Scriptwriting tasks present learners with a conflict and request to resolve it through composing a dialogue between fictional characters (Zazkis et al., 2013). Research has been arguing that these tasks allow scriptwriters to not only showcase the knowledge they developed from resolving the conflict but also raise issues that usually remain unarticulated in traditional problem-solution formats (e.g., Zazkis & Cook, 2018).

Gholamazad (2007) was one of the first projects to employ scriptwriting in the area of proof. Importantly to our investigation, Gholamazad drew on the commognitive framework, according to which thinking constitutes an "individual version of interpersonal communication" (Sfard, 2008, p. 81). Building on the same framework, Brown (2018) argued that by making students' envisioned interactions public, scriptwriting affords students to make their proof-related thinking visible and fosters reflection. Overall, scriptwriting has been acknowledged for providing opportunities to observe students' ways of seeing a mathematical proof (e.g., Zazkis & Cook, 2018).

COMMOGNITIVE FRAMING

Commognition construes mathematics as a discourse, associating its learning with an individual becoming a participant in certain activities (Sfard, 2008). One's starting to abide by the rules of the target discourse is an example of such an activity.

Proofs concern mathematical statements that can be rendered as either valid or not "according to well-defined rules" (Sfard, 2008, p. 224). The mathematics community has been defining and revisiting these rules throughout history (e.g., Kleiner, 1991). Notwithstanding, Sfard (2008) argues that "for today's mathematicians, the only admissible type of substantiation [of mathematical statements] consists in manipulation on narratives, and it is thus purely intradiscursive" (p. 232). We concur with this thesis in the case of a literate mathematical discourse and define RoPs as intradiscursive principles that underpin written proofs.

RoPs constitute a metadiscursive construct, prescribing what a written proof looks like. Sfard (2008) argues that such metarules “are the result of custom-sanctioned associations rather than a matter of externally imposed necessity [which] does not mean there are no reasons for their existence” (pp. 206–207). Sfard claims that even the most “objective” and commonly endorsed discursive rules that appear to be fully governed by inevitability and logical necessity are products of human choices that survived the test of time. In our case, this claim can be associated with modus ponents, modus tollens, predicate calculus, et cetera—i.e. RoPs that the mathematics community found useful and effective. The contingency of other RoPs might be more evident. For instance, a disciplinary tradition appears as the only reason for today’s mathematicians to mark the end of a proof with “Q.E.D” or the Halmos symbol.

In the context of proof learning and teaching, we distinguish between three types of discourses. In the first type, D_1, mathematical statements are endorsed without the issues of proof and proving being discussed explicitly. This could occur through the implementation of conventional procedures that are viewed not only as producing new narratives about mathematical objects but also as warranting the narratives’ validity (e.g., differentiation rules generate derivatives and ensure the resulting functions are derivatives of the original functions). In the discourses of the second type, D_2, statement generation and proving are separate from each other. D_2 discourses are proof-based versions of D_1 since most statements that are valid in D_1 remain valid, but the demonstration of their validity is expected to be different. This is where RoPs are purposefully *enacted* to demonstrate the (in)validity of a statement. Lastly, D_3 refers to a metadiscourse of D_2, i.e. a discourse in which the main objects are the rules of D_2. Such metadiscourses revolve around how statements in a specific D_2 relate to each other, why a particular narrative is valid when the other is not. On the D_3-level, RoPs are *endorsed* (cf. Sfard, 2008) through narratives that capture the rules in words (i.e. *rule-narratives* are generated). Within rule-narratives, we distinguish between *guiding* formulations that offer a direction by describing what a proof should do or look like, and *restricting* formulations, prescribing one what to avoid in a proof.

The distinction between the discourses is idealized, and the borders between the three are usually blurred in a classroom. The potential of this typology is in its capability to account for newcomers’ often-reported struggles with D_2 in non-deficit terms. Indeed, one’s violation of a particular RoP can be viewed as a rule that lingered from one discourse to another (e.g., the use of inductive reasoning, which is valid in D_1, to D_2 where deduction is expected). In such cases, raising to the level of D_3 appears necessary for a teacher to communicate the rules of D_2. Similarly, student engagement in a metadiscourse provides an opportunity to formulate, elaborate, and substantiate RoPs that are expected to be implemented in D_2.

METHOD

Our participants come from a special program in a large New Zealand university. The program is intended for mathematically motivated and academically inclined students in their final year of high school. Typically, the students are seventeen years old. As part of the program, they take a course that gives academic credit for a bachelor's degree in mathematics or engineering. The course is proof-based, and it covers selected topics in calculus, set theory, and graph theory. Proofs play a marginal role in New Zealand schools (Knox & Kontorovich, 2023), thereby, the first course lessons are dedicated to proof.

This study is a part of a larger developmental project – a co-learning partnership that educational researchers and university teachers formed to support students' transition to university mathematics. In this study, we collaborated with Patrick – a mathematician by training and a highly-acknowledged teacher with about a decade of experience in university mathematics instruction. As part of the project, Patrick revised his usual proof teaching to bring RoPs to the forefront of his first three lessons (50m each). In the first lesson, he introduced proof as "an argument that mathematicians use to show that something is true." Then, he presented three mathematical statements and led a whole-class discussion about what he dubbed as "broken proofs." In the following lessons, he illustrated how properties of real numbers can be used to derive additional properties (e.g., he used distributivity to prove "$-a = -1 \cdot a$"); he referred to these illustrations as "good proofs."

Several RoPs were discussed in the classroom, such as "examples do not prove universal statements," "each proof step must be mathematically valid," "a proof must start with a true claim and end with the assigned statement," and the rules of proof layout (e.g., "A proof needs to end with a □-symbol or 'QED'"). The rules were presented in seven episodes where Patrick generated narratives to present, explain, and substantiate the rules. Given the students' unfamiliarity with proof from their school studies, we expected the classroom rules to act as the main point of reference for students to lean on in the proof-centered activities that followed.

A scriptwriting task was collaboratively developed to provide students with opportunities to engage with RoPs (see Figure 1). The task was part of an individual homework assignment together with problems that are more typical to transition-to-proof courses (e.g., "Prove/disprove: If x and y are irrational, then $x + y$ is irrational"). The students had ten days to submit the assignment.

The analysis started with an overview of the collected 71 submissions to develop a general impression of whether students' scripts addressed RoPs. This process converged into 58 scripts. Other scripts were excluded from the analysis since they focused on difficulties one can experience in the proving process rather than on the rules proofs should abide by. Most scripts involved a Friend-character who shared an infelicitous proof attempt, and a Student-character who critiqued it. The critiquing utterances became the primary source of students' rule-narratives.

The question underpinning our analysis was *how do the students' RoPs (endorsed and enacted) compare to those discussed in the classroom?* At the first step, we mapped every RoP in each student's script to the rules Patrick emphasized in the classroom.

Then, we focused on the students' rule-narratives and examined the lesson transcripts searching for "the closest" thing Patrick did or said. This systematic comparison led to initial categories for the similarities and differences between the students' and Patrick's rule-narratives.

Writing proofs can be quite difficult, especially when you're new to them! One of the best ways to get better at writing proofs is to **look back** at past mistakes and think about how to fix them. This exercise is meant to help you do this, as follows:
(a) Come up with a mistake you've made (possibly while working on this assignment!) or seen others make when trying to prove something.
(b) Write a dialogue between you and an (imaginary) friend, where they make this mistake and you explain what's wrong with that idea and how to fix it: i.e. something like

Paddy's imaginary friend: So, I was trying that checkerboard problem above[4], and I figured that if I wrote that I tried three different ways to do it and none of them worked that would totally count as a proof.

Paddy: Well, actually it probably doesn't. You see . . .

(insert witty dialogue where the two characters realize the flaw in the above proof method)

Figure 1: Scriptwriting task

FINDINGS

RoPs discussed in the classroom were consistently enacted in eleven submissions. Most of the breaches pertained to semantic rules (e.g., finishing a proof with □), instances where the students used variables without defining them, and algebraic mistakes. Only six students attempted to endorse a universal statement with examples. In all but seven instances, the students aimed to prove valid statements and reject the false ones. For these instances, the students attempted to endorse universal statements with examples, used the target statements as part of the statement proofs, and provided evidence that were insufficient to reach the target conclusion. Breaches of multiple rules often went hand-in-hand, and these instances contained more rule violations than other submissions.

Next, we contrast the students' and Patrick's rules on a discursive level. We capture the contrast in terms of *similar*, *elaborate*, *extended*, and *new* rules. Similar rules emerged from 19 (out of 58) submissions, where students' rule-narratives overlapped with Patrick's to a significant extent. For example, Patrick contended that "when proving a statement, you should start off with something true", which one student echoed, writing: "If you want to show something is true, start with true statements."

Elaborate rules emerged from 20 scripts where students' rule-narratives came across as contextualized versions of Patrick's D_3-level rules or where students' RoPs endorsed aspects that were only enacted by the teacher. For example, in the classroom, Patrick explained that "You want to tell people what your variables are like." In the script, a Student-character spelled out that "If you represent rational x and y to be $^a/_b$ and $^c/_d$, you should define what a, b, c, d could or couldn't be." Accordingly, while Patrick offered a metamathematical version of the rule, the student's version was matched to the specificity of the proof in the focus of the fictitious dialogue.

Extended rules were found in 23 scripts where students' rule-narratives closely related to classroom RoPs, while introducing new aspects. For example, in the classroom, Patrick emphasized that "proof steps should be explained", in the sense that verbal explanations need to accompany the use of formulas and symbolic statements. In turn, one Student-character stated that "when you are trying to disprove a statement by using a counterexample that requires a proof itself, it must be proved, no matter how clear it is." This rule-narrative offers a combined guidance on disproving via a counter-example and ensuring its "counter-exemplary" status is established explicitly. This rule-narrative is not very far from suggesting that disproving is a type of proving, and thus, it is expected to abide by many of its rules.

Two new rules were identified in 9 scripts. The first one revolved around the invalidity of "a circular argument", i.e. the use of the yet-to-be endorsed statement in the statement's proof. This rule featured in eight scripts, where the Student-characters maintained that "proving a claim using what is required to be proved is clearly invalid" and "you're not actually proving anything [through circularity]. That argument contains no evidence that is distinct from its conclusion." Patrick emphasized the importance of including true statements in the proof, but students were the ones to identify circularity as a special sub-category and elaborate on its problematics. The script excerpt below is taken from a script where an Imaginary Friend (IF) and Student-character (Me) challenge a classroom rule. While Patrick highlighted that every proof step needs to be explained, the scriptwriter illustrates that an explanation can be followed by a request for another explanation, resulting in an endless process.

IF:	To solve 1d you need to know that integers are closed under multiplication.
Me:	You see, to prove that you would also have to explain what multiplication does and what the definition of an integer is.
IF:	Oh I see! So we must take some logical statements for granted else we would forever be asking questions.
Me:	Exactly! We best not think so deeply we will run out of paper!

DISCUSSION

We join scholars who maintain that it is unrealistic to expect newcomers to discover on their own how proof functions in mathematics (e.g., Hanna & Jahnke, 1996; Stylianides & Stylianides, 2022). Accordingly, we proposed that direct engagement with RoPs may be of didactical value for proof learners.

To trigger this engagement, RoPs were explicitly presented in the classroom first, and then, students were asked to script a fictional dialogue about a proof-related mistake. The analysis was consistent with our conceptualization: while the implementation of RoPs was evident in students' proofs (i.e. D_2), the scripts provided an arena for rule formulation, explanation, and justification (i.e. D_3). Indeed, most of our students used the voices of their fictitious characters to discuss the rules. These findings strengthen previous research on the potential of scriptwriting to advance proof learning and study this process (e.g., Brown, 2018). That said, we acknowledge that many facets of students' emerging discourses remained outside the scope of the submitted scripts. Hence, we concur with Zazkis and Cook (2018), proposing that scriptwriting may be a useful complementary way to study learners' transition to proof-based mathematics.

RoPs discussed in the explored classroom are not new to mathematics education. Indeed, studies have been reporting on students' struggle with these rules, as it emerged from students' infelicitous proof attempts (e.g., Selden & Selden, 1987). Our work adds to this body of knowledge by showing that after a relatively short period of explicit teaching, some students can operate with RoPs and also communicate about them on some metalevel. We acknowledge that our findings emerged from a special cohort that is hardly representative of a broader student population. Therefore, we call for further research into how learners come to grips with RoPs when these are taught explicitly.

Skeptics may argue that we should not "read too much" into the students' scripts as they could be "just" drawing on RoPs that the teacher demonstrated in the classroom. Indeed, some rules addressed in the students' scripts were similar to Patrick's. Moreover, the students' reliance on the classroom rules was possible since they had access to lecture captures. Even if so, we do not take the students' rule discussions for granted. First, from the commognitive standpoint, transitioning to a new discourse is impossible without a ritual phase, where learners imitate discourse oldtimers (Sfard, 2008). In other words, ritual participation is an unavoidable learning component. Second, an overlap between the students' and Patrick's rules draws attention to the impact of proof instruction on students' proof learning. In their comprehensive review of the literature on university proof-based courses, Melhuish et al. (2022) conclude that "we know less about how lecturers' actions influence students' learning. […] We do not understand the consequences of [lecturers' didactical] choices" (p. 11). The identified overlap suggests that the way a teacher formulates, substantiates, and implements RoPs may become a reference point for students' sequential use of the rules. Third, the students' deviations from the presented rules illustrate that explicit teaching of RoPs may be insufficient for students to follow them to the letter. Indeed, violations of the classroom rules were not rare, especially when students attempted to prove invalid statements and reject true ones. Teaching also does not explain myriad qualitative differences between the students' and teacher's rule formulations. The findings show that some students' rule-narratives expanded, elaborated, innovated, and even conflicted with the teacher's formulations. Colloquially speaking, the students' RoPs often appeared fuller, crispier, and more precise compared to what was taught. These findings make us believe that the relationship between teaching and learning of RoPs is more complex than may seem at first glance.

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Addressing the implementation problem in university teaching education: the case of study and research paths

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After more than fifteen years of research about study and research paths within the Anthropological Theory of the Didactic, we address the implementation problem of how to disseminate this inquiry-based instructional proposal to different university settings in first-year courses of statistics for engineering degrees. We adapt the methodology of didactic engineering as implementation strategy to create the didactic and pedagogical infrastructure that we consider necessary for carrying out an SRP by lecturers non-expert in didactics research. We present the first steps of the strategy that corresponds to a "proof of concept" project in process. The discussion raises new questions about how implementation problems can in turn nourish research in didactics.

Keywords: training of university mathematics teacher, novel approaches to teaching, implementation, study and research path, didactic engineering, infrastructure and superstructure.

INTRODUCTION

Study and research paths (SRPs) are inquiry processes proposed within the Anthropological Theory of the Didactic (ATD) that can be conceived as both instructional proposals and general models of inquiry (Bosch, 2018; Chevallard, 2015). The first SRP at university level was devoted to population dynamics and addressed to first-year students of a technical engineering degree. It was performed from 2005/06 to 2009/10 (Barquero et al., 2013). Since then, numerous implementations of SRPs occurred at several universities and following various modalities, as is recalled by Barquero et al. (2020, 2021).

In this context, a long-term investigative effort has been devoted to understanding the *ecology* of SRPs, that is, the conditions and constraints either favouring or hindering their implementation and successful dissemination, a critical dimension to address the implementation problem (Artigue, 2021). Given the importance of the teacher's role in development of study processes, SRPs were also developed to train future teachers, giving rise to what is called *study and research paths for teacher education* (SRPs-TE, Barquero et al., 2018). Along a third line of research, connected to the two previous ones, the process of implementing an SRP became more and more explicitly modelled as a piece of didactic engineering (DE). DE is an experimental methodology to validate and develop knowledge models in instructional settings for the analysis of didactic phenomena (Artigue, 2020).

Until recently, SPRs have always been implemented by teams of teachers including at least a researcher in didactics, expert in the ATD. From 2018, lecturers non experts in didactics began implementing SRPs in engineering, business administration, and management university degrees. Nowadays, their long-term dissemination and self-sustainability are questioned: *how to implement SRPs in university education, beyond the controlled conditions established by researchers? What conditions are needed? How to train lecturers to implement inquiry-based teaching through SRPs?* First experiences were described in (Florensa et al., 2018; Fernández et al., 2024), but many questions remain open.

This type of questioning can be located in the new field of research about the implementation of results in mathematics education, to which has been devoted a thematic working group in CERME since 2017 and the recently created journal *Implementation and Replication Studies in Mathematics Education* (Jankvist et al., 2021, Koichu et al. 2021). Different approaches are used in such studies, especially design-based research. As Artigue (2021) suggests, the ecological perspective and the proposal of SRPs (both linked to the ATD) might also be used as "internal theoretical resources for this field of study" (p. 33). The aim of the research presented in this paper is to progress in this direction, by considering the specific case of the dissemination of research findings about SRPs in university mathematics education. Our study can be linked to the issues opened by the last INDRUM2022 panel about innovation in university teaching based on research (Florensa et al., 2023).

The research strategy we propose, based itself on the *didactic engineering methodology*, focuses on creating conditions for the dissemination, from research in didactics to the teachers' practice, of the *didactic and mathematical praxeologies* (Bosch & Gascón, 2006) supporting SRPs. Our research questions are:

RQ1: How to model the conditions and constraints weighing on the dissemination of the pedagogical paradigm attached to SRPs?

RQ2: How to model the dissemination process in terms of didactic engineering, while including the necessary collaboration between researchers and lecturers?

FORMULATING THE IMPLEMENTATION PROBLEM WITHIN THE ATD

The dialectics between infrastructure and superstructure

When theorizing about SRPs, questions regarding both mathematical and didactical *infrastructure* and *superstructure* emerge quite naturally. Indeed, an SRP may be seen (schematically) as a process of inquiry that a group of students will undertake, aided by a (group of) teachers. This process is generated by a genuine question to which teachers and students must provide an answer. On the road to addressing this question, students will encounter and study already existing works and answers, searching through various pieces of media (books, articles, the Internet, experts, etc.). We can use the notions of infra- and superstructure to analyse the conditions needed to carry out the inquiry process:

> In ATD, a technique can be described as the association of a device and "gestures"; in particular, a *praxeological infrastructure* comprises devices, large and small, which are works, and which enable *superstructural activities* to be developed the execution of a given technique being based on this infrastructure. (Chevallard, 2009, p. 40)

For instance, searching for information on the Internet requires to carry out specific gestures (that is, a superstructural activity) on a given infrastructure, the web itself. A classroom in a school is also a necessary infrastructure to support teaching and learning processes (the superstructural activity), even if it can be replaced by other infrastructures, for example when online instruction becomes necessary. In the mathematical context, the infrastructures also exist in the form of a set of works prepared in the long run. For instance, in order to find the new coordinates of a point on the plane after a rotation by a given angle, complex numbers will prove to be an efficient infrastructure (by carrying out a superstructural activity based on calculations in $\mathbb{C}$).

The main value of the notions of infrastructures and superstructures is to point out that "there is a strong tendency, among individuals and institutions, to 'forget' infrastructure as a problem, while routinely exploiting it as a means. What prevails here is what we might call the 'silence of infrastructure'" (Chevallard, 2009, p. 41). Therefore, setting up a study process goes along with the preparation of an appropriate infrastructure. For instance, a learning management system such as Moodle may form part of such an infrastructure. However, and quite clearly, this cannot work without a matching superstructure on the students' side, closely related to their *praxeological equipment* (knowledge and know-how). The latter may either be already there or provided by the teacher. In any case, both the platform and a praxeological equipment would be needed, for students to perform a superstructural activity. In this respect, the ecological analyses may be seen as the study of the available pieces of infrastructure and the quest for potential sources to provide the missing elements:

> Health is when infrastructure is forgotten; it is when the superstructural illusion prevails, pushing aside the question of the infrastructural conditions and constraints of superstructural activities. (Chevallard, 2009, p. 41)

In the rest of this section, we will develop this model in the particular case of the dissemination of the SRP pedagogy. The thickness of the veil provided by this "superstructural illusion" should then appear even more clearly.

Implementation methodology and didactic engineering

As mentioned before, the dissemination methodology we rely on is that of didactic engineering (DE). As Artigue (2020) explains, the DE methodology was introduced in the Theory of Didactic Situations as an experimental epistemology of mathematics. We are considering here the format and phases proposed by Barquero and Bosch (2015) to address the problem of the conditions for disseminating SRPs to university lecturers. However, instead of using DE for the design, implementation, and analysis of an SRP (as it is usually done), it is here used to guide and analyse both the design, the implementation and the analysis of the dissemination of SRPs beyond the original designers. This strategy can also be interpreted as a second-generation DE (Perrin-Glorian, 2011), in so far as the result of a DE process (an SRP) becomes the centre of another DE supported by a collaborative process between researchers and teachers.

In this context, the four phases of didactic engineering may be seen as: (1) a preliminary analysis which aims at the delimitation of a didactic phenomenon, considering the ecology of SPRs and the consequent difficulties for their dissemination due to a lack of didactic and pedagogical infrastructures; (2) an *a priori* analysis of the dissemination strategy hinging on the design and use of appropriate infrastructures; (3) an *in vivo* analysis during the implementation of the dissemination strategy; (4) an *a posteriori* analysis of the strategy and its effects on the didactic phenomenon at stake. In this paper, we only address parts (1) and (2).

Infrastructure and superstructure of a dissemination process

Disseminating the pedagogy of SRPs using the methodology of DE puts at play three sets of infrastructural and superstructural elements, which we represent as layers:

The *outer layer* (*didactic* infrastructure) relates to a lecturer not expert in didactics teaching an SRP. Her students will develop a superstructural activity (an inquiry process based on an SRP) which will rely on a didactic infrastructure that the lecturer may arrange to some extent. However, the lecturer's gestures, in turn, rely on other infrastructures, which need to be accessible for the instructional process to take place.

The *principal layer* (*dissemination* infrastructure) relates to the dissemination process *per se*, which we address using the categories of DE (preliminary, *a priori*, *in vivo* and *a posteriori* analyses). That is, lecturers will develop the superstructural activity consisting in implementing their learning process to get acquainted with the pedagogy of SRPs. This should happen thanks to a disseminating infrastructure provided by us, the disseminators. It may include physical facilities (e.g. a website), but also immaterial ones (e.g. student-lecturers' personal heuristics on teaching and learning). As always, elements of this infrastructure will be out of our reach as disseminators. The description, justification and limits for this infrastructure is the object of investigation in the subsequent sections.

The *inner layer* (*scientific* infrastructure) regards the superstructural activity performed by researchers to create elements that would become parts of the previous infrastructure. This activity includes our theoretical framework, the ATD, and research results about SRPs' ecology and management (Barquero et al., 2021).

THE STRATEGY OF LABINQUIRY

To address the issues we identified, we consider the case of a dissemination project in process, LABINQUIRY, developed within the framework of the ATD. Results derived from previous projects have shown that SRPs have a strong transformative character and a positive impact on both students and teachers' performance and satisfaction (González-Martín et al., 2022). However, it is also observed that they are only viable and sustainable under regular university conditions as long as the teacher leading the inquiry is an ATD researcher or works in close collaboration with a team of researchers. The analysis of the ecology of the SRPs' brought to light specific types of institutional constraints that originated from the prevalence of the *paradigm of visiting works* in current university education (Bosch, 2018; Barquero et al., 2021; Jessen et al., 2019). These constraints explain the difficulties for teachers to manage inquiry-based instructional proposals and the barriers found to disseminating them beyond research-controlled settings (Dorier & García, 2013; Shpeizer, 2019).

On the bases of these results, the main aim of the project LABINQUIRY is to create two related prototypes for the transfer of SRPs to secondary schools and universities. The first prototype is composed of an online platform (Moodle or Google Classroom) to support the design, implementation, and management of inquiry-based teaching proposals specific to the paradigm of questioning the world. Its main objective is to create good conditions for teachers "launching" SRPs thanks to a didactic infrastructure that provides them with controlled conditions that are resilient to the institutional constraints identified. The second prototype is LABINQUIRY-Community, an online social network for teachers and researchers to manage the interactions generated by LABINQUIRY. It aims at generating insights from the interactions and use them to integrate teachers into a community of practice to share experiences, pool data and resources from the instructional processes, receive live advice from other colleagues and researchers, and even implement inter-school and inter-university SRPs. The community of teachers is decisive for the dissemination and implementation of inquiry-based teaching proposals on a large scale and under different institutional conditions.

The choice of using a pilot SRP

We identified two main difficulties when it comes to implementing an SRP in ordinary teaching conditions: its *a priori* preparation and its *in vivo* monitoring. The preparation includes a quite thorough epistemological analysis, including the design of a generating question and its analysis in terms of a questions-answers dialectic, and the study of its conformity with the official curriculum. To make the implementation easier, we decided to provide lecturers with a proposal of such an analysis, which already implies many pieces of knowledge and know-how (see next section). Concretely, we chose a "pilot SRP", which we experimented for the past four years, to make sure its epistemological basis and pedagogical design are sound. This SRP has been taught by a lecturer in statistics, non-specialist in the ATD, and is generated by a question about air quality in given Low Emission Zones around Barcelona (Fernández et al., 2024; Verbisck et al., this issue). It was improved throughout the years, until its current state of development which contains the following innovations. First, there are well-identified phases of the inquiry process and introduction of labels to refer to them. Secondly, there is an explicit role of the questions-answers dialectics. Third, there exist an external instance who presents the generating question of the SRP and to whom its answer will be addressed. Fourth, a more stable methodology for the students' logbooks (diaries) and intermediate reports is used, which are structured in a standardised way. Last but not least, there is a clearer organisation to report and debate on partial or temporary results and a more precise frame for their discussions in the classroom.

Preparing the SRP to be implemented so extensively might favour the dissemination process for several reasons. First, it is a fully developed example of what the pedagogy of SRPs looks like. Second, it can precisely arouse questioning about this early stage of the process, and so incite lecturers to try it by themselves on another topic. Finally, it allows the lecturer to focus on the in-class SRP's management, which is an inescapable stage of the whole procedure, and which itself poses numerous difficulties, some of which we will now further develop.

A PRIORI ANALYSIS OF THE DISSEMINATION PROCESS

Elements of the ecological analysis

As mentioned before, numerous studies discussed the ecology of SRPs in experimental conditions. However, we must now consider the ecology of the dissemination of SRPs, which could be a whole other story. The last implementations with lecturers who are non-experts in the ATD show that, among the main constraints limiting the implementation of SRPs, the workload they suppose for lecturers is one of the most important ones. Since Spanish degrees in Engineering are mainly organised according to the European Higher Education Area (EHEA), SRPs are totally aligned with the guidelines promoting a student-centred and competence-based instruction. Even if this significantly reduces the curriculum constraint, the pedagogical changes it supposes remain almost under the sole responsibility of the lecturers. The lack of appropriate pedagogical and didactic infrastructures leads to overwork for lecturers, especially in time and dedication. So, we might face rather strong constraints at this level of organisation of teaching duties.

The ecological problem is then formulated in terms of how to create the missing didactic infrastructure and how to make it available to lecturers. Significant conditions and constraints will arise given the kind of teaching and learning practices which already exist in the academic institution: we need new superstructures to put the new infrastructures into play. On the one hand, the very idea of didactic training for lecturers is not shared in the academic world, as may be seen through the pervasive absence of training to teach at the university level in many European universities. This is another symptom of the historical discredit of teaching, still regarded as a *semiprofession* rather than a "true" profession (Etzioni, 1969). On the positive side, and to the difference of many secondary school teachers, university lecturers are more familiar to the paradigm of questioning the world due to their research experience. Indeed, it is closer to their daily practice to manage a study process whose only *raison d'être* is a research question, carry it out through a dialectics of questions and answers, or search for books and study new works.

A praxeological model for the pedagogy of SRPs

The construction of our reference model for the practice of teachers in the SRP's pedagogy can be supported by its praxeological analysis, that is, a description of the know-hows (sets of tasks performed using given techniques) backed by pieces of knowledge of a more theoretical kind (be they heuristic or scientifically grounded). To build such a praxeological model, we relied on two sets of sources. First, a theoretical characterisation of the SRP's pedagogy (Bosch, 2018; Chevallard, 2015). Then, a substantially developed practice of reference taking place within our research team, whose members design and teach SRPs since 2005.

Without going too much into details, let us simply say that we model partial or full praxeological organisations attached to didactic tasks of the kind: "proposing a generating question to the students", "generating a questioning process in the classroom", "discussing the relevance of a set of derived questions", "organising the search for already available answers and their validation", "summarising the students' proposals of intermediate questions and partial results", "organising and defending the final answer of the class". The types of tasks described are the ones considered the most helpful to monitor the inquiry process. All in all, the ecological and praxeological analyses provide us with a clearer view of both the existing infrastructure and the superstructural activity which should take place within it. Based on both, we inferred the specifications of the infrastructure we needed to develop to strengthen (or merely make possible) the implementation process.

An infrastructure congruous with the superstructural activity to be developed

As most universities today use online learning management systems, it appeared natural to transfer the content and material of the pilot SRP to lecturers as a Moodle or Google Classroom page. In addition, this might help teachers not to be overloaded with the design of the SRP, concentrate on its managing and, if necessary, adapting parts of it. Such a Moodle page can then play the role of an online infrastructure to support students' superstructural inquiry activities.

However, the course management of an SRP does not go without a body of know and know-how on the part of the lecturer. This is why the Moodle platform will come together with a website providing numerous texts, videos, research papers, etc. for lecturers to get acquainted with the theory and practice of SRPs. If we might formulate a generating question for lecturers learning how to implement SRPs, this could be: "How to teach the pilot SRP given in Moodle in my particular institutional context?". Consequently, the structure of the website should anticipate possible derived questions from this generating question, provide partial answers (or at least, offer a self-sustained media) to facilitate a rich and shared *milieu* between lecturers, researchers, and educators. That is, the dialectics of crucial questions and praxeologies is at the core of the construction of the proper infrastructure, for the implementation process to take place. Nevertheless, the infrastructure cannot solely consist in a website and a Moodle platform, as they would rapidly become too short to cover each aspect of the forthcoming dialectics of questions and answers. This is why these two pieces of material infrastructure go together with the LABINQUIRY-Community. The latter's purpose is to provide an interacting community for lecturers to share, confront and validate their practice in the making. From a scientific perspective, the community will also be a valuable vantage point on the conditions and constraints concerning the dissemination of the pedagogy of study and research paths. The creation of this community can be considered as part of the paradidactic infrastructure supporting the dissemination process (Miyakawa & Winsløw, 2013).

CONCLUSIONS

Using the notions of infrastructures and superstructures opens the way to approach the study of the conditions and constraints weighing on the dissemination problem (RQ1), with the proposal of considering several layers: a didactic, a dissemination and a scientific infrastructure. This is a way to take the infrastructural problem into account, avoiding the "superstructural illusion". The first aim of LABINQUIRY is to provide a didactic infrastructure to help lecturers implement SRPs starting from a pilot case rooted in an online teaching platform. To disseminate the didactic gestures of the corresponding superstructure, a disseminating infrastructure is developed following the stages of the DE (RQ2). The DE concerns both the infrastructure and the personal activity of lecturers, which then collectively learn about the pedagogy of SRPs thanks to the companion community. This infrastructure is based on previous results from the designed SRPs and from the analysis of their ecology and the didactic praxeologies needed to manage them.

Last but not least, the study of this LABINQUIRY SRP case also illustrates how the elaboration of such an infrastructure can only provide a limited part of the "total" infrastructure for the paradigm of questioning the world to disseminate at the university. A project like LABINQUIRY may contribute to develop our scientific knowledge on research dissemination. Moreover, this type of implementation project is also crucial for the data it provides to further develop scientific knowledge about didactic phenomena, which is the ultimate infrastructure of our research activity.

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"You're a mathematician […] I'm more engineer": How a difference in educational background led to a commognitive conflict

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This study examines the important mathematical practice of defining in a class of master students. The research investigates the influence of those students' educational backgrounds, particularly their undergraduate degrees, on their proficiency in the practice of defining. Specifically, the students completed a task in which they had to define and describe 3D geometric objects, and their discourse was later analysed through the commognitive framework. Focusing here on a group of three students, we show how their different educational backgrounds led to the appearance of a commognitive conflict and an opportunity for learning that seemed to be utilised. The results of the study carry broader implications, since they may offer guidance for university instructors.

Keywords: commognitive framework, defining, teachers' and students' practices at university level, teaching and learning of specific topics in university mathematics.

INTRODUCTION

Mastering practices like defining and proving is fundamental for mathematics education at the university level (Fernández-León et al., 2021; Mejía-Ramos & Weber, 2019; Ouvrier-Buffet, 2011; Tabach & Nachlieli, 2015). The practice of defining stands out specially, since learning mathematics includes learning definitions (Tabach & Nachlieli, 2015), whose importance at all educational levels is also underscored by many researchers (Fernández-León et al., 2021; Gilboa et al., 2023; Martín-Molina et al., 2023; Ouvrier-Buffet, 2011; Torkildsen et al., 2023).

Defining is necessary for proving geometric theorems (Ouvrier-Buffet, 2011), allows proposing novel definitions that generalise existing mathematical entities (Martín-Molina et al., 2018), is essential for mathematical exploration and problem-solving (National Council of Teachers of Mathematics, 2000), etc. Despite its fundamental importance, defining geometric objects often poses challenges for both students and teachers (Tabach & Nachlieli, 2015). Indeed, common misconceptions, such as the belief that there exists only one correct definition for a concept, further complicate the learning process (Torkildsen et al., 2023). Consequently, addressing these challenges becomes pivotal because students need to learn to articulate precise definitions and to use them appropriately.

In this study, we were concerned with the reasons behind these misconceptions and, particularly, whether the students' educational backgrounds could motivate their appearance. Mathematics education literature has occasionally examined the extent to which students and teachers' educational backgrounds could explain differences in their performances (Anantharajan, 2020; Arnal-Bailera & Oller-Marcén, 2020; Ko & Herbst, 2020; Nortvedt & Siqveland, 2019).

In this regard, we proposed a task to master students in which we asked them questions related to defining 3D geometric objects. Focusing on their discourse while dealing with those questions, this research endeavours to examine the influence of these students' educational backgrounds (understood as their undergraduate degrees) in their defining practices.

THEORETICAL FRAMEWORK

In this research, the commognitive framework (Sfard, 2008) was used to study master students' mathematical discourse. This framework is useful because it permits the operationalisation of mathematical learning as a change in students' mathematical discourse. The commognitive framework considers mathematics as a specific type of discourse that involves mathematical objects. A mathematical object is defined as a pair of signifier and all the realisations of that signifier (Lavie et al., 2019), where signifier refers to words, symbols, etc. that the participants of the discourse use as nouns in their communication and a realisation is a "perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about [a signifier]" (Sfard, 2008, p. 154). For instance, a signifier could be the word "prism" and a realisation of it could be a picture of a prism or a physical model of it.

In this framework, the mathematical discourse is characterised by keywords, visual mediators, narratives, and routines (Sfard, 2023a). Specifically, keywords are mathematical words (e.g., solid, square). Visual mediators are the visible objects with which the participants identify the mathematical objects of their communication (e.g., the drawing of a square, the physical model of a prism). Moreover, narratives are of two types: object-level narratives and meta-level narratives. Object-level narratives are utterances (oral or written) about the mathematical objects or relationships about them (e.g., "this figure has 12 edges"), and meta-level narratives are expressions about activities with those objects (e.g., "to construct a definition for this figure, we first have to identify its properties"). The participants in the discourse can accept or reject the narratives. When accepted, they are labelled endorsed narratives. Finally, routines are "a set of meta-rules that describe a repeated discursive action" (Viirman & Nardi, 2021, p. 2). Meta-rules are rules that define actions in the discursive activity of the participants when they produce and substantiate narratives about mathematical objects. For instance, a meta-rule is "to add two numbers, we can first add the units, then the tens, then the hundreds, etc.", which produces and substantiates the object-level narrative "12+35=47 because 2+5=7 and 1+3=4".

Additionally, Sfard (2023b) points out that routines have two components: task and procedure. When a participant in the discourse tackles a task-situation, which is "any setting in which a person considers herself bound to act – to do something" (Lavie et al., 2019, p. 159), the task is the participant's interpretation of the task-situation. And the procedure is the set of actions that the participant performed in previous situations and considers useful to tackle the new task-situation (Sfard, 2023b). For instance, a task-situation arises when a student is asked to give a definition of a geometric object. For the student, the task could be to reproduce a definition for the mathematical object and the procedure could be to recall a previous definition of that object.

In the commognitive framework, learning is considered a change in the participants' discourse, which can be identified from changes in the participants' discursive characteristics. Powerful opportunities for learning arise when commognitive conflicts occur, i.e., when communication is hampered because the participants act according to different discursive meta-rules (Sfard, 2007). In particular, the narratives that cannot be endorsed according to the same meta-rules are called incommensurable (Sfard, 2023a).

In this study, we pose the following research questions: How do the students' educational backgrounds influence their discourse when defining? When does a difference in educational background lead to a commognitive conflict?

METHODS

Participants and context

The participants in this study were 33 students enrolled in a master's programme in Secondary Education which is compulsory to become a mathematics secondary school teacher. The master students were taking a course on educational research in mathematics whose aim was to introduce them to research on teaching and learning mathematics. The participants had completed different undergraduate degrees: 12 were mathematicians, 12 engineers, 7 architects and 2 physicists.

In one of the course sessions, the master students worked in groups of 3 or 4 students, forming a total of 9 groups (called here M1, …, M9). Students with different educational backgrounds were present in all the groups. In this study, the students of each group were labelled S1, S2, S3 and S4, irrespective of their group.

Data collection instrument

The data collection instrument was a worksheet comprising fifteen questions related to three geometric solids (see Figure 1). Specifically, there were five questions concerning each of the three geometric solids. Questions 1.A, 1.B, 1.C (where 1.A referred to solid A, 1.B referred to solid B and 1.C referred to solid C) asked to give a definition for the solids; Questions 2.A, 2.B and 2.C asked if the other solids were examples of their previous definitions; Questions 3.A, 3.B and 3.C requested that the students describe the solids; Questions 4.A, 4.B and 4.C asked to give a second definition for the solids and, finally, Questions 5.A, 5.B and 5.C requested that the students select a definition for the solids from several options.

Data collection

Each of the nine groups of students was provided with a paper copy of the worksheet, where they had to write down the agreements they had reached after discussing the questions in their group, and with an audio recorder to access their discussions. Furthermore, the three geometric solids were provided as a GeoGebra file on a laptop (Figure 1) to five of the groups, and as a physical model to the other four groups.

The study's data consisted of 9 worksheets containing written responses from student groups, along with audio recordings (approximately 1 hour per group) and their corresponding transcripts.

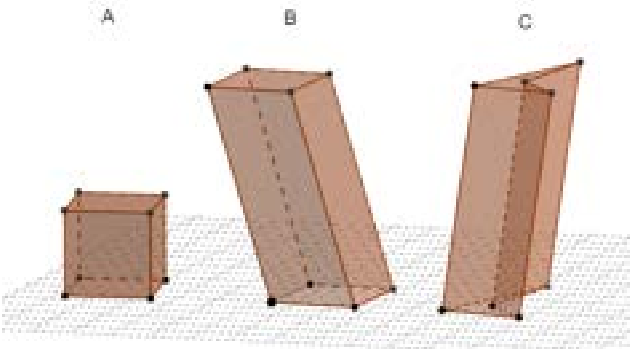


Figure 1: The three geometric solids constructed in GeoGebra

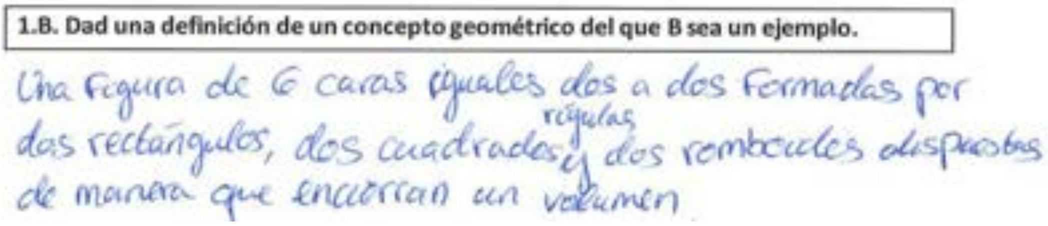
Analysis

We analysed the data of each of the nine groups in four steps. First, we identified which parts of the data referred to each of the fifteen questions of the worksheet. Second, the data were analysed to identify the four discursive characteristics. In order to infer routines, we carefully analysed the discourse to determine how the students had interpreted the task-situations and what procedures they had applied. Third, we analysed the identified characteristics to determine whether there were differences in the mathematical words, narratives, procedures or tasks used by the students. If there were, we checked whether those differences could be motivated by the influence of the students' educational background. This was done by studying if the differences appeared when students from different backgrounds or from the same background were conversing and by analysing the transcripts to check if the background could have influenced those differences. Finally, in a fourth step, those differences were also analysed to determine whether the students' narratives were incommensurable, i.e., whether there were commognitive conflicts.

RESULTS

In this communication, we focus on one of the nine groups, M5. We showcase this group because it had both mathematicians (S1) and non-mathematicians (S2 and S3) and their discussions show a discussion that is similar to others that appear in other mixed-background groups. Group M5 was provided with the physical model of the three geometric solids.

In M5, a task-situation arose when its students were discussing how to answer Question 2.B, where they had to decide whether the solids A or C were examples of the definition they had constructed for solid B in Question 1.B (see Figure 2).


1.B. Dad una definición de un concepto geométrico del que B sea un ejemplo.

Una figura de 6 caras iguales dos a dos formadas por dos rectángulos, dos cuadrados y dos rígidas romboides dispuestos de manera que encierran un volumen

Figure 2: M5's answer to Question 1.B, which could be translated as "A figure with six rigid faces, pairwise equal, composed of two rectangles, two squares, and two rhomboids placed so that they enclose a volume"

Researchers had designed Question 2.B as a task that required the students to check whether A or C were realisations of the narrative given as a definition in Question 1.B. Indeed, researchers expected students to reflect on whether the definition given for one signifier (of which solid B is a realisation) could serve them as a definition for other realisations (solids A or C), which meant checking whether solids A or C met all the requirements of the narrative of Question 1.B.

In group M5, the students interpreted, as researchers had envisioned, that Question 2.B required them to check whether A or C were realisations of the narrative given as a definition in Question 1.B. Their procedure for performing the task consisted on checking if A or C satisfied all the requirements that appeared in the narrative they had written as an answer to Question 1.B. However, the difference in educational background of the members of the group (S1 was a mathematician and S2 and S3 were engineers) led to a commognitive conflict because of incommensurable narratives related to the existence of inclusive definitions in geometry. By inclusive definition, we refer to a definition that allows an object to belong to more than one category. For example, an inclusive definition of rectangle would consider a square as a particular case of rectangle. Exclusive definitions are those that are not inclusive (e.g., a definition of a rectangle that excludes squares).

In the following, we present some conversations that took place when the members of M5 (S1, S2 and S3) were discussing whether solids A or C satisfied the definition they had given in Question 1.B. We highlight which narratives are incommensurable, indicating the existence of a commognitive conflict, and how the resolution of this conflict was an opportunity to learn for S2 and S3. The following excerpt shows the beginning of this discussion:

458. S1: I read Question 1B: a figure with six faces, pairwise equal, composed of two rectangles… A square is a rectangle.

459. S3: Sides…equal sides.

460. S1: Different sides. Is a square a rectangle?

461. S2: Yes.

462. S3: No.

463. S1: Sure?

464. S3: Yes.

465. S1: I think that a square is a rectangle.

466. S3: No, no, because otherwise it'd be the same. Two things are equal when they're called the same, when they're the same.

467. S1: No, no, no, no, you're wrong. A cat's an animal, but an animal doesn't have to be a cat.

468. S3: But they aren't the same, of course.

469. S1: No, but I'm not saying that they're the same.

S1 began recalling the narrative they had given as definition of solid B (line 458, Figure 2) and, when he reached the signifier "rectangle", he stopped and added that "a square is a rectangle" (458). This object-level narrative is the product of a meta-rule that S1 explicates at a later moment (505): in mathematics, definitions are inclusive. However, although S2 initially agreed, S3 rejected the narrative and there was a discussion about the inclusivity of classes (462-469). Indeed, S3 stated that the narrative "a square is a rectangle" cannot be true because "two things are equal when they're called the same, when they're the same" (466). This shows that, for S3, definitions could not be inclusive because only mathematical objects that are the same can have the same signifiers. Therefore, there was a commognitive conflict between S1's narrative (a square is a rectangle) and S3's (a square cannot be rectangle) because both narratives could not be endorsed according to one meta-rule. When S1 became aware of the conflict, he tried to convince S3 with a non-mathematical narrative that asserts that it is possible for a definition to be inclusive (467) and later added that he was not saying that a square and a rectangle were the same (469). However, in the next excerpt, we can observe how the conflict continued.

481. S1: Definition of rectangle: a figure with four sides that are pairwise parallel.

482. S2: Rectangle.

483. S1: I think that is the definition of rectangle.

484. S3: Then, a rhombus is a… a rhombus is a… a rhombus is a square.

485. S1: A rhombus is a square? A rhombus is a square? ... No, you're telling me that a square is a rhombus, that's what you mean, isn't it?

486. S3: Or a rhombus is a square, it's the same to me, if they're the same…

487. S1: No. A rhombus isn't a square. They're not the same, S3. A included in B is not B included in A, that is, we're saying the… the answer… the question is Is a square a type of rectangle? Is it? Is a square a type of rectangle? There are many rectangles: those that don't have equal sides… The square is a particular case of rectangle which has all the sides equal.

488. S3: Yes.

489. S1: Yes or no?

490. S3: Yes.

[…]

494. S2: I didn't know that.

495. S3: I'm like that.

496. S2: I didn't know that. I don't know if… I thought that all were rectangles except the square, which is the only one that has all its four sides equal.

S1 tried again to make the other members of the group see that a square is a rectangle and, therefore, that solid A satisfies the property given in the definition of Figure 2. For this, S1 resorted to giving an (incomplete) definition of rectangle (481). S3 seemed to interpret S1's previous narrative (about the square being a rectangle) incorrectly, thinking that S1 meant that a rectangle is always a square and that, therefore, a rhombus is also a square (484). S1 rejected S3's narrative in line 484, adding "no, you're telling me that a square is a rhombus" (485). S3 clearly did not understand the difference between both narratives, since he added "or a rhombus is a square, it's the same to me, if they're the same" (486). This is more evidence of S3's meta-rule, which forbids the existence of inclusive definitions. After this, S1 seemed to have become aware that the problem was S3's meta-rule about inclusion, because he shared a narrative about the inclusion of sets not being reciprocal and asked the others whether a square is a "type" of rectangle (487). He then answered his own question: "there are many rectangles: those that don't have equal sides… The square is a particular case of rectangle which has all the sides equal" (487). S1's last proposed narrative seemed to convince S3, who finally accepted that a square is a rectangle (488, 490), thus endorsing S1's narrative and seemingly resolving the commognitive conflict. S2, who had been silent for almost the whole discussion, seemed to also endorse S1's narrative and added that he had not known that previously (494, 496).

Later, the members of this group discussed if a square or a rectangle are rhomboids, since, in the definition of Figure 2, they had mentioned that two faces are rhomboids and the faces of A are squares. In this case, everyone accepted that the square is a rhomboid, and S1 explicitly mentioned his meta-rule in line 505:

505. S1: I think so, I think so. In mathematics, […] I'd say that there aren't exclusive definitions of […]

506. S3: Maybe.

507. S1: But not here anymore… [Laughter] [Profanity], S3.

508. S2: you're a mathematician, we're more... more practical.

509. S1: Yes, but… [Laughter].

510. S3: I'm more engineer.

Contrasting with S1's meta-rule, the narratives of S2 and S3 seem to imply that mathematicians are not practical people because they are concerned with the intricacies of definitions or the relationships between definitions.

In this discussion, misalignment between students' meta-rules led to incommensurable narratives, and thus a commognitive conflict, which is a possible reflection of the different undergraduate degrees that these students had completed. This conflict seemed to have been resolved successfully and S2 and S3 apparently learned from S1.

CONCLUSIONS AND IMPLICATIONS

The investigation into the influence of students' educational backgrounds on their discourse during a defining task has yielded some interesting results. To answer our research questions, "How do the students' educational backgrounds influence their discourse when defining? When does a difference in educational background lead to a commognitive conflict?", we focused here on the discourse within group M5, where a commognitive conflict was observed. This conflict seemed to stem from the different educational backgrounds of its members, who were a mathematician (S1) and two engineers (S2 and S3).

The commognitive conflict surfaced when evaluating whether solids A or C could be considered realisations of the definition provided for solid B in Question 1.B. The essence of the conflict was rooted in whether definitions should be inclusive in mathematics. S1, a mathematician, proposed an inclusive definition, which was at first rejected by S3, an engineer, who added that objects with different characteristics cannot share the same signifier. This fundamental difference in meta-rules led to incommensurable narratives and a subsequent commognitive conflict.

The resolution of the conflict unfolded through a series of discussions in which S1 attempted to persuade S3 of the inclusivity of definitions in mathematics. Eventually, S3 accepted S1's narrative, signifying a possible successful resolution of the commognitive conflict. Moreover, S2, who had remained silent initially, also endorsed S1's narrative, indicating that the learning opportunity may have been utilised. This meta-level opportunity for learning was possible thanks to the fact that S1, as a mathematician, was familiar with the rules of university mathematics discourse and thus could play the role of leader (Sfard, 2008).

This study highlights the impact of educational background on the formation of meta-rules and its influence on how individuals construct and use definitions. It complements previous works on how undergraduate students define 3D solids (e.g., Fernández-León et al., 2021; Martín-Molina et al., 2023) and emphasises the necessity for awareness of these differences to foster effective communication and learning. Future research could further explore strategies for facilitating effective communication and collaboration among students with diverse educational backgrounds.

Lastly, our larger study (of which the research in this communication is part) carries broader implications for the educational landscape. The detailed analysis of master students' defining practices offers invaluable guidance for university instructors and teacher training programs, which, armed with our insights, could refine their curriculum, emphasising defining skills as a cornerstone of mathematics education.

ACKNOWLEDGMENTS

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Students' perceptions of a Proof Assistant, in an *introduction to proof* course, in the first year of University mathematics

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We have introduced a proof assistant, Edukera, at University in a mathematics course aiming at teaching proof. We investigate the perceptions of the students of this proof assistant, based on data gathered through questionnaires. We are interested in the effects of using a proof assistant on learning proof, but also on the utility, usability, and acceptability of such an interactive learning environment. Our results reinforce the idea that proof assistants can contribute to the learning of proof, and bring recommendations for the introduction of proof assistants in teaching settings.

Keywords: Teaching and learning of logic, reasoning and proof, Digital and other resources in university mathematics education, Computer assisted theorem proving, Students' perceptions, Edukera

INTRODUCTION

Proof is at the heart of university mathematics, and the teaching and learning of proof and proving is central issue at the beginning of university in mathematics (Hanna et de Villiers, 2012, in particular part V). In this context, the role of logic, and the interplay between syntax and semantics have been shown as important (Durand-Guerrier et al, 2012, Durand-Guerrier, 2008, Selden and Selden, 1995). Difficulties in the learning of proof has led to explore the potential of technologies in its teaching (Hanna et al, 2019), and the development of computer science raises issues regarding the interactions between mathematics and computer science (Durand-Guerrier et al., 2019). In this specific direction, the use of *Proof Assistants* for teaching has developed. Originally, proof assistants are expert software that help build and automatically check proofs, in mathematics and computer science. In a mathematics *introduction to proof* course in University, we have introduced and used a proof assistant called *Edukera* (http://edukera.com/), conceived for teaching proof and mathematics, and developed as an online exerciser. We generally aim at exploring and analysing the potential of proof assistants for the teaching and learning of proof and proving at University. More particularly, in this paper, we analyse students' perceptions of a proof assistant, through questionnaires given during two consecutive years. We will rely on the concepts of *Utility*, *Utilisability*, and *Acceptability* used in the study of interactive learning environment (ILE) (see Tricot et al., 2003). *Utility* concerns the learning efficiency of the ILE, *Usability* concerns the possibility to use it (manipulation, interface...). *Acceptability* concerns the representations regarding its utility and acceptability, conditioning the decision to use it. First, we will succinctly present proof assistants for education, and previous research. Then we will describe Edukera, and the teaching setting. Afterwards, we will specify our research questions, and methodological aspects. Finally, we will present and discuss our main results.

PROOF ASSISTANTS IN UNIVERSITY MATHEMATICS EDUCATION: CONTEXT, RESEARCH QUESTIONS, AND RESEARCH HYPOTHESES

Geuvers (2009) gives a general presentation and overview on proof assistants. Here, we rely on the description of a proof assistant (PA) given by Bartzia et al. (2022):

> The term *proof assistant*, or *interactive theorem prover*, refers to a software tool allowing a user to interactively construct a formal mathematical proof. Some systems are designed to work in a specific domain such as geometry, logic or the analysis of computer programs, while others are general-purpose. Additionally, proof assistants used in the classroom can be sorted roughly in two categories: some are built by the community of educators and others are designed by specialists of interactive theorem proving for research or other professional purposes. (p. 254)

Many PAs have been developed, such as DEADuction, Lurch, Edukera for the first category, or Coq, Isabelle, Mizar, or LEAN for the second, and many of them are used or tested in educational purposes, in mathematics or computer science education. Although "we know almost nothing of [proof assistants'] potential contribution to other roles of proof, such as explanation, communication, discovery, and systematization, or how they now may become more relevant as pedagogical motivation for the learning of proof in the classroom" (Hanna et al., 2019, p. 9), it is increasingly admitted that using PA could contribute to the learning of proof.

Indeed, as tools that permit to build and check a formal proof, according to a logical system, PAs make explicit the formal rules governing the development of a proof, (such as the use of quantifiers and logic operators, and their manipulation in proofs). For instance, Chellougui (2020) shows that, out of a digital environment, introducing students to a logical system (Copi's natural deduction in this case), "contributes positively to the students' capacity to analyse mathematical proofs from the point of view of logical validity" (p. 319). But she identifies issues regarding the links that must be developed with the traditional forms of proofs in the mathematical course, and difficulties with the manipulation of the logical system for students. Using PAs could help overcoming these obstacles: as they offer an environment with (positive and negative) feedbacks in building a proof, and by taking in charge the checking of the validity of the steps of the proof, they permit to concentrate on the construction of the proof itself. As the tool focuses on aspects regarding the formal rules of the proof, they also allow to distinguish what concerns syntax from what concerns semantics, and could favour their dialectic in the development of proofs.

Although the idea of using PAs for teaching is not recent (e.g. Geuvers and Courtieu, 2007), there are still few experiments developed in a didactical research setting for the learning of proof in University mathematics. Thoma and Iannone (2022) compared the proofs produced by students engaged in a PA workshop (with LEAN) and other students. They observed two characteristics in the first group: accuracy of the use of mathematical language and proof writing resembling academic style, and division of proofs in goals and sub-goals. They hypothesize these characteristics to be an effect of using the PA. In a different perspective, Bartzia et al. (2022) analysed five different PAs (including Edukera) for teaching, through the lens of one same classical exercise for the first year of university mathematics. Their a priori analysis investigated possible effects on student's learning of proof, and characteristics that are likely to strengthen or hinder these effects. They identify that PA can support:

- the reading, understanding and appropriate use of definitions;
- the reading, and control (and writing in some cases) of formal statements;
- the focus on the current proof state (possibly to the detriment of the structure and other parts of the proof) – with an exception for Edukera, where the entire text of the proof is always visible.

They also underline the various feedbacks not available in a pen-and-paper proof, but mention that the automation of formal aspects of the proof can lead to a "tunnel effect" where students achieve a proof without completely figuring out what they did.

In the line of these studies, we have experimented the introduction of a PA in a first-year course in mathematics. In this paper, we focus on student's perceptions of the PA, through the dimensions of utility, usability, and acceptability of the ILE. We address two research questions:

Q1 In which way do the students consider that using Edukera helped them in learning proof? and, on what aspects of proof do they find Edukera helpful?

Q2 Considered as an interactive learning environment (ILE) for the learning of proof in mathematics, how is Edukera perceived by its users, the students?

Based on the above discussions, we formulate the respective research hypotheses.

H1. Using an AP in university mathematics, as a tool that makes explicit (and controls) the logical rules governing the proof, can help students in:

- Understanding these logical rules, and their meaning (status of the statements, links with the logical operators and quantifiers…), and progressively distinguishing *truth* and *validity* (Durand-Guerrier, 2008);
- Identifying the syntactic and semantic dimensions of proof, being able to articulate them, and particularly connecting the syntactic forms of proofs to the structure of the statement to be proved;
- Understanding the global structure of proofs, and from this, developing skills in analysing, understanding, and writing proofs.

H2. First, the control by the PA on the validity of the proof steps, and the justification of all the goals, permits feedbacks (impossibility of some tactics, unsuccessful paths, unachieved remaining goals…) that help students in developing proofs and learning about them; and this automatic logical control allows any valid proof to be accepted by the PA, which can contribute to the learning of proof. Second, the exerciser format enables students to organize their homework as they want, and to progress gradually, possibly trying various solutions, which can foster their learning of proof.

We defend that these hypotheses strongly depend on links and alignment between the goals, contents, and proof styles in the course and the ILE.

EXPERIMENTAL CONTEXT AND THE EDUKERA PROOF ASSISTANT

Due to our objectives and some constraints, and in particular the need for an autonomous work from students on the PA, we have selected the Edukera platform. Edukera (Rognier and Duhamel, 2016) has been built for an educational purpose, and relies on Coq, a specialists' PA developed at INRIA (https://coq.inria.fr/). Edukera is no longer maintained but still available in its current state. It takes the form of an online exerciser where students have to build proofs of given statement, with a graphical presentation of the whole proof and a point-and-click interface (see figures 1 and 2). It has an imperative approach to proof, that is, "the user orders changes to be performed on the proof state (the current set of declared variables and constants, assumed hypotheses, and goals) using a predefined set of orders (also called tactics)" (Bartzia et al., 2022, p. 254). It proposes various pre-implemented proof-exercises, in witch teachers can make a selection, organized in a progression of notions (from boolean logics to set theory and functions) and difficulty. It includes tutorial exercises, where each new notion or tool is introduced. Edukera allows to choose various presentations of the proofs. We have chosen the mode called "Fitch", which is the closest to traditional mathematics proofs (see figures 1 and 2). More technical details on Edukera an other PAs can be found in Bartzia et al. (2022). Figure 1 presents the user interface, and an exercise-proof for the theorem "the preimage of the intersection of two sets is the intersection of the preimages of the sets". Edukera displays the variables declaration and the hypotheses on the top of the proof in construction, and at the bottom, the goal "to be justified".

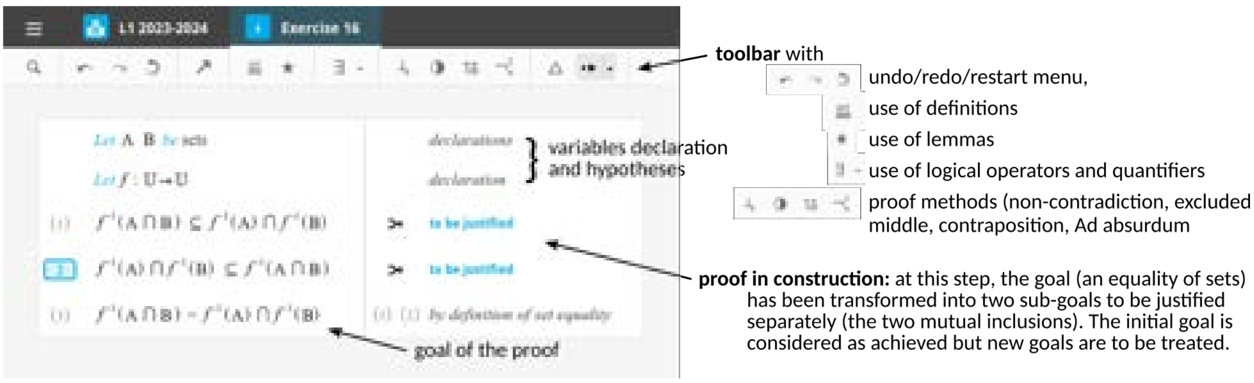

Figure 1: An exercise in Edukera, and the user interface.

As a first step, the user can work on the goal, using the definition of the set equality. Two new goals (the two mutual inclusions) appear as "to be justified" and the initial goal is considered as proved under those two new statements. The user can then decide to work on proving the second goal. Using the definition of the inclusion on this goal, and then the definition of the intersection on the hypothesis $x \in f^{-1}(A) \cap f^{-1}(B)$. Figure 2 shows what we get and possible following steps in the proof construction (right) in order to achieve it (no more statements to be justified).

This short example illustrates the way Edukera works, and the interactions a student can have with the proof in it. It will support our research hypotheses below.

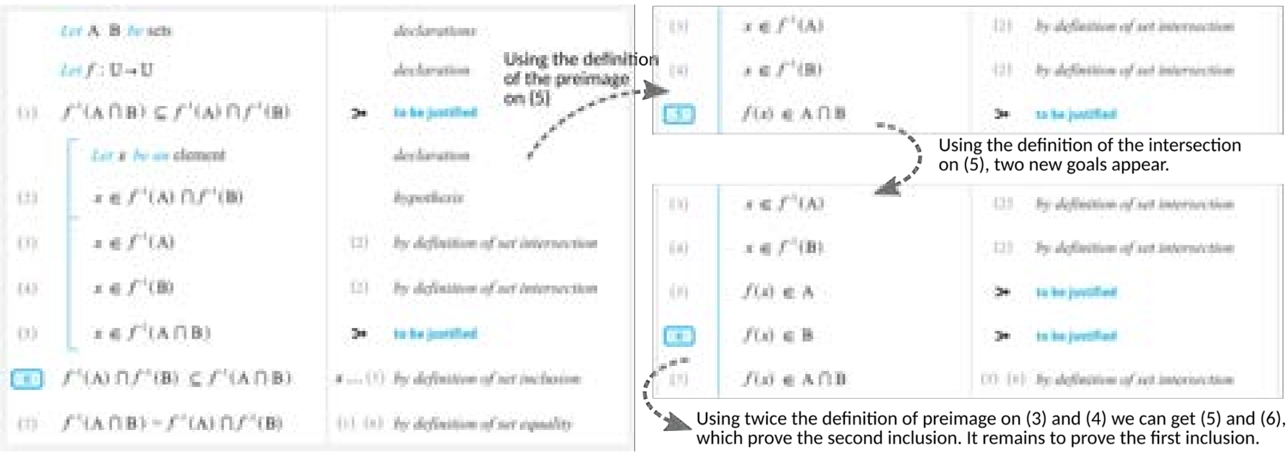

Figure 2: State of the proof in the following steps of construction (excerpts on right).

The use of the PA has been introduced in the first year of the bachelor of mathematics at the University of Montpellier (Kerjean et al. (2022) present various recent teaching experiences with PAs, including this one). The course, taught by the author of the paper, is a small course in the beginning of the year, called "Reasoning and set theory" which aims at introducing (before starting Algebra I and Analysis I) basics on the mathematical discourse, logics (boolean and first order logics), classical reasoning and proof techniques, notion of set (vocabulary and methods), and basics on functions (including injectivity, surjectivity, and bijectivity). The course attempts to make explicit the rules governing proofs, and aims to enable students to write proofs on elementary objects known since high school. This should enable them to better understand the presented proofs and to produce proofs in these courses.

The course is very small (9h lectures, 10,5h tutorial sessions) and concentrated on the three first weeks of the semester. Due to the limited teaching time, the PA has been incorporated as homework: it is introduced during the lectures in the beginning of the semester, and then the students are expected to work autonomously (the exerciser is structured to introduced features and concepts progressively, with tutorials). Two 1h extra lectures/demo/Q&A have been added in the semester. The PA has also been integrated as one of the three assessments, with the two first being traditional written tests. Students are expected to complete exercises regularly throughout the semester, and are evaluated on the number of exercises completed. We have made a selection of 170 exercises. Each proof-exercise can be attempted as many time as wanted, even when solved (while remaining considered as validated). This allows students to do (and redo) a lot of exercises. The grade obtained depends directly on the number of exercises validated (following a scale available in figure 5).

METHODOLOGY AND DATA COLLECTION

In order to study the perceptions of students according to these hypotheses, we designed a questionnaire to be completed by the students at the end of the semester, after the assessment. This questionnaire is composed of open questions, and Likert scale questions of agreement with some statements (Strongly agree / Somewhat agree / Somewhat disagree / Strongly disagree). The questions are presented in table 3.

Item	Questions	Type
1	Knowledge of Edukera and proof: I know how to use Edukera (for the type of exercises given this year) I know how to make proofs (for the type of exercises given in this course) I enjoyed using Edukera	 Likert Likert Likert
2	Give one or more **positive** aspects of Edukera:	Open
3	Give one or more **negative** aspects of Edukera:	Open
4	On learning proof in mathematics: Using Edukera helped me better understand the structure of a mathematical proof Using Edukera helped me better understand how to write a proof in mathematics Using Edukera helped me learn how to start a proof in mathematics Using Edukera has helped me find ideas when I need to produce a mathematical proof Using Edukera has helped me better understand when I read a proof in mathematics	 Likert Likert Likert Likert Likert
5	What Edukera **helped me understand** about proof in mathematics:	Open
6	What Edukera **did not help me understand** about proof in mathematics:	Open
7 and 8	Email if you agree to be contacted for research purpose, and free comment space	Open

Table 3: Content of the questionnaire

Item 1 permits to get auto-evaluation of students on proof, and the use of Edukera, and to know how much they appreciated working with the PA. Items 2 and 3 are designed to collect both positive and negative views on Edukera and its implementation in the course. Item 4 addresses the perception of benefits of Edukera regarding various dimensions of mathematical proof. Items 5 and 6 should give access more specifically to perceptions of difficulties and learning regarding proof. Item 8 can contribute in all those dimensions, by allowing extra comments.

This questionnaire was completed in December 2021 by 57 voluntary students (among 153) as a pilot study. Based on the preliminary results, it has been passed in 2022 in a mandatory format by 119 students (among 154). We will analyse here the answers from the two samples (n=176). We have also collected, for each year, the results of the whole students cohort (n=307), as a indicator of there investment in the PA and their achievements in it.

We will rely on quantitative and qualitative analyses, guided by our research questions and hypotheses. We will explore *Utility*, *Utilisability*, and *Acceptability* of the PA by using methods described by Tricot et al. (2003) as ILE's *evaluation by inspection* and *empirical evaluation*. Table 4 summarize the way we considered and evaluated the dimensions of utility, utilisability, and acceptability. Acceptability relates to the representations of utility and usability; in this sense, our empirical evaluation of students' perceptions informs particularly on acceptability.

RESEARCH RESULTS

Figures 5 and 6 summarize the principal quantitative data. In figure 5, we can see that more than 50% of the students solved 70 exercises or more in 2021 (mark 16/20 or more), and 60 exercises or more in 2022 (mark 15/20 or more). In 2021, only 17 students (11%), and 22 (15%) in 2022, did less than 25 exercises (marks less than 10/20), while more than 35% in 2021 (more than 25% in 2022) solved more than 90 exercises. We can notice that the two samples have quite similar profiles.

	Evaluation by inspection	**Empirical evaluation**
Utility – learning efficiency of the ILE, matching between the ILE and the learning goals. ***Focus on the learning of proof.***	In this paper, this evaluation relates mostly to previous works, existing research (like Bartzia et al., 2022), and *a priori analyses* of PA, leading to our hypotheses regarding (Q1) for the Utility, regarding (Q2) for the Usability, and regarding both (Q1) and (Q2) for Acceptability.	Students' perceptions of the learning benefits of the PA, mostly in items 4, 5, 6, and 8, but also 1 and students achievements.
Usability – possibility to use the ILE, its manipulation and its interface. ***Focus on the PA itself.***		Students' perceptions of Edukera, as a digital tool, mostly in items 1, 2, 3, and 8.
Acceptability – representations about the ILE regarding its utility and acceptability, and conditioning the decision to use it. ***Focus on the motivational aspects for using the PA.***		Student's perceptions of the learning benefits, and of the digital tool itself, and of the alignment between the course and the ILE. The motivational factors they mention. This can concern all the items. Achievements of the students (results and investment, see figure 5)

Table 4: Dimensions of the PA's evaluation, and contribution of the collected data.

It seems that the PA was rather usable: many students has involved in using it, and many exercises have been done. According to figure 6, in 2022, 80% of the students consider that they know how to use Edukera (73% in 2021), and 60% enjoyed using it (67% in 2021). This also contributes to the acceptability of the PA.

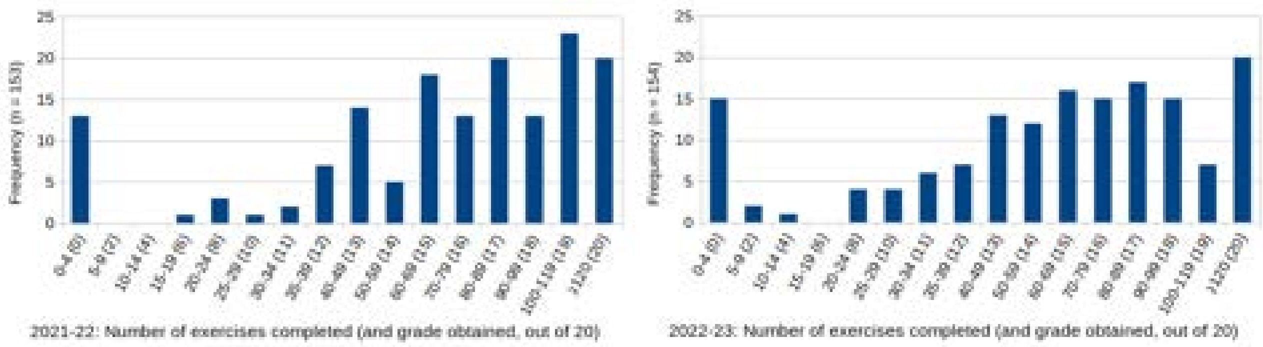

Figure 5: Distribution of the students by number of exercises treated (and marks).

Concerning Likert questions, we had quite similar distributions of the answers in 2021 and 2022, so we present here only the results for 2022 (higher number of answers). The answers to the diverse questions of item 4 are mostly positive, and rather few students strongly disagree. This supports the utility of the PA, and contributes to its acceptability. The two statements with the highest agreement are those related to understanding of the structure of the proof (70% agree, 29% strongly agree), and to the way to start a proof. Follows the statement about understanding when reading a proof. The two lowest (but still good) scores are for finding ideas, and writing a proof. This hierarchy appears coherent with **H1**. Only the score concerning writing proof looks contradictory (44% in 2021 and 45% in 2022 disagree). We will discuss this later, in the light of the qualitative analysis.

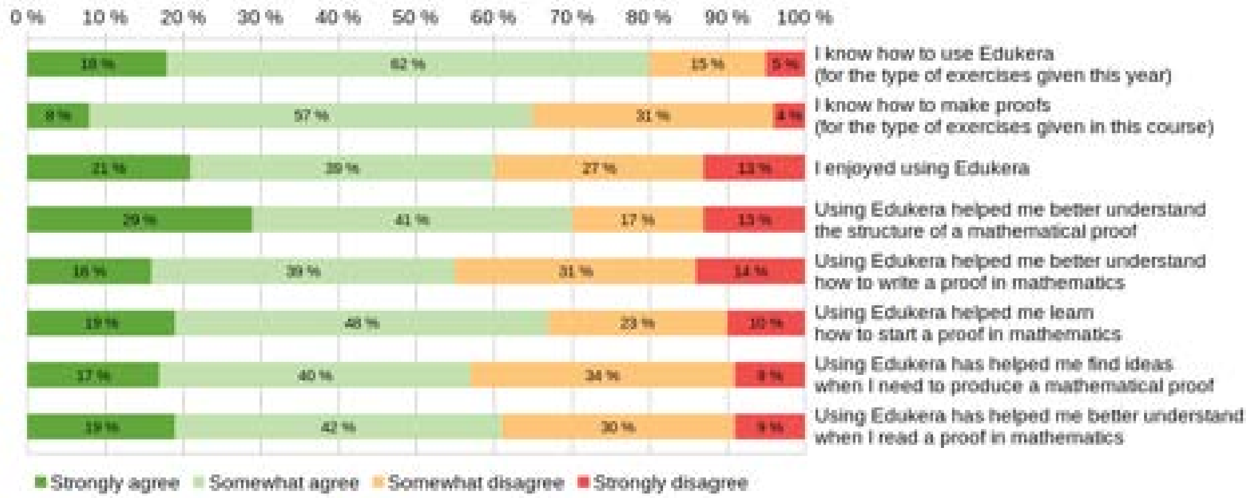

Figure 6: Students' answers in Likert scales in 2022, items 1 and 4, on 119 answers.

About the qualitative analysis of the data (answers from 2021 and 2022), for this paper, we will limit ourselves to present and discuss our principal observations. Regarding utility, many students recognize benefits or interest of the PA for learning contents in link with proof (logics, methods, rigour, writing proof…). Some point out the interest of disregarding the writing issue to focus on the logical arguments of the proof. Among the negative aspects of Edukera, some students didn't understand the mechanisms of the PA, or pretend that they knew how to make pen-and-pencil proofs of the exercises but did not manage to make the proofs with the PA, or consider that one can achieve many exercises by trial-and-errors without understanding. All these perceptions certainly contribute to the acceptability aspect.

When asked about what they learned from using the PA, students mention many aspects of proof: how to start a proof, the structuration of the proofs, how to write a proof, the meaning of operators and quantifiers, and the manipulation of hypotheses or other knowledge related to the deductive reasoning. Students even mention very precise notions, like scopes of the variables or specific type of proofs (inclusions, ad absurdum…). These results are coherent with **H1** and argue in favour of the utility of the PA. However, an element appeared more prominently than expected: many students say they learned about the role of the definitions in proofs, their importance, and the way they are used. A student express that he learned from the PA "the fact of going back to the definitions (and quoting them every time you use them, and that it is not trivial)". This was especially present in the conclusions of Bartzia et al. (2022). Indeed, this is also about the logical status of the statements in the proofs, mentioned in our research hypotheses, and known by research as an key issue in learning proof.

Difficulties in learning with Edukera mainly concern linking pen-and-paper proofs with formal proofs, as already mentioned, and specific tactics, or proof methods like excluded middle, or ad absurdum, which are known as innately difficult for students.

About usability, some students found the PA simple to use once they got familiar with it. Some enjoyed the interface and its design, or the exerciser aspect (progressive and autonomous work, possibility to retry many times…). This relates to acceptability, and many motivational aspects are also mentioned: playful aspect, liberty and mobility in the work, quantity of exercises… The form of the assessment is also mentioned as motivating. Usability is also the dimension that concentrates most of the negative comments. The principal problems perceived by the students are: the difficulty to familiarize with the PA, the lack of explanations, and the many bugs in the software. This adds to a negative evaluation of the acceptability. Another acceptability issue mentioned by the students is the distance between the course and the PA, and in particular the vocabulary of tactics, and the representations of proofs. The need for alignment between the PA and the course was one of our points. This can explain the weaker score of the "writing proof" statement mentioned above.

Although the PA was only used in homework, the perceptions of students contains many positive points. Most of the criticism (bugs excepted) concerns the place of the PA in the teaching setting. It seems possible and reasonable to plan more tutorials to familiarize with the PA, to connect better the course with the PA and its features, and to consider work on links between formal proofs in the PA and pen-and-paper proofs.

CONCLUSION

Our results support the learning potential of proof assistants. The role of definitions has appeared stronger than expected in students' perceptions, and must be further studied. One of our recommendations regarding the use of PA for learning proof is the need for a support to familiarize with the PA, and explicit links between the course and the PA. Our results are comparable to those of Iannone and Thoma (2023) on similar issues. They identified stronger difficulties than we did regarding the syntax of the PA (Lean). It can be due to the specificity of the PA, and research such as Bartzia et al. (2022) should be continued. This research is focused on students' perceptions, which is not enough to validate the benefits of PA in learning proof. Further research is needed to address the effects on students' proof and proving skills.

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To prove or not to prove? A case study of one university mathematics lecturer’s substantiation routines

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Lectures using “chalk talk” are a central means by which students are introduced, often implicitly, to standards of proof in university mathematics. In a case study of one lecturer, video recordings of lectures on algebraic geometry are analysed, with the aim of characterising ways in which statements are communicated as being true. The lecturer uses a range of substantiation routines less rigorous than formal proof, but still playing a key role in becoming convinced of mathematical claims. Although precise communication supports substantiation, low levels of precision are not always associated with low levels of endorsement. This adds new insights into where and how precision is communicated in chalk talk. We argue that decisions about what to omit and when to be imprecise are important features of mathematical discourse.

Keywords: Teachers’ and students’ practices at university level, teaching and learning of logic, reasoning and proof, substantiation, mathematical discourse, lecturing.

INTRODUCTION

Proving is a central part of mathematical practice. However, the concept of a “proof” does not itself have an accepted formal definition, but is rather socially defined, since expert mathematicians are confident that they can recognise a proof when they see it (Cabassut et al., 2012). Moreover, rather than relying solely on formal logical deduction, “working mathematicians insist on the informal and semantic components of proof” (ibid., p. 170). A key challenge for research is therefore to understand how this expert sense of what counts as proof is conveyed to students. Lecturing is still a crucial part of the teaching of university mathematics, not least in more advanced, so-called proof-based courses (Melhuish et al., 2022). Hence, it is important to understand how proving is presented in lectures. Lecturers’ presentations of proofs in university mathematics lectures can be seen not only as introducing students to mathematical content but also as a way of modelling mathematical thinking, including proving (Fukawa-Connelly, 2012). This study therefore aims to characterise ways in which one university lecturer communicates to students that they should accept particular mathematical statements as true, focusing on the presentation of proofs.

ASPECTS OF MATHEMATICS LECTURING

There is a certain uniformity to the outer form of mathematics lectures. In their influential study, Artemeva and Fox (2011) noted the universality of “chalk talk” as the primary means of teaching university mathematics. Two features of chalk talk are *running commentary*, where lecturers simultaneously talk and write on the board, and *metacommentary*, where they talk without writing, usually about what has already been written (or what is about to be written). Fukawa-Connelly et al. (2017) described in more detail what characterises board work and metacommentary, noting that the formal mathematics is presented on the board while the more informal mathematics (explanations, heuristics…) was communicated orally. However, they also found that students typically only make notes of what is written on the board, apparently placing less value on "informal" oral content. This raises questions about the purpose of metacommentary, and suggests a need for further, in-depth research about what is communicated through both running commentary and metacommentary.

Moreover, one aspect of "informal" lecture content that merits further investigation is the role of human agency and subjectivity. While substantiating mathematical statements through proof is often seen as impersonal and "humanproof" (Sfard, 2008), research into spoken English in academic settings has argued that use of personal pronouns is an indicator how the respective roles of speaker and audience are conceptualised (Fortanet, 2004). In the case of proving, pronoun use could perhaps shed light on the division of labour between the lecturer and students: who contributes what to the proof. However, the use of personal pronouns in mathematics can also be a way of articulating generalities (Rowland, 1999). The use of "we" and the generic or impersonal "you" potentially create a sense of distance between the speaker and the mathematical statements, presenting them as general, possibly invariant, rules. In contrast, "I" and the specific or personal "you" reduce the distance between the speaker and the statement, giving the impression of uncertainty or subjectivity. Hence, pronoun use in the presentation of proofs could also serve as markers of the implicit objectivity or subjectivity of the narrative.

THEORY

We take a commognitive perspective, viewing mathematics as a discourse and therefore mathematics learning as becoming part of a discourse community. A discourse is characterised by its use of words, visual mediators such as symbols and graphs, routine ways of performing tasks, and endorsed narratives, that is, statements viewed as acceptable within the discourse (Sfard, 2008). A central type of routine, substantiation is the process of being convinced that a narrative can be endorsed. "Being dependent on what participants find convincing, routines of substantiation are probably the least uniform aspect of mathematical discourses" (Sfard, 2008, p. 231). Similarly, routines are situational and person-specific (Lavie et al., 2018). In these terms, the aim of this study is to begin to understand the range of substantiation routines that are possible within the university mathematics discourse, and in particular within the discourse of proving, which is itself concerned with formal substantiation in the everyday sense of the word. To this end, we address the following research question: How can the substantiation routines used by the lecturer in connection to proof be characterised and classified?

METHODS

This research was conducted at a large Swedish research university and focused on one university lecturer who is an active research mathematician, has more than 10 years of experience as a lecturer and is the recipient of a university pedagogical award. Lectures formed part of the master's level course Commutative Algebra and Algebraic Geometry. There were 13 students enrolled in the course, about half of whom attended the lectures, which were taught in English, which is not the lecturer's first language. This fact contributed to the choice of this course for data collection, since the first author does not speak Swedish. Video recordings were made of three 90-minute lectures during April and May 2023 and subsequently transcribed, and photographs taken of board work. The topics of the lectures were the Riemann-Roch Theorem; modules, module sheaves and constructions on sheaves; and quasi-coherent sheaves, respectively. The sections of the lectures identified by the lecturer as showing proofs formed the primary data for analysis, with the remaining sections used for reference.

The overall approach to data analysis was examining the data in multiple ways, going backwards and forwards between transcripts, videos and photographs, looking for repeated patterns or routines. Analysis strategies included the following:

- Coding for repeating ideas or patterns. Some example codes were: *rhetorical questions*, *use of examples*, and *closing conditions for proofs*. Codes were also created for words used repeatedly by the lecturer, for example *careful*, *control* and *intuition*.
- Concept mapping the initial codes, looking for similarities and connections.
- Scanning for specific features. The data were re-examined several times with different foci to get a sense of which might be routine features across all three lectures and which were only employed occasionally or only in one lecture. Some examples were: comparing what was spoken and what was written, uses of personal pronouns, uses of the word "is", and metaphors.
- Diagrams of proof structures were generated to help visualise the overall structure and structural elements of how proofs were presented.
- Research journal. Alongside the other approaches, ideas were recorded in an ongoing written record. This served as a means of communication between researchers, as well as a way to form and develop thinking through writing.

The preliminary analysis was conducted by the first author, with regular meetings to discuss ideas. This analysis was then reviewed and refined by the second author. We believe this team approach strengthens the analysis through our complementary perspectives. The first author is an education researcher and mathematics teacher with experience of teaching in a variety of educational settings and phases in the UK. Having studied mathematics only to undergraduate level, she primarily brings a pedagogical perspective and is a relative "outsider" to the university mathematics discourse. This outsider viewpoint has enabled us to question aspects of the lectures that might be taken for granted by a more expert mathematician. The second author is a mathematics education researcher with extensive experience of using commognitive tools in analysing university lecturing practice. He has also studied mathematics at the graduate level and beyond, and has experience as a university lecturer in mathematics. He thus contributes more of an “insider perspective” to the analysis.

RESULTS

The lecturer in our study demonstrates a range of discursive practices for becoming convinced of a mathematical statement, at least for the purposes of the lecture course, including several that are less rigorous than formally proving. He typically compartmentalises statements that are not proved, justifying them in other ways using the omission and explanation routines outlined in Table 1.

Endorsement	**Substantiation routines**	
Low	Omission	1. Black boxes 2. Inviting verification 3. Exercises 4. Postponing
	Explanation	1. Examples 2. Intuition 3. The main idea
High	Proving	

Table 1: Substantiation routines, in approximate order of level of endorsement produced from low to high.

By implication, what is left are the things that *are* proved. Proving could therefore be seen as the default means of becoming convinced. In the following analysis, we use the term “proof” to refer to the entire argument and “proving” to mean the process of presenting a series of mathematical statements and logical deductions. Hence proving is one possible element within the structure of the overall proof, although our focus here is on that which is not fully proved. Notably, the lecturer habitually shifts between different levels of endorsement several times over during a single proof.

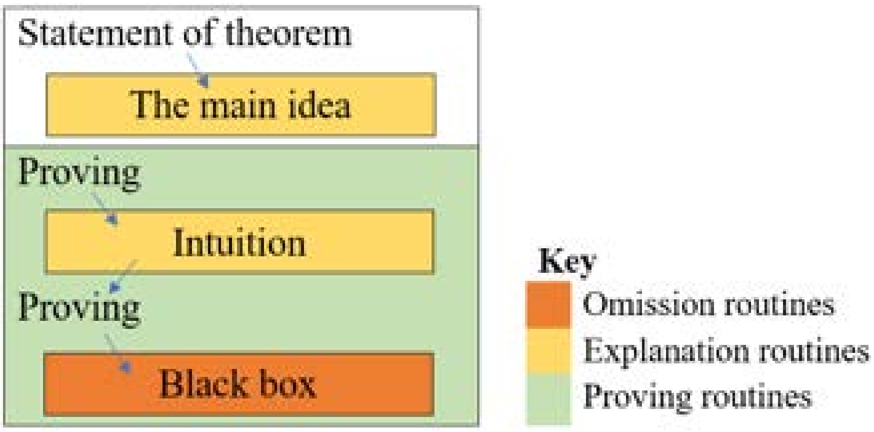

Figure 1: Substantiation routines within the proof of a theorem.

For example, in the proof represented in Figure 1, the lecturer first states the theorem and gives the main idea behind the proof (explanation). He then moves between proving and explaining the intuition behind an unproved "fact", finally citing "technicalities" that are not shown at all in the lecture (omission) to complete the proof.

Levels of precision

In our data, the *running commentary* and *metacommentary* appear to be associated with different levels of precision. In metacommentary, the lecturer frequently uses imprecise or vague language for both objects ("things") and processes ("somehow"), as well as employing metaphors from outside mathematics ("avatar . . . world"):

> [You] use the fact we had earlier that says that yeah *somehow* you can globalize *things* from stalks to sheaves. So by satisfying this universal property this is what we mean by F being the closest possible sheaf to F prime. Any map from F prime can be uniquely expressed as a map from F. So *F is the avatar of F prime in the world of sheaves*.

In addition, he frequently points out the imprecision within metacommentary:

> Somehow, I haven't really defined div f and so on for the constant zero function. I'm sweeping this under the rug, that's not the important part here.

What is perhaps more interesting is the precision of language during running commentary. It would be natural to assume that when the lecturer is simultaneously talking and writing, his speech will exactly match the writing on the board. Curiously, this is rarely the case. Since doing two different things at the same time is not easy, it seems reasonable to suppose that there are pedagogical reasons for these differences. In fact, the running commentary often seems to function to make the board writing more precise. For example, when the lecturer writes that $f \in K(C)$, in the commentary he emphasises that "f is *some* element of the function field", that is, that f should be taken as a general element rather than a specific one.

Not only are there differences between written mathematical notation and speech, but there are also differences between words that are written and spoken, such as saying "because" but writing "since". Occasionally, the lecturer corrects written words, for example rubbing out "means" and replacing it with "is equivalent to". This implies that written words are carefully chosen and therefore that the lecturer is aware of nuances of meaning in the words that he writes as much as in the notation. In general, both board writing and accompanying running commentary tend to involve high levels of precision while metacommentary is often less precise, although this is not always the case. As we move on to look at substantiation routines, we will consider where these different levels of precision and imprecision occur.

Omission routines (low level of endorsement)

The lecturer engages in various discursive routines for justifying the use of unproved mathematical statements, that is, treating them as if they were endorsed. In roughly ascending order of "strength" as a means of substantiation, these are: *black boxes*, *inviting verification*, *exercises*, and *postponing*.

In all the omission routines, decisions to exclude information are only communicated as spoken metacommentary. When talking about his decision to exclude information, the lecturer uses “I”, as in “*I* won’t go into details”, and when delegating proofs to the students, he uses the personal “you”. However, when talking about unseen proofs he uses generic terms like “one”, the generic “you”, and even “it”. Hence the decision on whether to show a proof is presented as personal and subjective, whereas the proofs themselves are given a sense of objectivity. Moreover, unproven statements are written precisely on the board with the usual running commentary and are often labelled “fact”, “exercise” or “technicalities” to distinguish them from statements that are proved. It therefore seems that levels of precision do not necessarily correlate with levels of endorsement, since unsubstantiated statements can be presented in precise terms.

1. Black boxes

On several occasions, technical details are omitted by simply stating that a proof is possible. Students are not expected to see the details of these proofs at any point in the course; the “black box” is never opened, certainly for the students and possibly for the lecturer as well.

> I think that a lot of technicalities are hidden because otherwise it would take too much time and energy and they don't give you much of an idea of this.

> You can show, but I'm out of board so I won't, that this is a linear map with inverse mapping a function g to g divided f. So the two vector spaces are isomorphic as vector spaces and the proof is complete.

2. Inviting verification

In some cases, the lecturer states that a proof is possible and also invites students to check the proof themselves, if they decide it is necessary. While this could just be a strategy for making a black box sound more plausible, it could also be a pedagogical device to encourage students to take responsibility for personally becoming convinced, rather than relying on the lecturer’s authority as a teacher and expert.

> Let me write this down quickly and leave it to you to verify that what I'm writing down is reasonable.

3. Exercises

Students are assigned some proofs to be completed as exercises outside of the lecture. In these cases, they will see the proof at some point in the future, although they temporarily have to take the statement on trust. Exercises and inviting verification routines are notable for their use of the personal “you”, involving the students directly in the proof, rather than including them indirectly through the use of the pronoun “we”, which is typical elsewhere.

> I leave this as an exercise. So there are two types of proofs I leave as an exercise: proofs that are simply too difficult and too tedious and proofs that actually are easy if you sit down

and work on them. This is somehow in the middle. It's not hard, it's easy to believe maybe. You need to work out the details.

4. Postponing

On two occasions the lecturer postpones a proof until later in the lecture, meaning that students will see the proof soon but are required to temporarily accept the unproved statement as if it were already endorsed. In fact, the proofs are only postponed by two and four minutes respectively, so the students do not have to wait long. Since this only happens twice, it is hard to conclude that this is a routine in this lecturer's teaching. However, we include it as it fills a gap within the categorisation of possible omission routines. It is distinct from setting exercises in terms of both the time frame (the omitted proof takes place within the same lecture) and the division of labour (the lecturer assigns the substantiation to himself rather than to the students).

> We will get there in a minute why this is true. If we accept that this is true, what have I done?

Explanation routines (medium level of endorsement)

In the omission routines just described, mathematical statements are introduced without any substantiation, at least at the time. In contrast, in explanation routines the lecturer offers justifications for endorsing mathematical statements that are less rigorous than actually proving them.

1. Examples

The use of examples within proofs suggests that the lecturer is aiming for understanding, not just reaching the conclusion of a logical process. Examples are generally written on the board, often using precise notation, and accompanied by running commentary. Examples are also often used in conjunction with other substantiation routines, to illustrate an intuition or even another example.

> And so this might look a bit abstract but what is the guiding example that we have, or a guiding, the guiding example.

2. Intuition

When the lecturer introduces an unproved fact, he often goes on to explain why the fact makes intuitive sense. The use of pronouns here is rather mixed, serving several purposes at once. "I" is used when talking about whether or not to show a proof. However, the intuition itself is sometimes presented as objective and sometimes as subjective. For example, "it's exactly what *you* expect" seems to be a generic "you", suggesting that this is the expectation that anyone would have. On the other hand, he also asks "what is *your* intuition?" as a genuine enquiry to the students in the room. These mixed pronouns perhaps help to convey that intuition is a lower level of endorsement than proving. Indeed, alongside appeals to intuition the lecturer also models a mistrust of one's first instincts, often using the pronoun "I" to emphasise the subjectivity of this form of endorsement.

> So I'm going to write something that looks obvious at first glance, wrong at second glance, but correct at third glance . . . at least, these were the three thoughts that went through my head when I wrote the proof two minutes ago.

The lecturer also employs both metacommentary to explain the role of intuition and running commentary when writing intuitions on the board. As with examples, intuitions are usually labelled to distinguish them from statements with higher levels of endorsement or are differentiated visually using quotation marks or different coloured chalk.

3. The main idea

The lecturer frequently states the main idea behind a proof, either before or after proving. This main idea is clearly not enough in itself to endorse a mathematical statement since it is presented in addition to proving. However, it is a key part of the process of becoming convinced as it supports students' understanding of the proof. The lecturer sometimes generalises the main idea, for example referring to what you would "usually" do or to "proofs of this type". Hence, explaining the main idea could also serve the purpose of increasing students' understanding of other proofs.

The main idea is frequently introduced using the generic "you", as well as (less often) the generic "we", with the implication that the actions are generalisable and could equally be carried out by anyone. In addition, it is usually communicated through relatively long chunks of metacommentary, frequently using imprecise and colloquial language. This again appears to break the link between precision and substantiation. In this case, the main idea is conveyed in an imprecise way but also plays an important role in becoming convinced.

> So why did I show you this proof? Well, the main idea here is to tweak your divisor. You want to prove something of the divisor and you tweak the divisor. So the divisor, definition of the divisor is a rather intuitive, how to say this? It's a flexible notion, a divisor is just a bunch of points plus each other. It's a formal sum. So since it's a formal sum, you can add whatever you want. And so you add some clever, easily controlled other thing so that the result is something you can say something about, and from this you deduce something about your original divisor. This is emblematic to proofs of this type.

DISCUSSION

A key finding of this research is that calling on forms of substantiation less rigorous than formal proving is a normal, embedded part of the practice of presenting proofs in lectures, at least for the lecturer in this case study. This adds to our previous observation that different lecturers appear to follow different implicit "rules" for endorsing mathematical statements (Viirman, 2021). It also corroborates the findings of Lai and Weber (2013) that some lecturers see lecture proofs and textbook proofs as serving different purposes, so that "for oral proofs given in lecture, they feel it is permitted or even beneficial to omit details and draw inferences from diagrams. In other words, the epistemic requirements of written proofs are relaxed for proofs given in lectures" (p.105). Lai and Weber’s data involved observing mathematicians preparing materials for a hypothetical lecture, so it is interesting to see similar findings within a real lecture situation. However, they go on to conclude that these omissions could be a source of misconceptions, since lecturers may expect higher standards of proving from students than those they use themselves in lectures. We suggest a possible alternative interpretation, that these omissions could in fact be part of the mathematical practice that lecturers model to students, sending the message that fully objective mathematical proofs only exist as an ideal, whereas in practice mathematicians are required to make decisions about what to prove and what to take on trust. The extent to which students do enact omission and explanation practices within their own proofs is a question for further research.

In addition, the current study builds on the work of Artemeva and Fox (2011) in understanding the nature of chalk talk, adding detail about the roles of precision and imprecision during running commentary and metacommentary. Our findings only partly agree with Fukawa-Connelly et al.’s (2017) assertion that most informal content in mathematics lectures is presented orally rather than written. While *the main idea* is generally presented as metacommentary using informal and imprecise language, *examples* and *intuitions* are often written on the board, with varying levels of precision. Similarly, omission routines frequently involve writing on the board with precise notation and running commentary, which could hardly be called “informal” despite the absence of rigorous substantiation. Indeed, spoken running commentary often adds to the precision of written statements. This use of different levels of precision raises questions about the extent to which precision can be considered a general mathematical behaviour, as we ourselves have previously suggested (Viirman, 2021). Additionally, given the ongoing debate about the purpose of the lecture format in university teaching across disciplines (French & Kennedy, 2016), understanding what the spoken content of lectures adds to students’ learning is an important area for continued research.

CONCLUSION

Our study of one mathematics lecturer identified discursive routines for substantiation relating to both different levels of endorsement and different levels of precision. Given the small scale of the research, this is by no means intended as a comprehensive framework but rather is presented as a starting point for further reflection among university educators around the standards of proving and precise communication that are modelled to students. Given the prevalence of chalk talk in mathematics teaching across countries, gaining an in-depth appreciation of how this genre works in practice is crucial to understanding and improving university mathematics teaching.

Furthermore, this research raises questions about whether not only the lecturer, but also the audience of the lecture affects how proofs are presented. The next stage of the research will therefore seek to compare the teaching of proving to mathematics students and to prospective mathematics teachers, with the aim of increasing understanding of the relationship between the disciplines of mathematics and pedagogy at the university level.

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Comparative judgement and its impact on the quality of students' written work in mathematics

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In this experimental study, we investigated the use of comparative judgement as a way to facilitate learning through comparison. Secondary and undergraduate mathematics students (N = 24) evaluated peer work on the topic of solving rational inequalities where solutions were presented to them either in pairs or one-at-a-time. Presenting solutions in pairs resulted in a greater improvement in performance outcomes than presenting solutions individually. Students who compared solutions were more focused on how mathematics was communicated, rather than the final answer, and were more likely to implement 'good' features into their own work. Comparative judgement may therefore be effective for improving performance outcomes because it facilitates the noticing and implementation of 'good' features.

Keywords: Novel approaches to teaching, assessment practices in university mathematics education, comparative judgement, evaluative judgement, learning by evaluating.

INTRODUCTION

In our daily lives, the act of comparison is integral to the decision-making process. This fundamental mechanism extends to the realm of learning, including the context of mathematics education (Alfieri et al., 2013). There are several ways to incorporate comparison within mathematics education, for example, engaging students in discussions where they share problem-solving strategies (Boaler, 1998), using direct instruction that presents side-by-side examples with explicit emphasis on comparisons (Begolli & Richland, 2016), or offering students pairs of worked examples to facilitate self-guided comparison of strategies (Star & Rittle-Johnson, 2009). This study introduces an alternative approach known as comparative judgement, where students assess pairs of peer responses to a question. Responses are presented as pairs and students judge which of the two is 'better'. Students complete multiple rounds of comparisons enabling the responses to be ranked from 'best' to 'worst'.

This study builds on empirical findings which highlight improved performance outcomes for students using comparative judgement. Bartholomew et al. (2019) demonstrated that middle-school students who engaged in comparative peer-based assessment while designing travel brochures outperformed those who participated in a face-to-face peer feedback exercise. In a study with first-year design students generating Point of View statements, Bartholomew et al. (2022) observed that a brief 20-minute comparative judgement task resulted in higher-quality Point of View statements compared with students who did not engage in comparison but were instead given extra time to complete their work. In the context of English, Bouwer et al. (2018)

found that students engaging in comparative judgement produced higher-quality essays compared to those who evaluated peer work sequentially using marking criteria.

While these studies show some promise for improved learning outcomes in a variety of subjects, there is a gap in research addressing whether similar effects are observed in the context of mathematics.

LEARNING BY COMPARING

Since a theoretical framework for learning through comparative judgement has not yet been established, we draw upon the research tradition of learning from worked examples to explain why comparing might be useful for learning. In this paper, the term *worked examples* refers to examples written by educators for the purpose of learning. We use the term *worked solutions* to refer to solutions written by students for the purpose of assessment. Worked solutions may include traits such as scribbling out, hard-to-follow layout, or poorly worded explanations, possibly making them more difficult for students to learn from than a carefully designed worked example.

Learning from worked examples involves providing learners with a problem, the steps that were taken to reach a solution, as well as the final solution. Presenting students with multiple worked examples simultaneously is more effective for learning than providing the same worked examples sequentially (Alfieri et al., 2013). Comparing worked examples facilitates the recognition of underlying structures by enabling learners to notice commonalities across multiple examples which can then be applied to future problems with similar features. According to variation theory, the ability to identify these distinctions is necessary for learning (Marton, 2015). In order to generate new knowledge, one must notice a new aspect which can only occur when it is contrasted against previously noticed aspects in a pattern of variation. The process of comparison becomes a useful means to generate such variations.

While current research argues that comparing worked examples is better for learning than learning from examples one-by-one, it remains unclear whether this holds true in the context of comparative judgement. In comparative judgement, the emphasis is on *evaluating* worked solutions rather than *understanding* worked examples. Additionally, while including variation in worked examples is beneficial, too much variation might exist across student-produced worked solutions. When too much variation is present and both relevant and irrelevant elements vary simultaneously, it can be more difficult for learners to discern relevant information whilst simultaneously ignoring irrelevant information (Marton, 2015).

That said, certain aspects of comparative judgement do align with recommended pedagogical practices associated with learning from worked examples. First, learners are unlikely to notice similarities and differences across multiple solutions without a prompt to do so (Alfieri et al., 2013). Comparative judgement requires students to explicitly compare two solutions, thereby increasing the likelihood that they recognise similarities and differences between solutions. Second, during comparative judgement, students complete a number of comparisons which can generate variety across solutions. Increased variety should increase the chance of students noticing similarities and differences and exposure to multiple approaches should increase procedural flexibility, that is, the ability to select and apply different procedures effectively (Große, 2014). Lastly, Seery and Canty (2018) argue that comparative judgement can support self-reflection and self-regulation. By comparing multiple pieces of work, students can position their own performance against those of others, providing them with a better understanding of the quality of their own work. This requires students to establish their own criteria for proficiency, especially if marking schemes are not provided, which helps students build an understanding of what it means to be capable.

CURRENT STUDY

The current study extends previous literature by examining learning through comparative judgement in the context of mathematics. While our focus is on comparing the effectiveness of different instructional methods in mathematics learning, our approach aligns with the broader goal of comparative judgement research wherein our interest lies in investigating the impact of comparative judgement on the overall quality of students' work. In the context of mathematics, this will likely relate to mathematical proficiency, ability to communicate ideas, and appropriateness of solution methods.

For the current study, students participated in a peer review activity and were randomly assigned to one of two groups. The first group evaluated other students' solutions to a rational inequality problem presented in pairs (*compare* group) while the second group evaluated the same set of solutions one-at-a-time (*sequential* group). We hypothesised that the compare group would outperform the sequential group as current literature argues that comparing worked examples is better for learning than studying worked examples one-at-a-time. Additionally, prompting students to compare worked examples was shown to be effective for learning, and we hypothesised that asking students to select which of two solutions was 'better' would have the same effect as an explicit prompt to compare.

Despite evidence that comparing examples is effective for learning, it is unclear whether this approach extends to the context of comparative judgement. When learning from worked examples, educators carefully create examples to facilitate understanding, while in comparative judgement, solutions created by other students lack intentional instructional design. Additionally, when comparing examples, educators purposefully select complementary examples that make similarities and differences more noticeable. During comparative judgement, when pairs are selected by a computer, pairings are not selected with purposeful variation in mind. For the current study, solutions were not deliberately paired to highlight discernible differences between pairs. Consequently, any potential benefits from comparisons may be negated by the use of worked solutions not designed for instructional purposes or solution pairings that do not emphasise key elements.

METHOD

Participants

The study included 24 participants of which 15 were Year 10 and 11 students from a select entry secondary school (10 female, 5 male) in Victoria, Australia. Students in Year 10 were accelerated students studying mathematics one year ahead of their peers. All Year 10 and Year 11 students were studying the same Year 11 mathematics subject. The remaining nine participants were undergraduate students (1 female, 8 male) who were studying undergraduate mathematics. Secondary school students were unfamiliar with rational inequalities while undergraduate students would not have been shown how to solve such problems recently, if at all.

Design

We used a pretask-intervention-posttask design. The think-aloud method was used to capture students' thoughts during intervention. For the intervention, students were randomly assigned to one of two conditions, the compare and sequential condition, with 12 students in each condition (4 undergraduate students in the comparative group; 5 undergraduate students in the sequential group). Students in the compare condition were shown samples of worked solutions in pairs and asked to judge which of the two they felt was 'better'. Students in the sequential condition were shown the same set of worked solutions one-at-a-time and asked to assign each a score out of 5.

Instruments

Pre- & post-task: Students solved two tasks, a pre-task and a post-task. The pre-task required students to find the set of real numbers, x, such that $\frac{x+1}{x-7} > 3$, where $x \in \mathbb{R}\backslash\{7\}$. The post-task was similar in level of difficulty and required students to find the set of real numbers, x, such that $\frac{5x-2}{x+5} > 6$, where $x \in \mathbb{R}\backslash\{-5\}$. For both tasks, students were asked to write a solution as if submitting it for assessment.

Nine assessors, who were members of the research team or volunteer mathematics lecturers, ranked the pre- and post-tasks. This was done using an adaptive algorithm where assessors evaluated students' work as pairs and asked to judge which of the two they thought was 'better'. In total, assessors made 445 comparisons. This produced a ranking of students' work from 'best' (a score of 100) to worst (a score of 0).

Intervention: Students evaluated a set of eight worked solutions for the same pre-task problem. Solutions included (1) correct and incorrect answers which incorporated both minor errors and conceptual misunderstandings; (2) a range of methods and solution approaches from algebraic to graphical; (3) both high- and low-quality work; and (4) neat and messy solutions. All students were shown an identical set of eight solutions and in the same order for their respective experimental conditions.

Semi-structured interview: Students' pre- and post-tasks were placed side-by-side and students were asked to comment on any aspects of their solution they had chosen to keep the same or any they had changed and explain why.

PROCEDURE

All data collection occurred in a single problem-based interview lasting between 45 and 60 minutes. Students were given unlimited time to complete all activities. Students first completed the pre-task and did not receive feedback. Next, students evaluated the worked solutions, either in pairs or one-at-a-time. Students who compared solutions were asked to form a judgement for each pair using three prompts: (1) Which solution do you feel demonstrated better mathematical understanding? (2) Which solution do you feel was better at communicating their thinking? and (3) Which solution do you feel was better overall? Students in the sequential group were asked (1) How well do you feel this solution demonstrated mathematical understanding? (2) How well do you feel this solution communicated their thinking? and (3) How well do you feel this solution did overall? and were then asked to give a score out of five for each prompt. Students were asked to think-out-loud while evaluating the worked solutions and were informed that there was no one correct evaluation strategy and that the way in which they either chose one solution as better or allocated their marks was up to them. Students were not provided with marking schemes or correct answers. Following evaluations, students completed the post-task without access to previous solutions. Finally, students compared their pre- and post-task solutions and commented on any changes made.

The think-aloud method (Ericsson & Simon, 1993) was used as the primary tool to access students' conscious thoughts during intervention. Interviews and think-aloud data were audio recorded and transcribed. Despite criticisms of think-aloud, notably its incompleteness in capturing underlying unconscious processes, the elicited data, though not exhaustive, remains valuable and informative about students' conscience cognitive processes.

ANALYSIS AND RESULTS

Knowledge gains from pre-task to post-task

Performance gains were measured by taking the difference between students' post-task and pre-task ranking scores. Data were screened for normality. Performance gains were normally distributed with skewness and kurtosis values within acceptable ranges; skewness ranged from -0.40 to 0.34, and kurtosis ranged from -0.22 to 0.83. The Levene's test of determining homogeneity of variance was not violated ($p = 0.578$). A two-sample t-test indicated significant differences in performance gains between groups, with students in the compare group (M = 22.1, SD = 16.7) found to have greater performance gains than those in the sequential group (M = 8.2, SD = 12.3); $t(22) = 2.32$, $p = 0.03$, $d = 0.95$.

Number of comparisons

Using students' think-aloud utterances, we examined the relationship between students' post-test outcomes and the number of comparisons made during intervention. A comparison was considered to be an instance where a student directly compared the characteristics of one solution to another solution (e.g., “They both communicated their reasoning well” or “This one used a quicker method”).

Students in the compare group made a total of 364 comparisons as opposed to 164 comparisons made by those in the sequential group. On average, students in the compare group (M=30.3, SD=12.8) made more comparisons than students in the one-at-a-time condition (M=13.7, SD=13.2), which was statistically significant $t(21) = 3.14$, $p = 0.005$, $d = 1.3$.

We questioned whether the act of making a comparison might be one reason for improved performance outcomes and examined the relationship between students’ performance gains and the number of comparisons students generated. To explore this, a general linear model was used with the number of comparisons as a predictor variable and condition as a factor. Making more comparisons was not found to be predictive of performance gains, $F(1, 23) = 0.07$, $p = 0.790$, $\eta^2 < 0.01$. In short, even though students in the compare group outperformed those in the sequential group, it is unlikely to be because they made more comparisons.

Changes between pre- and post-task

Changes between pre- and post-tasks, as judged by the research team, were analysed to explore differences between groups. When analysing changes in students’ work, the focus was not on the *quality* of students’ changes, but rather on identifying *whether* a change had been made. Thematic analysis techniques were used to group instances of comparisons into codes. The data generated three categories: (1) Accuracy: making an improved attempt at solving the problem; (2) Communication: changing the amount of written explanation, including a heading, adding or removing visual components, changes in the choice of set notation; and (3) Method: changes to the choice of method such as changing from an algebraic to a graphical approach. Subjective qualities such as neatness, quality of the explanation, or whether one method was better than another were not included as the intent was to identify instances of change rather than assess whether these changes resulted in improvement in students’ work.

As an example, Figure 1 shows the pre- and post-tasks for a student in Year 11. At pre-task, this student had found only the partial solution for the rational inequality, did not include any words or written explanation, and included algebraic manipulations only. At post-task, they had included the written annotations “If denominator is positive” and “If denominator is negative” as well as two number lines. This was counted as two changes under the category of communication. Additionally, the student changed their algebraic procedure by attempting to consider when the denominator might be positive or negative. Although the final answer was not correct, this was regarded as an improvement in understanding as the student showed awareness of the need to consider when the denominator is positive or negative. This was counted as one instance under the category of accuracy.

Results for the number of changes identified by the research team are displayed in Table 1. Students shown solutions in pairs made 34 changes between pre- and post-

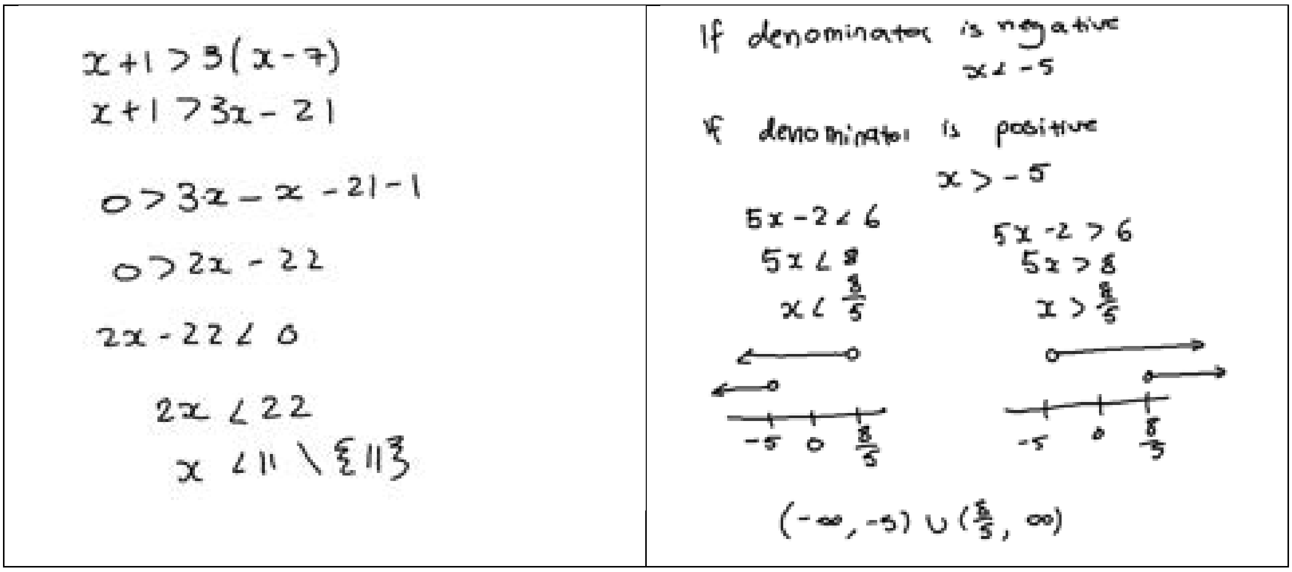

Figure 1: Sample pre-task (left) and post-task (right) completed by a Year 11 student

tasks while students shown solutions one-at-a-time made 13 changes. A Kruskal-Wallis test indicated this difference was significant, H(1) = 6.60, $p = 0.010$. For those in the compare group, most changes were to do with communication.

Next, the types of changes between pre- and post-task students reported during the interview stage were investigated. These comments were categorised using thematic analysis techniques similar to those described above. Elements students reported retaining/changing were grouped under the following five categories: (1) Accuracy: getting the final answer correct; (2) Communication: comments regarding the layout, amount of writing, headings, use of columns, etc.; (3) Method: changes in the approach used; (4) Presentation: making their solution neater; and (5) No changes made. The data are summarised in Table 2.

Elements	Experimental group	
	Sequential	Compare
Accuracy	6	10
Communication	5	23
Method	2	1
No meaningful change	4	-

Table 1 No. of changes made between pre- and post-task as judged by research team

A Kruskal-Wallis test indicated no statistically significant differences between conditions and the number of elements students reported retaining/changing, H(1) = 2.71, $p = 0.100$. While no effect was detected, we note that those in the compare group commented on more elements to do with how information was communicated than other categories, and that these students noticed more elements overall than those in the sequential group.

Elements	Experimental group	
	Sequential	Compare
Accuracy	7	6
Communication	4	11
Method	1	5
Presentation	1	1
No meaningful change	2	1
Total changes	13	23

Table 2 No. of elements retained/changed between pre- and post-task as reported by students

DISCUSSION

The present study investigated the impact of evaluating peer work comparatively on performance outcomes and sought to examine underlying factors contributing to any observed positive effects. Results indicate that students who evaluated peer solutions comparatively experienced greater performance gains than students who evaluated the same peer work one-at-a-time. It has long been known that comparing worked examples prepared by educators in mathematics classroom is beneficial for learning (Star & Rittle-Johnson, 2009). This paper shows that evaluating peer work comparatively is also useful, and that comparative judgement appears effective in the context of mathematics.

Why comparing solutions was more effective than presenting the same solutions individually is still not clear. Existing literature emphasises the benefits of comparing in learning. It is plausible that the improved performance outcomes when comparing is because students in the compare group were explicitly instructed to compare solutions and, as such, generated more comparisons, suggesting that these students had more opportunities to discern structural similarities and differences across solutions. However, we found no significant relationship between the number of comparisons made and performance gain, consistent with previous research (Star & Rittle-Johnson, 2009). Hence the underlying factor is unlikely to be the *quantity* of comparisons students generate, but rather, the substance and quality of these comparisons. Further research is needed to investigate the nature of these comparisons and their relationship with learning.

Furthermore, comparative judgement appears to influence the types of changes students made to their own work. Students who compared solutions were more likely to modify how their work was communicated, made more changes overall to their work, and were better at verbalising these changes. In contrast, students evaluating worked solutions individually did not make as many changes to their own work and were less likely to verbalise these changes. This supports the claim made by Kimbell

(2020) that the reason comparative judgement results in improved performance outcomes is because it enhances students' ability to articulate elements that constitute high-quality work. For the current study, students who engaged in comparison were more adept at identifying and expressing changes in the overall quality of their work with less emphasis on the correctness of their final answer. The implication for educators is that comparative judgement may be a useful tool when wanting to direct students' attention beyond simply solving a problem correctly and towards more holistic elements of quality.

LIMITATIONS

The current study has some limitations. First, it was conducted in a laboratory setting which may have magnified the demonstrated effects compared to a naturalistic classroom environment (Alfieri et al., 2013). Future researchers may wish to see whether the findings from this study are replicated where comparative judgement is used as an authentic classroom activity. Second, this study included a small sample size. Statistical analysis should be interpreted with caution as this may have weakened findings. Future research with larger cohorts is warranted to validate and extend these findings.

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The concept of study and research inquiry: A foundational approach

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This paper outlines the concept of Study and Research Inquiry (SRI). It emphasises how SRIs facilitate the modelling of didactic praxeologies and how the dialectic of media and milieus is central to knowledge creation and validation. The study scrutinises the institutional treatment of questions, highlighting a hierarchy that often marginalises emergent inquiries. It also reconceptualises the notion of authorship in knowledge creation, promoting a model where inquirers are recognised as legitimate authors. Addressing the fragmentation of knowledge, the paper suggests that SRI can mitigate the epistemic deficit caused by the traditional treatment of students as non-authors, fostering a culture of inquiry where knowledge is not only constructed but also critically examined and democratically validated.

Keywords: Dialectic of media and milieus, novel approaches to teaching, study and research inquiry, teachers' and students' practices at university level, the anthropological theory of the didactic.

INTRODUCTION

This research study is being conducted within the framework of the anthropological theory of the didactic, as outlined for instance by Chevallard (2019). The study aligns with the extensive compilation of research encapsulated by Barquero et al. (2022), which has thoroughly examined the design and implementation of study and research paths over the past two decades. Two points need to be made clear from the outset. The first has to do with vocabulary. In the context of the ATD, it has become customary to speak of study and research *paths* (SRPs), a vocabulary that was originally introduced by the first author of this study (see e.g., Chevallard, 2015). Here, we will rather use the phrase "study and research *inquiry*" (SRI), to distinguish between the action (the inquiry) and the effect of the action (the path): when someone inquires into some question, that inquirer opens up a path which, as a general rule, is not charted in advance. Inquiry and path are dialectically linked: at each moment, the progress of the inquiry depends on the path already completed, and the continuation of the path depends on the point reached by the inquiry. The inquiry is the means, and the path the end, which will enable the inquirer to arrive at an answer to the question being studied. Inquiry and path are like the two sides of the same coin.

The second point is even more fundamental. The word "inquiry" that we use will perhaps lead the reader to relate the concept of inquiry to what is known today as *inquiry-based learning* (see e.g., Dorier & Maass, 2020). Although this is not unreasonable, we must be careful not to equate one with the other. The concept of *study and research inquiry* does not refer specifically to a particular study technique, or, rather, to a particular study praxeology. It is a generic tool for modelling study praxeologies of all kinds. Our examination centres on the following research question: "*From a broad anthropological point of view, what are the reasons for developing didactic research on study and research inquiries?*"

THEORETICAL TOOLS

In this presentation, we reduce the elements of the ATD drawn upon to an essential minimum (refer to Chevallard, 2019, for further details). The starting point is the notion of *institution*. Remember, then, that a classroom (with its students and teacher) is an institution, and so is a school, a family, a ministry, a married or unmarried couple, etc. In each institution, there are various *positions* that can be occupied by the *subjects* of the institution: there is the position of student and the position of teacher, the positions of father and mother, of minister and minister's chief of staff; etc. It is worth noting that, generally speaking, a position can exist without being occupied by a person at a given time. Again, in a general sense, the "anthropological" character of a notion or statement is indicative of the fact that this notion or statement refers to any possible institution (in the previous sense). In the framework of the ATD, every human individual is a *person*, that is to say is subjected to a host of changing institutional positions from the cradle to the grave. Finally, we will use the term "instance" to refer to either a person or an institutional position.

Study and research inquiries

To limit the symbolism used, we will often consider the case of a simplified class, noted $c = [\hat{X}, \hat{y}]$, where $\hat{X}$ is the set of students $\hat{x}$ and $\hat{y}$ is the teacher. The fact that c *inquires into* a question Q, that is, *studies* the question Q, will be noted as follows: $(c, Q) \rightarrow A^{\blacktriangledown}$, where $A^{\blacktriangledown}$ is the answer to the question Q that the class c will arrive at. The generic technique of inquiry τ_I described by the ATD is based on what is known as the *Herbartian schema*, which, in its so-called semi-developed form, is written as follows: $[(c, Q) \Rightarrow M] \rightarrow A^{\blacktriangledown}$. Here, M, called the *milieu*, is the set of "tools" on which c will draw to fashion $A^{\blacktriangledown}$.

What are these "tools"? A first category of them is made up of the answers that c may find ready-made in the institutions of society to which c can have access. Such an answer is denoted by $A^{\diamond}$, so that M can be written as: $M = \{A_1^{\diamond}, A_2^{\diamond}, \dots, A_n^{\diamond}, \dots\}$. Let us note here that this model of SRIs is applicable not only to the extensive inquiry conducted by a doctoral student (and his/her supervisor) over several years. It also applies to the simplest inquiry imaginable, such as for example one that seeks to answer the question "What are the first 20 decimal places of $\sqrt{2}$?", which would merely involve copying down what some powerful calculator "answers," that is to say: 41421356237309504880.

Alongside the "ready-made" answers $A_i^{\diamond}$ (in the case they exist), the milieu M also includes what are known in the vocabulary of the ATD as "works," W_j. These may be works of all kinds: mathematical theorems and theories, experiments, historical studies, etc. Finally, the use of all these resources generates questions Q_k that further enrich the milieu M, as do the answers (which are works of a kind) given to these questions during the inquiry and the works used to produce them. The milieu M has the following general form: $M = \{A_1^\lozenge, A_2^\lozenge, \ldots, A_n^\lozenge; W_1, \ldots, W_m; Q_1, \ldots, Q_p\}$.

All this raises an overarching question: What is the "validity" of the answer $A^\heartsuit$ arrived at by the inquiring instance? (A related question, as we shall see, is the following: What is the shareability of $A^\heartsuit$?) The answer of the ATD to the question of the validity of $A^\heartsuit$ consists in what we call the *dialectic of media and milieus*. Here, *media* are any instances that emit messages, while a *milieu* is a system whose reactions to certain types of "disturbances" are governed by laws (i.e., its reactions are not random). A calculator is on the one hand a *milieu* which, when "disturbed" by being asked what the value of a numerical expression (for example $2^3 - 7.54839$) is, reacts by *calculating* this value. On the other hand, it is a *medium* which communicates the calculated value by *displaying* it (in this case, the display is 0.45161). The dialectic of media and milieus aims to compare, through the intervention of media, the reactions of milieus to definite disturbances. The power of the media/milieus dialectic depends on the universe $\mathcal{U}$ of the inquiry, that is, all the institutions (including that of the inquiring instance itself) which provide the answers $A_i^\lozenge$, the works W_j, and the questions Q_k making up the milieu M of the inquiry. When the universe $\mathcal{U}$ is enriched, then the inquiry can be continued (or restarted); for that reason, the result $A^\heartsuit$ is always *partial and provisional*.

Spectators, actors, and authors

To shed light on the foundations of the theory of inquiries developed in the ATD, we now need to introduce a triplet of key concepts. In the framework of the ATD, for any object σ, any instance $\hat{\imath}$ has a certain *relation* to σ, denoted by $R(\hat{\imath}, \sigma)$. This relation is the set of what $\hat{\imath}$ knows or thinks about σ, of what $\hat{\imath}$ can do with σ, etc. (This set is empty if $\hat{\imath}$ "does not know" σ at all.) The kind of objects we are looking at here are the *activities* a that take place in a given institution, for example in a classroom, in a family, etc. A person who is the subject of this institution can have a relation to a of one of the following three types: this person can be a *spectator* of the activity a, he or she can also be an *actor* of a, and finally he or she can be an *author* of a, the word "author" being taken in a sense we will specify later. When a teacher lectures, the students are spectators in an activity in which the teacher is usually the only actor. When students try to do an exercise under the supervision of their teacher, the students are actors, while the teacher is more often than not a mere spectator.

It is crucial to challenge the commonly held, yet overly simplistic, view that being a spectator to an activity is inherently "passive;" for example, the teacher who observes his students concentrating on solving an exercise is not necessarily passive. The same applies to students attending a presentation by the teacher. Of course, we do not know what silent spectators do (think, etc.) during an activity. This is one of the situations where, particularly in a classroom, a differentiation may appear between those who learn a lot and those who learn little or almost nothing (because they "do" nothing).

The fact remains that, as a general rule, observing those who act is an essential means of learning to act.

We now arrive at the third concept announced, that of *author*. Latin *auctor* means literally "one who causes to grow." It derives from *augere* "to increase" from a root that we also find in the verb "to augment." An author within a given institution is any instance that "augments" the material that feeds the life of that institution. When the institution is a class $c = [\hat{X}, \hat{y}]$, the teacher may at times be a spectator, at others an actor, but above all he or she is an author, for example when giving a lecture or giving the students an exercise: in this way he or she nurtures the life of the class. In the same way, teachers make a contribution to the life of the class—they "augment" it—when they give a test to their students, mark the students' work and provide oral or written comments.

Two points need to be made here. The first is that calling someone an author is not in itself commendatory: *everyone* is an author in a number of institutions. A mother or father who decides what goes into the evening meal is an author. The owner of a business who gives his employees a day off is an author. An endless list of examples could be given here. The second point to emphasise is now perhaps apparent: in a given institution, not just any subject can act as an author. For this to happen, one needs to be recognised as having an authority within the institution, an attribute that is not necessarily held by all. But here we come to a turning point in our study.

THE NEED FOR STUDY AND RESEARCH INQUIRIES

Students as non-authors

Let us suppose that the class $c = [\hat{X}, \hat{y}]$ is a traditional class in the sense of the paradigm of "visiting works" (Chevallard, 2015). Normally, it is up to the teacher to deal with the topics on the class programme. In doing so, the teacher has a relation of author to the activity of the class. Students play their part as actors: they make notes in their notebooks. Of course, it is not impossible for a student to give a presentation on a given subject. This presentation may be marked by the teacher. But it will not be validated as a class reference, a role reserved for the teacher's presentations. In fact, it seems that, in almost all cases, subjects given to students to work on in this way are seen as non-essential and might just as well not have been dealt with, even if the other students, and even the teacher, as "mere" spectators, can learn from such presentations. In this case, a student is not an author (in the classroom). This extends to other classroom activities. For example, if the students have to do an exercise, a student's solution cannot become the "class solution," which will be the solution given by the teacher. Of course, if a student works outside the classroom with a few classmates, he or she can be recognised as an author by that small institution. However, in a traditional classroom, a student will never be seen as an author. Nor will students have to defend their "answer" (their presentation, their solution, etc.), which will be marked or at least judged *without discussion* by the teacher. The result is that students are not educated to be authors. This will lead to an *epistemic deficit* in the life of institutions, as we shall see.

Authoring and inquiring

There is a direct link between authoring and inquiring. The act of authoring is always the result of a decision—giving the students this exercise to do, feeding the children this dish, and so on. Decision-making is always the result of more or less conscious questioning: "What am I going to tell the students about the concept of concavity?" the mathematics teacher may wonder, or "What should we make for dinner?" parents might consider. In essence, being an author means responding to a question Q with an answer A, a principle that holds whether "author" denotes a writer or not. Therefore, being an author in a given institution about some activities can be seen as synonymous with raising questions about these activities and providing them with answers. Or, to put it more succinctly, it involves asking and investigating questions. The author is, de facto, an inquirer. So, what is the problem?

To save on the symbolism used, let us take up the one introduced above and generalise it a bit: by $\mathcal{C} = [\hat{X}, \hat{Y}, \hat{Z}]$ we now mean a *collective* $\mathcal{C}$ whose members $\hat{x} \in \hat{X}$, who can be students or teachers, have to act as co-authors in $\mathcal{C}$ to produce an answer $A^{\heartsuit}_{\hat{X}}$ to a given question Q. The members of $\hat{Y}$ can be, as usual, teachers if $\hat{X}$ is a set of students and teacher educators if $\hat{X}$ is a set of teachers (remember that, in any case, it can be that $\hat{Y} = \emptyset$). The members of $\hat{Z}$ are spectators of $\hat{X}$ (they can be parents or other relatives if $\hat{X}$ is a set of students, or colleagues if $\hat{X}$ is a set of teachers, etc.). The authors-inquirers' answer $A^{\heartsuit}_{\hat{X}}$ should prevail in $\mathcal{C}$, so that we can write ($\mathcal{C}$, Q) ➥ $A^{\heartsuit}$, with $A^{\heartsuit} = A^{\heartsuit}_{\hat{X}}$. This supposes that $\hat{X}$ has inquired into Q (under the supervision of $\hat{Y}$), that is to say that we have [($\hat{X}$, Q) ➡ $M_{\hat{X}}$] ➥ $A^{\heartsuit}_{\hat{X}}$, the answer $A^{\heartsuit}_{\hat{X}}$ having withstood the dialectic of media and milieus to which $\hat{X}$ (supervised by $\hat{Y}$) duly subjected it. But is this enough for $A^{\heartsuit}_{\hat{X}}$ to be "democratically" acceptable to $\hat{Y} \cup \hat{Z}$? Does a regime of what we will call, for lack of a better term, *epistemic democracy* or instead a regime of *epistemic tyranny* will prevail in $\mathcal{C}$? Common observations of human activities in ordinary institutions point to the fact that, without any malice whatsoever, the ordinary regime is one of "petty tyranny." The members of $\hat{X}$, whoever they may be, tend to impose their views on the members of $\hat{Y} \cup \hat{Z}$, already for this reason that the members of $\hat{Y} \cup \hat{Z}$ have not really inquired into the question Q. People who have constantly been treated in this way will know little else but this way of doing things and will become small-time tyrants in the positions they occupy. What are the praxeological roots of this uncooperative relation to knowledge and truth? This is certainly a hard question, which we will now try to tackle.

The fate of study and research inquiries

The first fact we can observe is what we will call the *repression of questions*. Why, in the "ordinary" life of institutions—and classrooms in particular—are so few questions explicitly raised? One of the reasons for this phenomenon has to do with *society* and, even more so, the *civilisation* in which it is immersed, where the right to raise questions is generally reserved for a chosen few. This fundamental fact would require an in-depth study which, for lack of space, we will not go into it here, but that we will try to summarise with one example, that of the 19th century French writer known by his pen name Stendhal (1783–1842), who had lived his youth in the wake of an unprecedented development of "public instruction." Around 1836, he wrote:

> Today, children are taught that "equus" means horse; but they are careful not to teach them what a horse is. Children, in their indiscreet curiosity, could end up asking what a magistrate is, let alone what a magistrate should be. (Stendhal, 1930, pp. 124–125) [1]

We can imagine that there were still fond memories of the renovation of public education represented in particular by the short-lived "Normal School of Year III," which for a few months (from 20 January to 19 May 1795) brought together 1,400 "students" of all ages to attend classes given by often illustrious "professors"—in the case of mathematics, Joseph-Louis Lagrange (1736–1813) and Pierre Simon de Laplace (1749–1827). The lessons were written down and distributed to the students, so that they could prepare for the debates scheduled between students and teachers, where teachers considered questions raised by students. At the first debating session, a student called Placiard asks for clarification on the reasons for the order of arithmetic operations, which are done from right to left, except for division, which begins with the left. Lagrange, then "the first mathematician in Europe," answers first. Here is his answer:

> The difficulty you propose is very good. I confess to you that I have thought about it more than once, and that it seemed to me that indeed, at least for the correspondence, we should have started subtraction also with the left; for, we know that division is only a subtraction, and that multiplication is only a repeated addition. … As for division, we feel that we could not do it any other way, because we must start by doing the opposite of multiplication. In multiplication, we start by multiplying the units, then the tens and hundreds. In division, the opposite of multiplication must be done. That is the reason to start the operation from the left. It is possible there are other reasons; I thought about it and found nothing satisfactory. (Dhombres, 1992, Notes 49–52)

Then it is Laplace's turn to speak, which he does in a very different style, responding a little dryly, even if he has to somewhat contradict his eminent colleague:

> The operations of arithmetic must be ordered in such a way that the sequence of these operations does not influence the figures already written; and that is what happens in the way these operations are performed. The same disadvantage would occur in the other arithmetic operations if they were performed in the opposite order to the one adopted. (Dhombres, 1992, Notes 53–54).

What we want to emphasise here, is that Lagrange and Laplace's answers (to the question of the order of arithmetic operations) illustrate a very general phenomenon relating to the *fate of questions* and the fortunes of inquiries. There is in fact a hierarchy in the institutional treatment of questions. In a given institution, there are, first of all, questions that are *not raised*, even though they sometimes exist silently through the answers that are received, and taken for granted, in that institution—answers that are no longer seen as such, since the questions they answer are ignored. Very close to that, at a second level in this hierarchy, questions that are not recognised as open to discussion or challenge because the inquirer considers the knowledge in question as "natural." At this level, certain knowledge is so deeply entrenched in the institution that it is not only unquestioned but also perceived as inherently unquestionable. This phenomenon has been explored by Strømskag (in press) through the lens of student teachers' study and research inquiries into Norwegian upper secondary calculus education. The student teachers deemed it inconceivable to venture beyond the available mathematics textbooks and the official curriculum in their quest to discern and substantiate the key components fundamental to an introduction to differential calculus at the upper secondary school level. At a third level, a question is raised, sometimes recurrently, but is not studied—it is merely recalled from time to time. The inquiry, if it can be called an inquiry, is at a standstill indefinitely. At a fourth level, a question is explicitly posed, but is immediately followed by a "definitive" answer, which seems to render any further inquiry pointless. As far as we can see, this is typically the treatment that the question of division receives from Laplace in the debate with students at the Normal School of Year III. In truth, this seems to be the fate of many of the questions considered in educational institutions (Chevallard, in press). At the fifth level, the question is examined, an inquiry takes shape, but it soon fails to find a conclusion, peters out, and vanishes. This, it seems to us, is the case with Lagrange's answer. The fact that Lagrange was among the finest mathematicians of his era, known for his lengthy contemplation of a question, suggests that the discontinuation of inquiries, which frequently end before they fully develop, is a rather commonplace phenomenon. What can we do about it? Is there a sixth level in the hierarchy of the institutional treatment of questions?

Beyond opinion: An education for inquiry

From a scientific point of view, the vanishing of inquiries and questions needs to be explained: this is a huge research question, which we will only touch on briefly here. What occupies their place? For the most part, ready-made answers to questions, will become the *opinions* of persons or institutions. When such a relation to knowledge prevails, the concern is no longer with truth, but with *one's* "truth," which we choose more or less freely, in order to form our *own* opinion. So, the question is: how can we escape the tyranny of opinion and outline a different relation to knowledge, one that allows in particular for what the ancient Greeks called *epoché*, which means "suspension of judgement" or "withholding of accent," and leads to objectively "shareable" answers, both within and between institutions? In the framework of the ATD, the best approximation to this shareable relation to knowledge is, it seems to us, the one that both commands and is generated by the theory and practice of study and research inquiries as summarised by the Herbartian schema.

What can be done to educate students of all ages to question the world? Here we assume a collective $\mathcal{C} = [\hat{X}, \hat{Y}, \hat{Z}]$. First of all, $\mathcal{C}$ needs questions to inquire into. These questions must arouse the interest of $\mathcal{C}$ and, in particular, of the authors-inquirers $\hat{x} \in \hat{X}$. It should be noted here that, in the episode at the Normal School of Year III, the question raised by the student Placiard was one that probably did not really interest the great mathematicians Lagrange and Laplace (this seems obvious in the case of Laplace). This is undoubtedly one of the reasons why the inquiry whose results are presented to the students is far from complete. (Note that, for example, Laplace's reply does not answer the question posed by Placiard: it is true that calculating the quotient from left to right satisfies the criterion that Laplace sets out; but why is this the case, when, in the other three operations, the *same* criterion leads us to operate from right to left?) Contrary to what an egoistic attitude might suggest, it does not seem appropriate that the choice of the question Q that will generate the inquiry process should be a question that the $\hat{x}$'s "ask themselves"—in many cases, if I ask myself a question, I will be the sole judge of the answer, which I will not have to share with anyone else. It should rather be a question that objectively *confronts* them. Clinical observation suggests that, from this perspective, an optimal system is one in which an instance ϖ, independent of $\mathcal{C}$, proposes a question, in the same way that a "client" proposes to a team of "experts" $\check{X} \subseteq \hat{X}$ his or her problem to be solved. In a classroom, the "client" ϖ may simply be a *programme of questions* to be studied, set by a higher authority (the ministry, etc.).

Once a question Q has been chosen, the inquiry proper can begin. Here again, clinical observation (of student teachers, for example) highlights a serious obstacle: most people tend to rely on what they already know, or have already studied, that is, they adopt a *retrocognitive attitude*, rather than they seek out and study tools that might be relevant to their inquiry but are new to them, which would be a procognitive attitude. This retrocognitive "reflex," no doubt triggered by traditional school education, explains the quick loss of impetus to be observed in many inquiries. But on this point, the analysis must go one step further. To this end, we will indulge in a concrete example, that of an inquiry into the following question Q: "When the driver of a car, driving on a horizontal road, brakes to stop the car, how can we predict the distance that the car will travel before it comes to a complete stop?" This question aligns with the work previously explored by Strømskag (2023, pp. 54–63). Here, we will take as our object of observation—and not as a scientific publication with which we would have to argue—the "report" given in the Wikipedia article entitled "Braking Distance" (2024). The question under study is first specified as follows:

> **Braking distance** refers to the distance a vehicle will travel from the point when its brakes are fully applied to when it comes to a complete stop. It is primarily affected by the original speed of the vehicle and the coefficient of friction between the tires and the road surface, and negligibly by the tires' rolling resistance and vehicle's air drag. The type of brake system in use only affects trucks and large mass vehicles, which cannot supply enough force to match the static frictional force.

From the outset, then, certain parameters ("the tires' rolling resistance," "the vehicle's air drag," and the type of braking mechanism) of the system under consideration are excluded from the intended modelling process. The proposed story is as follows. If the vehicle of mass m is travelling at speed v when braking begins, its "kinetic energy" is $E = \frac{1}{2}mv^2$. The kinetic energy of the car will be cancelled out by braking, that is, by the work W of a certain force F created by braking. Here, this force is assumed to be constant, proportional to the mass m of the vehicle, the acceleration of gravity g and a certain "coefficient of friction," μ, leading to the equation: $F = \mu mg$. If d is the braking distance we are looking for, then $W = \mu mgd$. As we "must" have $W = E$, we arrive immediately at: $d = \frac{mv^2}{2\mu mg} = \frac{v^2}{2\mu g}$. We see that, in particular, we have: $d \propto v^2$. Until then, a basic knowledge of physics suffices. However, evidently, the inquiry is far from over. How, for example, can the assumptions made about F be justified? And what is the coefficient of friction, μ? How is its value determined? Here, we enter another field of knowledge, and the account provided by the Wikipedia article almost stops at this point. This alone shows that the inquiry is far from completed. For more on the limitations of such an inquiry into braking distance, see Strømskag (2023).

How can this phenomenon be interpreted? This question leads us to yet another fact, which stems from the dominant relation to knowledge in modern societies, deeply shaped by the institutional *fragmentation of knowledge* and the corresponding *epistemic self-isolation* of most inquirers. In such a paradigm, knowledge is divided into "disciplines" (mathematics, physics, technology, chemistry, biology, history, sociology, etc.) and often seems confined within these boundaries, hence impeding free interdisciplinary inquiry. Questioning and the use of disciplinary knowledge beyond one's certified field is not universally accepted. The more or less conscious refusal of free procognition most certainly results from this. In this we encounter the need to contribute to the emergence, in contemporary societies, of a new politics of knowledge, in agreement with what we have called an epistemic democracy. We believe that the development of study and research inquiries both presupposes and entails such a politics of knowledge. To illustrate the imperative of integrating knowledge from another discipline into an inquiry, the preceding example has been deliberately crafted to incorporate knowledge elements from physics.

CONCLUSION

Let us return to the question formulated at the beginning of this study: *From a broad anthropological point of view, what are the reasons for developing didactic research on study and research inquiries?* The main reason highlighted in the foregoing is as follows: by making everyone a potential inquirer, the theory and practice of inquiry developed within the framework of the ATD makes it possible to raise and study the problems linked to the crucial project aimed at changing the authors that we all are in many respects into citizens capable of justifying, explaining and sharing the answers we make, within all the institutions in which we are called upon to live.

By articulating the foundational principles of SRI and its emphasis on modelling didactic praxeologies, as well as the critical role of the dialectic of media and milieus in knowledge creation and validation, this study is a contribution to the inquiry-oriented instructional strategies advocated by Barquero et al. (2022). Our work endeavors to yield relevant insights and strategies for developing an inquiry-based, epistemically democratic, and sharable education in mathematics and beyond.

NOTES

1. Extracts from texts published in French have been translated into English by the first author.

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"The cost of mistake is nothing" - Exploring the development in student's activity using an interactive theorem prover

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This paper explores the development of one undergraduate mathematics student's activity while using an interactive theorem prover (Lean) to write two proofs. The data are from a larger study exploring students' activity when using Lean. We collected data through video recorded interviews using a think aloud protocol in which the student was asked to share his screen and prove statements in the Natural Number Game, a learning resource designed to introduce Lean coding. We explore the different ways in which the student engaged with the immediate feedback he received, the actions he took based on that feedback, and discuss differences in his activity based on his coding of two proofs, which illustrate the development of his ways of working with Lean.

Keywords: Lean, Interactive theorem prover, errors, proof writing, instrumental approach.

PROGRAMMING AND INTERACTIVE THEOREM PROVERS

The teaching of programming in mathematics degrees in the UK is now more common than it was ten years ago, and it is mainly present in computational and applied mathematics and statistics modules (Iannone & Simpson, 2022). Interactive theorem provers (ITPs), which have been used for decades in mathematics research as a tool by some researchers in the pure mathematics, are recently starting to be used in undergraduate mathematics teaching (Avigad, 2019; Thoma & Iannone, 2022).

The ITPs are different in the language they use, the mode of interaction, the user interface, the proof structure, and the visualisation of the proof state (Bartzia et al., 2022). More specifically, depending on the language used the ITPs are categorised in imperative, where users use rules (tactics) to manipulate the set of given variables, hypotheses and the current goal, and declarative languages, where users express mathematical assertions and their justifications – similar to the pen and paper proofs. One of the most commonly used ITPs in the UK is Lean (https://leanprover-community.github.io). As other ITPs, Lean has an imperative language, provides users with line-by-line instant feedback on the logical coherence of the proof and the symbolisms used. In our study, we will explore students' engagement with the Natural Number Game, a game designed to teach Lean[1] coding. The Natural Number Game interface is divided in the main section, where the user writes code (right hand side) and receives feedback from Lean in the form of changes in the context (at the top left)

[1] The version of Natural Number Game used in this study is based on Lean 3.

and error messages (bottom left), and the menu on the left hand side which builds up with proven results as the user goes through the Natural Number Game to provide information on the tactics and theorem statements used (Figure 1). The creators of the Natural Number Game also made some changes in the way that Lean works in that environment meaning that in some cases tactics are simpler or more elaborate aiming to assist the user who is learning Lean through that environment.

Studies so far have suggested the potential impact Lean could have on students' understanding of mathematics (Avigad, 2019) and explored Lean's features which could support students' difficulties with proof (Hanna et al., 2023). The limited, until now, empirical evidence suggests that there are benefits in using Lean concerning the way in which students structure proofs and how they introduce the main mathematical objects involved in the proofs (Thoma & Iannone, 2022).

Students are faced with difficulties when starting to use Lean (Iannone & Thoma, 2023). These difficulties are sometimes similar to students' difficulties in coding which have been widely documented in computer science education (e.g., Qian and Lehman, 2017). Brown and Altadmri (2014) explored students' errors in Java and provided a classification examining syntactic, semantic and type errors. Their results showed that most of the errors were related to semantic issues except for one syntax error using mis-matched brackets. A different approach is taken by McCall and Kölling (2019) who explored not only the frequency but also the difficulty of the errors. In their study on novice programmers' errors, the authors identified the five most severe categories: variable not declared; incorrect variable declaration; simple syntactical error; incorrect method declaration and semantic error. Finally, in their review of the literature, Qian and Lehman (2017) discussed the main difficulties students face when coding grouping them in syntactic, conceptual and strategic difficulties. As the use of Lean in teaching is becoming more widespread its important to further explore the coding activity of students. In this paper, we aim to provide further insights into how students utilise the feedback and error messages provided by Lean, as well as explore the development of their activity as they become more familiar with this tool.

In what follows, we discuss the theoretical considerations of our study and our research question; describe the context of our study; and, then provide the findings of our analysis.

THEORETICAL ASPECTS

Recent studies exploring the use of programming at university level have employed the instrumental approach to investigate students' activity (e.g., Gueudet et al. 2022). In this approach an instrument is considered as the pairing of an artifact and a scheme. More specifically, artifacts are products of human activity with particular goals (in our case Lean), and schemes are illustrating the organisation of the human's activity for a specific goal (in our case the organisation of the coding activity). In our use of the instrumental approach, we adopt Vergnaud's definition of scheme (2009) which comprise four aspects: goals (considering the intentional aspect of schemes); rules of actions (focusing on the generative aspect of schemes); operational invariants (the epistemic aspect of schemes) which are conceptualised as concepts-in-action and theorems-in-action; and possibilities of inferences (exploring the computational aspect). The instrumental approach allows insights in how the students "simultaneously act and learn, along their activity mediated with artifacts" (Gueudet et al., 2022, p.357).

We are interested in investigating student's coding activity (schemes) using Lean (artifact) in the context of the Natural Number Game. Our previous work (Thoma & Iannone, 2023) explored the rules of action governing the activity of two different students when proving the same statement in the Natural Number Game. We now turn to explore how schemes develop as students continue using the artifact. For the purposes of this paper, we focus on one aspect of the schemes, the rules of action, "the sequences of actions, information gathering, and controls" (Vergnaud, p. 88), governing students' coding activity in Lean. Specifically, we investigate the development of schemes of one student focusing on the rules of action or the sequence of rules of action which highlight the student's scheme adaptation and development. Our research question is: What are the rules of action that govern a student's coding activity in two related tasks and how these illustrate the development of schemes in using Lean?

CONTEXT AND METHODS

The undergraduate students who participated in the larger study upon which our paper is based were enrolled in a first-year mathematics module at a UK research-intensive university. Lean was introduced by the lecturer as an optional opportunity to be taken up either in students' own time or in optional sessions. The lecturer also provided learning resources in the form of lecture notes and other interactive resources. The students were approached through a questionnaire and asked to consider participating in interviews via video conferencing. In this study, we focus on one of the students – whom we call Ben - who was interviewed after the teaching period ended. Ben was interviewed seven times and each interview lasted about one hour. For the purposes of this paper, we explore Ben's coding of two levels of the Natural Number Game: level 6 of addition world and level 9 of multiplication world, in his first and seventh interview respectively.

In level 6 (Figure 1), the students are asked to prove that "For all natural numbers a, b and c, we have a+b+c=a+c+b". It is important to note that within the text of the level, the students are told about the use of brackets "the convention is that if there are no brackets displayed in an addition formula, the brackets are around the left most + (…). So the goal in this level is (a+b)+c=(a+c)+b" (text from level 6). In level 9 of the multiplication world, the task to be proven is "For all natural numbers a, b and c, we have a(bc)=b(ac)". We chose these two levels as the statements to be proven are similar and the student engaged with these in his first and last interview which would allow us to show the development in his way of working with Lean.

Figure 1: Screenshot from the Natural Number Game Level 6 – Addition World

In our analysis, we considered both the audio and video data which allowed us to capture Ben's activity in these two levels. The transcribed audio data were analysed taking into account the lines of code that Ben attempted, the movements of his mouse and his highlighting either verbally or visually of different aspects of the proof or the Natural Number Game level text. For each of the levels, we created a table (e.g., Tables 2; 3) which provided information on the written code, capturing the order in which the code was provided as often multiple attempts were made prior to finalising the code; and the relevant transcript with further information on actions captured in the video (e.g., the student highlighting or pointing particular sections of the screen). The data, as well as the transcript from the student's reflection, were then analysed separately by the two authors also in relation to data from other levels for this and the other students.

The focus of our analysis in this paper is on the observed rules of actions governing the student's coding activity in Lean. Examples of these are: focusing on using information provided by the current proof state (e.g., *I am aware of the goals that I have to prove* (*RoA_goals*); information provided in the text of the particular level (e.g., *RoA_I engage with the information from the level*); feedback provided in the form of an error message (e.g., *I am receiving an error message, I should check why* (*RoA_engage with error message*)) etc. In what follows, we present in detail Ben's actions in the two levels and showcase the rules of action governing his activity.

RESULTS

In this section, we describe Ben's actions in Level 6 from Addition World and Level 9 from Multiplication World, while highlighting instances which show how he interacted with the error messages or feedback provided by Lean. Ben's final proofs are provided in Table 1. Also, we illustrate with numbers on the right hand side of the proof the order in which Ben wrote the particular line of code as this showcases potential changes or alternations that took place while writing the Lean proof. For example, the first line in the final coded proof (*rw add_assoc* in Table 1 – left-hand side) was the 8th attempt that Ben made in order to progress with that proof. Similarly, the first line of code (*rw* *mul_comm* in Table 1 – right-hand side) was written by Ben in his first attempt to progress the proof.

As mentioned in the introduction, Lean uses an imperative language meaning that the user utilises tactics to manipulate the context (e.g., the variables and hypotheses) and the current goal. Ben's proof features the following tactics: rw (rewrite) and refl (reflexivity); and use the following theorem statements, which were proved in previous levels: add_assoc (associativity in addition); add_comm (commutativity in addition); mul_assoc (associativity in multiplication); and mul_comm (commutativity in multiplication). These tactics and theorem statements are provided to the user of the Natural Number Game in the left-hand side drop-down menu of the levels (Figure 1). Further tactics and theorem statements are added to that drop-down menu as the user progresses in the game.

Proof :
`begin`
`rw add_assoc,`
`rw add_comm b c,`
`rw add_assoc,`
`refl,`
`end`
	[8]
	[10]
	[12]
	[13]
		Proof :
		`begin`
		`rw mul_comm,`
		`rw mul_assoc,`
		`rw mul_comm c a,`
		`refl,`
		`end`
			[1]
			[2]
			[4]
			[5]

Table 1: Screenshots from Ben's final proofs to Addition World level 6 (left-hand side) and Multiplication World level 9 (right-hand side)

Ben took more time to solve level 6 (Figure 1), compared to the previous levels of this world focused on induction, associativity and commutativity of addition, and successors. In table 2, we show some instances[2] which illustrate the written code, the transcript and the analysis of the rules governing Ben's activity. The rules of action that govern the activity when coding this proof show that Ben is aware of the goals in his proof; he is considering the process of writing a proof in pen and paper but is unclear how to translate this in the Lean language and syntax. The latter aspect is closely related to the language input that is used by Lean which is imperative rather than declarative. Additionally, Ben often used the immediate feedback and error messages to guide his code.

[2] Due to space limitations we chose to highlight particular moments of the coding of the proofs. These are chosen to showcase how Ben arrived to the final proofs (illustrated in Table 1).

Line of code	Corresponding interview transcript	Rules of action
[1]rw add_comm,	I would love to use commutativity on these two terms on b c and c b. But I think if I just write it add_comm it won't work like this because. Yes it won't work on this because not the right things are getting…	I am aware of the goals that I have to prove (*RoA_goals*). I am aware of how I could do this on pen and paper but I do not know exactly how to do this in Lean. I am not sure what will happen if I write this line of code (*RoA_guess*). I am receiving an error message, I should check why (*RoA_engage with error message*).
[7]rw add_comm b c,	Ok, that's fun, they say that it didn't find instance of the pattern but…there is an instance (highlights the section on the expression of the task). So, maybe I should present it somehow differently (re-reads the text).	I am aware of the goals that I have to prove (*RoA_goals*). I am receiving an error message, I should check why (*RoA_engage with error message*). I engage with the information from the level.
[8]rw add_assoc,	Ok, I think the problem is that b+c is not in brackets and a+b is in brackets. Therefore it does not see b+c. So, if we write something with associativity. Yeah we will get b+c (highlights the relevant section in the error message)	I am receiving an error message, I should check why (*RoA_engage with error message*). I am not sure what will happen if I write this line of code (*RoA_guess*).
[9]rw add_comm	And then I can write (receives error message)	I am receiving an error message, I should check why (*RoA_engage with error message*).
[10]rw add_comm b c,	Yes, perfect.	I need to revise my code to make it work.

Table 2: Part of our analysis of Ben's coding activity of Level 6 in Addition World.

After he completed this level Ben noticed:

> "sometimes you write these theorems in Lean and without clear understanding what will happen, you have sort of idea that it will be useful and it will work. […] The cost of mistake

is nothing so basically you can try writing it and look at what will happen because it is (a) programming language and therefore when you get the idea that might work and it seems good enough, you just write it without thinking it until the end”

Our analysis of this level illustrates that the main sequence of rules of action that govern Ben’s activity is *RoA_guess* and *RoA_engage with error message*, this is also further supported by his reflection as to how he uses the immediate feedback and the error messages he receives. The engagement with the immediate feedback focuses sometimes on revising his code (e.g., the move from [9] to [10] in which he is changing b+c to b c which is what Lean expects to receive) or on a change in theorem statements (e.g., the move from [7] to [8], where he is initially using rw add_comm and then changes to rw add_assoc). In his reflection he shares *“you just write it without thinking until the end”* showcasing how he views this process of writing the Lean proof where due to the limited impact that a mistake has, he is attempting possible coding choices without carefully considering the impact that these choices would have on the proof process. This is linked to one of the features that Lean has which is providing immediate feedback to the user’s input.

We now turn to a different proof which Ben worked on during his seventh interview. It is important to note, that there were fewer lines of code that Ben used to provide this proof compared to the one in his first interview (Table 1). In the table below, we show some of the initial rules of action that govern Ben’s activity, highlighting the development of his coding between the first and seventh interview.

Line of code	**Corresponding interview transcript**	**Rules of action**
[1] rw mul_comm,	Yes, I see the way how I would solve this with pen and paper. The problem is that here I think I will need to apply rewrite tactic to particular things, and I’m not sure because it it is always the hard part for me. OK. So we start with mul_assoc thing and change… Oh no, we start with mul_comm thing and we change a and bc. I don't know how it will work (explores the right pane). Yeah, it actually works as I want it.	I engage with the information from the level. I can predict the outcome of this line of code (*RoA_predict*). I am aware of the goals that I have to prove (*RoA_goals*). I know how to do this in pen and paper and that guides my code. I am not sure what will happen if I write this line of code (*RoA_guess*).

[3] rw mul_comm,	Yes, and here we use (highlights the goal state – b(ac) specifically the inner bracket). Commut…yeah it is commutativity thing. Its… I think it's going to change code. Uh, yeah. Basically we need to use mul_comm on this thing (highlights ca from the goal state). So let's just try (types – glances at right hand pane)	I am aware of the goals that I have to prove (*RoA_goals*). I can predict the outcome of this line of code (*RoA_predict*).

Table 3: Part of our analysis of Ben's coding activity of Level 9 in Multiplication World.

Our analysis of the whole level illustrates that the main rules of action governing Ben's activity are: *RoA_goals*; *RoA_I know how to do this in pen and paper and that guides my code*; *RoA_predict*; *RoA_I engage with the information from the level*. In his reflection afterwards Ben commented:

> I had this plan from the beginning, so I just. Yeah, mul_comm and then associativity and then again mul_comm for the two elements that were in brackets for the c and a. And yeah, and also I remembered how actually to rewrite tactics with particular elements. Yeah, I guess what I didn't know like a month ago that you basically need to input variables of the of the lemmas to rewrite tactics. So yes.

Our analysis of the rules of actions that Ben utilised in this level finds a clear awareness of the goals guiding the overall proof plan, awareness of the specific order of the use of theorem statements (mul_comm and mul_assoc), further understanding of the use of tactics (e.g., how to specify elements in the rewrite tactic), and leveraging prior Lean coding patterns. These are different compared to the rules of actions governing Ben's activity in Level 6 of the Addition World which were more focused at finding the code without necessarily having a clear plan as to how the code will work on the given proof state. His reflection also supports this claim as it showcases that he "*had this plan from the beginning*" and shows awareness and further familiarity in manipulating the different tactics.

DISCUSSION AND CONCLUSION

In this paper, we focused on exploring the coding activity of one student in the learning environment of the Natural Number Game. We selected two levels and explored the rules of actions that govern Ben's coding activity. The two levels chosen for analysis were relatively similar in terms of their statement and Ben's proofs used similar tactics and proof statements (Table 1). However, we observe that Ben's coding activity is different in these, illustrated by our analysis of rules of action (Table 2 and Table 3) and Ben's reflection after each of the levels. In general, the rules of action that govern the student's coding activity as identified in our analysis explore particular aspects of the coding activity: engaging with the language and syntax of Lean; utilising prior experience and information provided as part of the level; student's awareness of the pen-and-paper proof; engagement with error messages and goal changes; and guessing or predicting whether particular lines of code would work in the given situation. These add to our previous analysis (Thoma & Iannone, 2023) and resonate with some of the rules of action reported in Gueudet et al. (2022).

Additionally, our analysis illustrated the development of Ben's schemes as the rules of action governing the activity in the two levels are different. In level 6 (Table 2) the analysis showed that Ben used often guessing and adapting his code based on Lean's response via the error messages. Whereas in Level 9 (Table 3) his attempts are more informed (from engaging with the goals; the level information; his idea of this proof in pen and paper) and this can be also seen in the number of coded lines in Table 1 (e.g., [1]-[5]) with most of them being part of his final proof.

Our analysis of the student's interactions with the Lean error messages is not focused on providing a classification of the different types as for example the studies in computer science education (e.g., Brown and Altadmri, 2014; McCall and Kölling, 2019). We take a different approach exploring how the interaction with the error messages shapes and informs the student's coding activity. It is important to further explore this interaction in further Lean levels especially when the student is faced with using new tactics.

In our next steps, we aim to further consider the other aspects of schemes and how for example operational invariants shape and inform the rules of action and thus impact on the schemes. Furthermore, explore in more detail how the other student's schemes developed in order to highlight further the development of schemes when coding in Lean.

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Integración de la perspectiva de género en cursos de matemáticas a nivel licenciatura

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Palabras clave: Perspectiva de género, cultura en el aula, resolución de problemas.

INTRODUCCIÓN

Por las características propias de la enseñanza aprendizaje de las matemáticas se han realizado diversidad de estudios sobre sus diferencias. Fuentes y Renobell (2020, p. 65) señalan que el "sexo es una de las variables con mayor capacidad de predicción del éxito en el rendimiento matemático", en donde en la mayoría de los estudios los hombres obtienen mejores resultados que las mujeres, así como mejores actitudes hacia su aprendizaje. En esta investigación se considera que integrar la perspectiva de género en el aula puede tener un impacto positivo para disminuir estas diferencias. Además, dicha integración debe estar en armonía con los métodos y estrategias de enseñanza que sustenta el modelo de aprendizaje de la institución y el profesorado. Las preguntas de investigación son: ¿Cuál es el efecto que tienen las actividades diseñadas con perspectiva de género en el aprendizaje de conceptos matemáticos del alumnado? ¿Qué modificaciones se dan en los discursos y argumentaciones entre el estudiantado, así como entre el estudiantado y el profesorado al implementar actividades para el aprendizaje de las matemáticas con perspectiva de género?

PERSPECTIVA DE GÉNERO EN EL AULA

Diseñar un curso desde la perspectiva de género implica revisar detalles tanto de los contenidos como de las dinámicas que se dan en el aula. La presente propuesta se basa en el documento *Matemáticas. Guías para una docencia universitaria con perspectiva de género*, de Irene Epifanio implementada en la Universitat Jaume I, en específico sobre la incorporación de la perspectiva de género en la docencia en matemáticas como la gestión del aula, contribución de las mujeres a las matemáticas, lenguaje no sexista, trabajar en valores, relaciones interpersonales, entre otros (Epifanio, 2020).

METODOLOGÍA

Por las características de la investigación se propone el estudio de caso en sus dos modalidades, caso único y casos múltiples. Lo anterior debido a que se realizará un estudio de caso por cada profesor/a-curso participante y múltiples casos al contrastar lo obtenido en cada caso analizado (López, 2013).

Los participantes en la investigación será el profesorado interesado en integrar la perspectiva de género en sus cursos de matemáticas del Departamento de Matemáticas, del Centro Universitario de Ciencias Exactas e Ingeniería (CUCEI) de la Universidad de Guadalajara (UdeG).

El proyecto iniciará con una prueba piloto en un curso de Probabilidad y Estadística para Ingenieros Civiles, en donde se piloteará un cuestionario que será aplicado en todos los cursos del profesorado interesado en participar en el proyecto, así como las actividades específicas del curso en cuestión.

PRUEBA PILOTO

Como parte de la didáctica en el aula se trabajará bajo el enfoque de resolución de problemas estadísticos de Chatfield (1995), la cultura estadística (Batanero, 2013) y la perspectiva de género (Epifanio, 2020). Bajo estos enfoques se diseñaron tres problemáticas, con hechos reales o que pueden ser considerados como reales por el estudiantado, que serán el eje en las cinco unidades del curso:

1. Analizar las pruebas de compresión e intemperismo de un tipo de cemento puzolánico.
2. Concurso de ladrilleras.
3. Análisis de la Encuesta Nacional sobre el Uso del Tiempo

Las actividades diseñadas con estas tres problemáticas siguen las directrices de Epifanio (2020) en cuanto al género de los actores principales, en donde los dos primeros son neutras y en el tercero la protagonista es una mujer.

Además, las actividades fueron diseñadas para que el estudiantado se familiarice con las estrategias para resolver problemas estadísticos, más que con el uso de técnicas estadísticas, en donde aparecen varias soluciones y el alumnado tiene que utilizar su juicio para dar sus recomendaciones con uso de lenguaje estadístico (Chatfield,1995).

Al final del curso se aplicará un cuestionario anónimo diseñado con el objetivo de conocer la percepción del estudiantado sobre la implementación de la perspectiva de género en el aula.

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Some undergraduate students' conceptions of proof

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Keywords: teaching and learning of logic, reasoning and proof; students' practices at university level proving process; conception of proof; questionnaire.

INTRODUCTION

In the literature, there is a convergence on the students' difficulties with proof and proving processes between tertiary education and in university, and their causes (e.g. Selden & Selden, 2012). Among the results, students have difficulties in grasping proof as an object, as opposed to proof as a problem-solving tool (Balacheff, 1988). Their difficulties with proof also lie in the different roles of examples, the semantic, syntactic and pragmatic aspects of a proof, the writing of proofs, and the use of inappropriate reasoning. In order to circumscribe how researchers in education have investigated the undergraduate students' conceptions about proof, Ouvrier-Buffet (2023) has conducted a bibliographical study aimed at identifying the nature of the mathematical domains involved, the methods and the theoretical underlying backgrounds used in existing surveys (interviews, questionnaires, analysis of students' works etc.). She points out that there is room to design a device to identify undergraduate students' conception of proof and proving, trying to avoid the difficulties that are specific to mathematical contents of university, and then to answer to the following research question: What are undergraduate students' current conceptions of proof?

METHODS AND NATURE OF THE DATA

Ouvrier-Buffet (2023) has designed a questionnaire to achieve this aim, exploring "basic" components of deductive proofs with the following choices: choosing contents where the obstacle of mastering the involved concepts is minimised, problems outside of formalism (to avoid this obstacle) and outside of curricula (to avoid obstacles or ready-made results and processes). As justified in Ouvrier-Buffet (2023), the questions and the theoretical frameworks to analyse the results are inspired by Stylianou et al. (2015), Healy and Hoyles (1998) and Balacheff (1988). The questionnaire is available online (https://hal.science/hal-03987587). It has several sections in order to: describe the abilities of students to check the hypotheses of a given theorem by themselves and to specify the instances of this theorem, as well as their abilities with formal statements (Part I); evaluate the students' abilities to write "simple" proofs (Part II); identify the students' conceptions when evaluating proposed proofs in various mathematical domains, familiar and unfamiliar to students (arithmetic, geometry, graph theory, combinatorial geometry - Part III). Part IV explores the students' global conception of proof (the way they think about proof) with open questions.

Although a broader public is targeted, the first data were collected in February 2023 at the University of Namur (Belgium) from 36 students enrolled in the first year of university in mathematics or physics. These students have already completed one semester of courses at the university, with a similar curriculum (except that the mathematics students have a specific course of introduction to the mathematical approach). The students had 2 hours (included in their class schedule) to complete the questionnaire. However, these hours were not linked to a specific course in their programme, which minimised the effect of didactic contracts on students' responses. The data were encoded into a sheet, following the *a priori* analysis of the possible answers and proof processes. This encoding makes it possible to cross-reference the typologies of students' proofs (Part II) and the proofs classified as rigorous, correct, incomplete, etc. (Part III). This cross-reference can be broken down according to the domain in which the proofs are proposed.

CONTENTS OF THE POSTER AND PERSPECTIVES

The poster presents the underlying theoretical tools used to design the questionnaire and analyse the data [e.g. Balacheff 1988's characterisation of arguments (i.e. naïve empiricism, generic argument and intellectual proof) and Healy and Hoyles' criteria revisited by Stylianou et al.], the *a priori* analysis of the questions, and the first results illustrated with excerpts of the students' work. For instance, if the students interviewed have difficulties in making conjectures, they are more likely to express an opinion on the evaluation of the proofs proposed in Part III of the questionnaire. New questionnaires are expected to be completed by the new cohort of students at the University of Namur in February 2024. We are also trying to disseminate the questionnaire more widely by diversifying the target groups (teacher trainers, high school students, mathematics teachers).

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Digital STACK tasks and exam results

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Keywords: Digital resources in university mathematics education, digital STACK tasks, motivation, economics students.

DIGITAL TESTS IN UNIVERSITY STUDIES

Tasks have been designed with STACK for almost two decades (Sangwin, 2007). STACK enables new didactic features compared to tasks on paper like immediate feedback to directly encounter errors and repetition effects through rrandomizing tasks. Possible disadvantages, however, lie in the input of mathematical expressions, which is not always easy for beginners.

Implementing STACK tasks as regular elements of large courses may cause problems, as syntax difficulties arise, the calculation paths can usually not be inserted. Moreover, the question design must be adequate to STACK syntax and needs to be suitably randomized. While there is a growing body of literature on the design and implementation of STACK tasks (e.g. Speer & Eichler, 2022), there is rather little work in the literature on how STACK can be used in large courses and how students accept these tasks and cope with digital tests. As we implemented digital tests mostly based on STACK tasks in a large course on service mathematics, we aimed to explore the following questions:

RQ 1: How does the participation in the digital tests evolve during the semester?

RQ 2: How do students cope with the format?

RQ 3: How does students' performance in the digital tests relate to their performance in the exam?

METHODS

The course at hand was a large mandatory course for first semester economics students at Paderborn University with 1073 officially enrolled students, 769 of which participated in the final exam. The course consisted of lectures, tutorials with in-class exercises, a large tutorial for a Q&A session and seven bi-weekly voluntary online tests consisting of 8-10 tasks, 80% of which were STACK tasks that were mostly randomized while the rest were multiple-choice tasks. These tests had replaced the weekly homework from previous semesters, so there were no homework assignments besides the online tests.

The test could only be submitted as a whole and the whole test could be repeated arbitrarily often within a fixed frame of two weeks. The tests used deferred feedback, the individual STACK feedback was only shown after submitting the whole test.

Students could receive a bonus point for the final exam for each test in which they achieved 70% or better, with a limit of 5 points. The points were only awarded if the exam was passed without bonus points. To answer RQ 1, each test had evaluating questions about the test as a whole (including possibly non-STACK tasks).

FINDINGS AND CONCLUSIONS

For RQ 1, we analyzed participation numbers. In the first test, 795 students took part, and then the number fell to 661 in the second test and finally 389 in the seventh test. Such patterns of decreasing participation are well known in this course. Surprisingly, the participation in number of tests was U-shaped: Large portions of students took either no test (193 students) or all tests (210 students) with the lowest number for three or four tests (83 and 82 students). Future research should clarify students' reasons. Possibly, not everyone had access to shared social practices based on the new digital technology that need to become established for good learning (Viberg et al., 2023).

For RQ 2, we found that 84 % of the students could cope well with the input (answering at least "3" on a 5-point Likert scale from very poor to very good). Unlike our expectation, this proportion did not decrease during the semester. Thus, not the system in general but content-specific aspects like the notation seem to cause the problems.

For RQ 3, we compared students' performance in the online tests to the exam scores with 769 students. The data show a significant positive correlation between the number of online tests taken and the exam score explaining 21% of its variance. An even stronger correlation was found for the total number of online tasks passed (0-7) and the exam result, explaining 28 % of its variance. However, this might not just be the positive effect of learning by solving the task. In general, more capable students could take more tests and have a higher pass rate.

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IMPLEMENTATION OF SERVICE-LEARNING PROJECTS IN MATHEMATICS. FORMATIVE ASSESSMENT

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Keywords: Assessment practices in university mathematics education, novel approaches to teaching, service learning, learning trajectory.

INTRODUCTION AND THEORETICAL BACKGROUND

Service-learning (SL) is an educational approach that connects community service with intentional learning for the formation of professionals (Anderson 2003). Although there are numerous experiences in different areas of knowledge, few have been carried out in the field of undergraduate studies in Mathematics (Carducci 2014). Further research is needed to consider the epistemology and the mathematical task itself.

This poster presents the implementation of Service-Learning projects in Mathematics at the University level, focusing on one case: the development of the bachelor thesis of the Mathematical Engineering degree. It has been important to qualify elements of the field of Mathematics Education and very specifically to provide a didactic tool that qualifies the formative evaluation phase of the Service-Learning methodology using the notion of Hypothetical Learning Trajectories (HLT) (Wilson et. al, 2013).

RESEARCH QUESTION, METHODS & POSTER DESIGN

The research questions are: How is the evolution of mathematical competences in students who develop their bachelor thesis of the Mathematical Engineering degree focused on SL projects? What are the limits and possibilities offered by the *actual learning trajectory (ALT)* in contrast to *HLT* to develop a LS Project in bachelor thesis?

The case described is based on the development, during an academic year, of the LS project "Your city near you", a collaboration between two entities, the UCM university and the Youth Council of a city near Madrid (CJA) (Gómez-Chacón et. al, 2020).

The HLT begins with the statement of a problem that has a social objective and, in order to respond to this need, a service will be carried out that enhances specific mathematical competences through a series of tasks that will be stated as the steps of the LS project progress. In contrast to the HLT, there is the actual learning trajectory (ALT) construct that corresponds to the learning trajectory that actually occurs, i.e., the trajectory that the student has followed in the context of the development of the SL Project. ALT is inferred from the data collected because it is not possible to directly measure student' actual learning (Dierdorp et. al, 2011). The competences of the Mathematical Engineering degree are considered by the mentor in order to give suggestions and tasks to the student. The data considered are how the student activates these competences when solving the different tasks to achieve the objectives of the project, together with the errors and difficulties she may face.

The generating idea of the LS project is based on the social need presented by the CJA to make young people (16-30) aware of the leisure possibilities and resources offered by the city. In view of this, we formulated to the student the following task: *To develop a mathematical (algorithm-based) tool to improve the plan of activities initially proposed by the CJA to make young people aware of the leisure resources offered by their city*. In order to achieve this objective, the following questions for enquiry were proposed to the student: *How to formulate the real problem in terms of mathematical optimization? What mathematical and computational models can be used?*

The problem to be solved is modelled mathematically using graph theory. A vertex routing problem is identified, to find a Hamiltonian cycle in a network such that the total distance travelled is minimal, i.e. a Travelling Salesman Problem (TSP) (Applegate et. al., 2007). The model is solved through an exact method implemented in GAMS and heuristic methods implemented in PYTHON.

Qualitative research was conducted, we have also developed an observation with participatory intervention, which has consisted of the design, experimentation, and analysis of a didactic device, which we call HLT for LS projects, based on the study and progressive modelling of questions that arise in the field of LS. The poster will outline the theoretical framework of the project and give an overview of the method study design. Also, the results of the case study will be displayed. A mix of texts, schematics, and diagrams will be used to visualize the project.

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Transitions and crisis in mathematical enculturation

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Keywords: Transition to, across and from university mathematics; Teachers' and students' practices at university level; Mathematical enculturation; Crisis in secondary-tertiary transition; Transformative education

INDIVIDUAL CRISIS IN SECONDARY-TERTIARY TRANSITION

In recent years, significant research in mathematics education has centred on the challenges that students experience during the secondary-tertiary transition (STT). A growing but still limited number of research looks at this transition from a socio-cultural and affective perspective (Di Martino et al., 2022). One such approach is known as *rite of passage* (Clark & Lovric, 2008). It describes STT as a life crisis as well as the necessary adjustments on the part of the student, which arise due to the major changes caused by the transition to university mathematics.

In the spirit of this approach the PhD project and the corresponding study presented on this poster aim to comprehend the individual transitions of mathematics students due to mathematical enculturation in their first semester of studying mathematics. Especially the question is asked how the relation between the students and mathematics change due to individual experiences of foreignness and crisis which occur in the interaction with the newly learned subject matter and the new institution. For this purpose, STT is understood as the transition from a school-based to a university-based mathematical community and is seen as a transformative educational process (Koller, 2023). In the new academic environment, students face crises that question their existing relation to the mathematical world (subject and studies), to others in it (e.g. peers and teaching staff) and to themselves (Günther, 2023). They react to these crisis in either a rejecting, assimilating, or productive way, leading to either rejecting or embracing the culture of university mathematics and the adaptation of the aforementioned relation (Günther & Hochmuth, 2023 (in print)).

DESCRIPTION OF THE STUDY, RESULTS, AND IMPLICATION

The mentioned theoretical concepts guided a qualitative interview study. It was carried out with nine first-year students at Leibniz University Hannover studying subjects related to mathematics, such as mathematics or physics, in either the Bachelor of Science programme or the two-subjects teacher education program (B.Sc./B.A. → M.Ed.). During the semester, each student participated in five to six individual one-hour narrative interviews.

The initial interview took place in either the second or third week of the semester and focused on the student's overall perception of mathematics as a subject and as part of their learning experience. Additionally, the individual relations to others in the new social context were discussed. The final interview, which was conducted after the final exam, also examined these topics to identify any differences and transformations within the semester. The intermediate interviews, which were distributed evenly throughout the semester, focused on the current state of learning, and studying, especially difficulties, irritations, and experiences of foreignness related to mathematics and the new environment of university.

In the ongoing analysis of the interviews, on the one hand, it is searched for the individual relation to the mathematical world, to others, and to themselves, on the other hand, for implicitly expressed crises and responses that explain the transformation between the beginning and end of the semester in more detail. From this analysis, categories of transformation are formulated that outline typical courses of the transition.

As the evaluation of the study has not yet been finalised at the time of this proposal, no premature implications could be formulated here. The poster will present the then current state of the analysis and its implications.

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A novel approach to teaching and assessing students' critical thinking in university mathematics

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Keywords: novel teaching, critical thinking, mathematics.

SUMMARY

This poster draws attention of mathematics education community to the importance of explicit teaching and assessing students' critical thinking skills while teaching university mathematics courses. In recent years, fake news, conspiracy theories, misinformation and deep fakes are getting more sophisticated and more common in our society. Therefore, abilities to recognise mistakes and think critically are crucial nowadays. To enhance students' critical thinking skills, it is proposed to include so-called provocative or 'impossible' questions in teaching and assessment in mathematics. Such questions look like typical routine questions but in fact that have a catch – they are *deliberately* designed to mislead the solver. Often a catch is based on a restricted domain or indirectly prompts the use of a rule, formula, or theorem that is inapplicable due to their conditions/constraints. University mathematics lecturers' attitudes towards the suggested pedagogical strategy are presented in the poster.

INTRODUCTION

There are many definitions of critical thinking. The vast majority of those definitions use such words as analysing, evaluating, examining, reasoning, especially in the context of active learning, mathematical thinking and problem-solving. This poster deals with a specific aspect of critical thinking – recognising mistakes. In particular, the ability to question the question, that is to recognise a flaw or mistake in a question or problem *before* trying to solve it. Attitudes of school mathematics teachers towards the use of provocative questions in teaching and assessment from the original study by Klymchuk (2015) were discussed in Klymchuk & Sangwin (2020). This study extended the above studies to the university level. The research question was: What are the reasons of university lecturers for adopting or otherwise of the use of provocative mathematics questions in their teaching and assessment.

THEORETICAL FRAMEWORK

The concept of the Inquiry-Based Mathematics Education (IBME) introduced by Artigue and Blomhøj (2013) was used in the study as a theoretical framework. Laursen and Rasmussen (2019) consider inquiry as a branch of active learning with the distinguishing characteristic of offering "students and instructors greater opportunity to develop a critical stance toward previous, perhaps unquestioned learning and teaching routines" (p. 132). According to them, IBME is based on the following four pillars: "student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building." (p. 141).

THE STUDY

This (pilot) study was conducted with eight very experienced university mathematics lecturers from four universities. All lecturers had a PhD either in mathematics or a related field. A combination of judgement and convenience sampling methods was used to select the participants. All lecturers have been familiarized with the suggested pedagogical strategy. They were also given several examples of provocative questions from calculus. Two such provocative questions are below.

Question 1. Find the derivative of the function $y = \ln(\ln(\sin x)$. Note: although it looks like a routine question on differentiation techniques using the Chain Rule, the rule is not applicable as the function has an empty domain.

Question 2. Sketch a graph of a function that is differentiable on the interval (a,b) and discontinuous at least at one point on (a,b). Note: any sketch would be incorrect as the task is impossible - a function differentiable on interval (a,b) is continuous on it.

The participants were given a questionnaire on the use of provocative questions in their teaching practice. A thematic analysis was used to analyse their responses.

Six out of eight lecturers (75%) reported that they would include provocative questions in their teaching. They gave very positive comments on the students' benefits from such questions in terms of enhancing their critical thinking. Three out of eight lecturers (38%) reported that they might try including provocative questions in the assessment. The other five lecturers (62%) expressed concerns about a possible negative impact on their own career (lower pass rate; possible complaints from students as such questions might be perceived as "unfair" or "abnormal" by some students; possible negative comments from the university management).

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Emotions when studying mathematics at university

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Keywords: Transition to, across, and from university mathematics, Teachers' and students' practices at university level, Emotions

EMOTIONS IN THE LEARNING OF MATHEMATICS

During the last years affective variables such as motivation, interest, value, beliefs, and emotions, have been widely discussed in terms of their impact on mathematics learning and teaching. It is known that besides cognitive variables, they play an important role in successful participation in mathematics; they can predict for example study retention, study satisfaction, and performance (Hannula et al. 2019). Most affective variables have been operationalised as states more than as traits, considering them as mostly stable over time. However, emotions are known to be often situated and volatile, which makes them less accessible in time. In many studies, they have thus been surveyed either retrospectively or prospectively (Schukajlow et al., 2023). Most researchers differentiate positive (e.g., curiosity, hope, joy) and negative emotions (e.g., anxiety frustration, helplessness) which mostly show either positive or negative relations to participation and performance in mathematics (Schukajlow et al., 2023).

However, taking a closer look at the first year of mathematics at university, emotions are rather under-researched. This seems surprising, given that emotions are usually very present in students' experiences during the transition and often occur in interview settings (see e.g., Gildehaus & Liebendörfer, 2021; Göller, 2020; Lahdenperä et al., 2022). Furthermore, the transition comes with some kind of turning point for many students. While they mainly experienced positive emotions in mathematics in school, entering mathematics at university is often related to negative emotions, such as frustration and helplessness (Göller & Gildehaus, 2021). For many students, these new negative emotions can be intense and closely related to identity tensions (Gildehaus & Liebendörfer, 2021), motivation, and self-regulated learning (Göller & Rück, 2023; Lahdenperä et al., 2022).

We thus aim to take a closer look at students' retrospective reflections on their emotions and pose the following exploratory research question: What emotions, in general, do first-year mathematics students describe, and to what do they refer them?

THEORETICAL FRAMEWORK

Following our explorative approach, we do not aim at pre-defining specific emotions per se, but orient ourselves along those mentioned in Hannula et al. (2019). We are further defining "reference group" as something the emotion is objected to (Schukajlow et al., 2023). This seems to be of great relevance, given the transition situation in first-year university mathematics, where emotions can quickly change.

METHOD

We conducted several interviews with mathematics students during their first year of study. Those stem from different projects and contexts (see Gildehaus & Liebendörfer 2021; Göller, 2020; Lahdenperä et al., 2022) which additionally provides a comparative perspective. Based on our theoretical framework, we conceptualized emotions and reference groups and used content analysis to structure the data material as well as discourse analysis to delve deeper into specific situations in our interviews.

PRELIMINARY RESULTS AND DISCUSSION

Since we have not finished our analysis process yet, we can only provide preliminary insights, mainly those of structuring our data material, but not yet of discourse analysing it. While there seems to be a wide variety of different emotions being identified (e. g. anger, anxiety, fun, hope, hopelessness, joy, surprise), negative emotions, appear to be dominant in students' descriptions. Furthermore, the emotions are referred to rather opposing objects on different hierarchy levels, e. g. to oneself, to a specific mathematical content, to mathematics in general, to studying, to tutors, to specific exercises, or to peers. Comparative perspectives show that specific emotions (e. g. hopelessness) seem to be given only in specific contexts, which we aim to further analyse. Additionally, we are taking a closer look towards the situations the emotions occurred and in line with that, discuss theoretical and practical implications.

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Computational thinking, artificial intelligence and empowerment in university mathematics education

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Keywords: computational thinking, programming, artificial intelligence.

INTRODUCTION

We outline a project studying how computational thinking and programming together with artificial intelligence might play a role in the future mathematics education at university level. We use the theoretical notion of mathematical and digital empowerment, and through interviews we study how students that are in the midst of or have finished their master´s degree in mathematics education reflect on how computational thinking, programming and artificial intelligence influence their understanding of mathematics and how new technology in the future might change mathematics education. The emergence of computers having capability of artificial intelligence (AI) has for a long time been used to assist humans in all kinds of fields, from picking out optimal focus points for cameras, to assisting medical doctors when the decide whether a tumor is malign or not. However, with growing computational power in personal computers, and rapidly changing technology in AI language models in the last few years, AI, specifically AI-powered chat-robots are available for the masses. The close connection between mathematical modeling, computer science and programming make AI tools for generation of computer programs especially effective. One particular research question that we try to address in the project is *"Can the currently emerging AI-chat-robot (AICR) technology serve as a reliable discussion partner for the students learning problem solving in their mathematics education, in particular in computational thinking and programming?"* This research question can be answered in different ways, and at the current stage of the project, we collect data by interviewing students about their own experiences of computational thinking, programming and artificial intelligence in mathematics education.

THEORETICAL FRAMEWORK

We aim to study students' reactions to the emergence of computational thinking, programming, and artificial intelligence in their mathematics education. We use the concept of mathematical empowerment which was introduced to the research literature by Paul Ernest. Ernest´s concept of mathematical empowerment is divided into three domains, which could be overlapping: epistemological, social, and mathematical empowerment (Ernest, 2002). Furthermore, Tissenbaum et al. (Tissenbaum, 2018) use the terms Computational identity and Computational action in a similar manner as Ernest, however in the field of computational education, which is more akin to informatics than mathematics education. However, the field of computational education and the field of mathematics education can in our setting be seen to be more connected than what has been the case earlier, because of the entrance of computational thinking in the new curriculum. The two similar but still different theoretical frameworks of Ernest and Tissenbaum combined gives a more nuanced and powerful view in our setting where computational thinking and programming has become part of the mathematics curriculum. The term Computational thinking (CT) was popularized in a short article by Wing in 2006, but the meaning of the term has a long history stemming from the constructionist research community of Papert et. al, however with slightly different meanings (Wing, 2006, Papert, 1980, Papert, 1991).

METHODOLOGY

Empirical data is collected through semi-structured interviews of students who are in the midst of, or who recently finished their master´s degree in mathematics education. We have currently (pre-conference) interviewed three students. The students have at various stages of their bachelor-degree met courses in programming, and their programming experience has been an integrated part of their degree through mathematical modeling and computational thinking.

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What does lecturer's teaching practice for students' mathematical meaning making entail?

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Keywords: Teachers' and students' practices at university level, Teaching and learning of analysis and calculus, tool mediation, meaning making.

I investigate the relationship between lecturers' teaching practice in first year undergraduate mathematics courses and students' mathematical meaning making (understanding). Teaching practice is "what teachers do and think daily, in class and out, as they perform their teaching work" (Speer et al., 2010, p. 99). This is an area with reported dearth of research in mathematics education (Melhuish et al., 2022). It is an important area as it offers insights into the craft of teaching mathematics in the first year of the university and enhances knowledge of the opportunities for students to make mathematical meaning. Such opportunities can help alleviate students' difficulties with first-year university mathematics and contribute to reducing students' high dropout rates from universities.

RESEARCH QUESTIONS

The questions I ask in the study are:

What is the nature of lecturer's teaching practice with first year undergraduate mathematics courses at a university in the UK?

What is the relation between lecturer's teaching practice and students' mathematical meaning making?

THEORETICAL UNDERPINNINGS

I look at university mathematics teaching practice through a Vygotskian lens, using the notions of 'tool mediation' and 'dialectic' (Wertsch, 1998) to characterise it. This characterisation is in terms of lecturer's actions with teaching tools, which have a dual material and intellectual nature. For example, the mathematical notation of a definition is presented on the board (materiality) and convey the meaning of the mathematical object (intellect). The lecturer's actions with teaching tools mediate the student's mathematical meaning making and the lecturer's development of teaching and meaning making of the mathematics.

METHODOLOGY

Over three academic semesters at a research university in the UK, I observed and audio-recorded the teaching of twenty-six lecturers, and I discussed with them about their underlying considerations for teaching. The analysis in this study focuses on the characterisation of one (of the twenty-six) lecturer's teaching, which I observed for more than one semester. This lecturer had 20 years of experience with teaching mathematics at the university level. During her classes, the students articulated the mathematics in written form on the board or orally. I took a grounded analytical approach (Glaser, 1998) to observational data (transcripts from audio data of observations). Throughout the analysis, I constantly compared excerpts that were coded with either the same open code or the same theoretical code, the latter being a concept derived from the research literature.

FINDINGS

On the one hand, in my characterisation of the lecturer's teaching, I identified a variety of tools drawn from the context of mathematics, such as heuristics like 'sketch a graph' and 'consider special cases'. On the other hand, I found tools for contextualising the mathematical content for students, for instance, metaphors. In the presentation, I will document and explain these sets of tools, contributing to the research literature first with a way for researcher analysis of university mathematics teaching practice and second with specific teaching tools used for the students' mathematical meaning making.

DISCUSSION

The findings of this research shed light on the relationship between lecturers' teaching practice and students' mathematical meaning making at the university level. As one of the few observational studies of this kind, this research offers a fine-grained level of detail of how the interaction between a lecturer and a group of students makes meaning of the mathematics discussed. The knowledge produced from this study can be used for teaching interventions aimed at students' mathematical meaning making and for the design of professional development workshops for lecturers.

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Propuesta didáctica para la enseñanza de la prueba de hipótesis basada en la resolución de problemas estadísticos

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Palabras clave: resolución de problemas estadísticos, enseñanza de la estadística, prueba de hipótesis

INTRODUCCIÓN

La educación estadística se ha basado en enfoques que promueven el uso de algoritmos y conceptos estadísticos, dejando al estudiantado con conocimientos fragmentados y desarticulados, lo que obstaculiza su comprensión del entorno y su capacidad para participar críticamente en la sociedad (Alpízar, 2007). Chatfield (1995) sugiere que el estudiantado debe emular la labor del y la estadista en su vida diaria, enfrentándose a situaciones que estimulen el pensamiento estadístico, incluyendo la comprensión de conceptos fundamentales, la competencia en el análisis de datos y el razonamiento estadístico (Batanero, 2013). Gal (2002) destaca la cultura estadística como parte del pensamiento estadístico, involucrando habilidades para interpretar y evaluar información, así como para discutir y comunicar sus implicaciones. Sin embargo, su desarrollo se ve limitado por la enseñanza determinista de la estadística. La Guía para la Evaluación e Instrucción Estadística (Bargagliotti, 2020) propone la resolución de problemas estadísticos para promover el ciclo de investigación estadística, destacando la importancia de utilizar datos de situaciones reales para guiar al estudiantado a través de todas las etapas del ciclo.

SECUENCIA DIDÁCTICA

Con base en las consideraciones anteriores, se diseña una secuencia didáctica para promover el aprendizaje de la prueba de hipótesis en cuatro sesiones, con una duración de tres horas cada una. La planificación detallada se puede encontrar en el siguiente enlace: http://bit.ly/414PWty. Los datos utilizados provienen de la Encuesta Nacional del Uso del Tiempo 2019 (ENUT19), realizada cada cuatro años a nivel nacional en México para personas mayores de 12 años.

Con fines educativos, se realiza un tratamiento a la base de datos en RStudio, que se puede consultar en el enlace anterior. Se consideran tres categorías de trabajo de la ENUT19: trabajo remunerado, trabajo dedicado a la producción de bienes exclusivos para uso doméstico y trabajo no remunerado. El rango de edad se restringe de 18 a 26 años, y las horas de trabajo total por semana de 8 a 126.

Las actividades de las primeras tres sesiones utilizan la misma base de datos y siguen la siguiente estructura. Inicialmente, el alumnado responde individualmente a preguntas sin conocimiento previo de la prueba de hipótesis. Posteriormente, las discusiones en equipo conducen a conclusiones provisionales. A continuación, se lleva a cabo una primera plenaria de equipo para reforzar la necesidad de análisis inferenciales y consolidar el conocimiento previo requerido. Luego, la profesora introduce nuevos conceptos relacionados con la prueba de hipótesis, seguidos de su aplicación en equipos. Finalmente, una segunda plenaria ayuda a reforzar conceptos y abordar las dudas de los estudiantes. Durante las sesiones, el estudiantado se apoya del programa StatGraphics para realizar las actividades.

La última sesión se dedica a la evaluación individual final, con el objetivo de evaluar las habilidades estadísticas adquiridas según el marco de Gal (2002) y la aplicación del ciclo de investigación estadística introducido de manera holística en sesiones anteriores. Los datos utilizados para la evaluación se recuperan de la sección de Tiempo Libre de ENUT19 (Instituto Nacional de Estadística y Geografía, 2019).

CONCLUSIONES

El uso de datos reales es deseable y adecuado para el desarrollo de las habilidades del estudiantado propias del sentido estadístico bien encaminado. Además, permite que el alumnado imite el trabajo de un o una estadista, delimitando la pregunta de investigación estadística que desea responder, aplicando su conocimiento previo y relacionándolo con el conocimiento introducido por el profesorado. Así, no solo se establece la relación fundamental entre la estadística descriptiva y la inferencial, sino que se provee a los datos de significado, característica inherente a las problemáticas que rodean al alumnado en su día a día.

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Problemas verbales en los que subyacen los números racionales: procesos de resolución de alumnos universitarios

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Palabras Clave: problemas verbales, números racionales, alumnado universitario, resolución de problemas, procesos de resolución.

INTRODUCCIÓN

La resolución de problemas se considera una tarea matemática clave que, según diversas investigaciones y organismos internacionales, mejora la comprensión y el desarrollo lógico-matemático de las y los estudiantes, teniendo aplicaciones interdisciplinarias. En particular, los problemas verbales son de las primeras actividades que experimenta el estudiantado durante su etapa escolar. Sin embargo, en el proceso de resolución de un problema los discentes enfrentan serias dificultades, más aún, cuando implica el uso de los números racionales (Herreros-Torres et al., 2022). Además, persiste la idea de que esas dificultades perduran en todos los niveles escolares, incluso en el universitario. Por ello, el objetivo de esta investigación es caracterizar el proceso que sigue para resolver problemas con números racionales un grupo de estudiantes que ingresan a ingeniería en el sistema universitario mexicano, a fin de identificar posibles dificultades asociadas a las fases del proceso de resolución de problemas descritas por Puig y Cerdán (1988).

MARCO CONCEPTUAL

Según Verschaffel et al. (2020), un problema verbal se entiende como una descripción narrativa de una situación que involucra datos numéricos, los cuales son sometidos a operaciones matemáticas para derivar respuestas a preguntas. Al proceso de resolución de un problema verbal, Puig y Cerdán (1988) lo entienden como “la actividad mental desplegada por el resolutor desde el momento en que, siéndole presentado un problema, asume que lo que tiene delante es un problema y quiere resolverlo, hasta que da por acabada la tarea” (p. 8). Así, tomando en cuenta ideas de investigadores como Polya, dichos autores propusieron seis fases en el proceso de resolución de un problema, estas son: lectura, comprensión, traducción, cálculo, solución y revisión-comprobación.

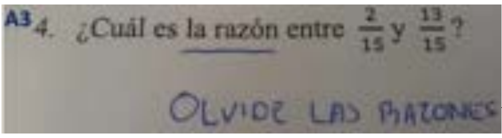
Para el diseño de los problemas verbales que se aplican en esta investigación, se toma como concepto matemático principal a los números racionales. En particular, en México, durante la educación básica y media superior se propone la instrucción de fracciones, porcentajes, números decimales y razones. De acuerdo con Kieren (1976) todos estos son interpretaciones de los números racionales. Por lo anterior, para el diseño de los materiales se toman en cuenta aspectos relacionados con dichas interpretaciones.

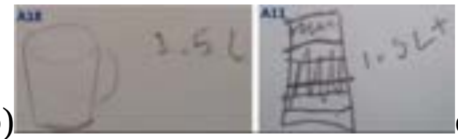
MATERIAL Y GRUPO DE EXPERIMENTACIÓN

La experimentación se hizo en dos momentos. Una vez con un grupo de 18 estudiantes y otra vez con 30, ambos de nuevo ingreso a ingeniería mecánica. Para la recolección de datos se diseñaron dos cuestionarios. En el primero se planteó un problema verbal que implica el uso de la fracción como parte-todo y como operador, así como el uso de porcentajes, notación decimal y razón. El segundo cuestionario evalúa los mismos conceptos, pero desprendidos del contexto del problema verbal.

RESULTADOS Y CONCLUSIÓN

Al comparar los resultados de éxito entre los dos cuestionarios, se observa, en términos generales, mejor rendimiento cuando se resuelven las tareas fuera de un contexto que implique la resolución de problemas verbales; excepto el caso en que se pide "calcula 1/5 de 2/3". Este resultado indica que los estudiantes no asocian la palabra "de" a una multiplicación, sino que usan otras operaciones. Además, en ambos cuestionarios hay evidencia de que los alumnos no recuerdan qué es una razón (Figura 1a), hay una tendencia en confundir la razón con la sustracción. Con respecto a los procesos de resolución de problemas, los alumnos emplearon representaciones gráficas o pictóricas como ayuda en las fases de comprensión y traducción (Figura 1b). Sin embargo, muestran falta de comprensión en expresiones relacionadas con partes de partes. Por ello, en las fases de lectura, comprensión y traducción del problema enfrentan mayor dificultad, aunque en la fase del cálculo también tienen dificultades (Figura 1c).

a) 

b) 

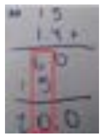
c) 

Figura 1: Respuestas de las y los estudiantes.

Aunque se identificaron procedimientos adecuados asociados a las interpretaciones de los números racionales, es crucial mejorar su enseñanza, especialmente en la comprensión para resolver problemas. De lo contrario, las y los estudiantes podrían enfrentarse a otras dificultades en otras áreas de las matemáticas universitarias.

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Generalizing from examples: an epistemological contribution

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Keywords: teaching and learning of proof, transition to university mathematics, analyzing experts' practices, process of generalization, use of examples.

In mathematics courses, the use of examples is often reduced to illustrating concepts (Bills et al., 2006). Since more than four decades, the international didactic community has investigated the help provided by examples for expressing one's proof (Balacheff, 1988; Zaslavsky, 2018). Considering the difficulties regarding proof and proving experienced by students at the transition to university mathematics (Selden, 2012), such use of examples is promising insofar as it "reduces the level of abstraction and suspends or even eliminates the need to deal with formalism and symbolism" (Zaslavsky, 2018, p. 290). Although flourishing, the research has not stabilized the vocabulary (interchangeable use of the terms *generic example*, *generic proof*, *generic argument* and even *proof by generic example*) and it is often unclear what the authors speak about (Dogan and Williams-Pierce, 2021). The three levels for analyzing the generality and necessity of a proof carried on a generic instance defined in (Trouvé, 2023) help. An acceptation within this polysemy echoes to Steiner's stance that "it is not, then, the *general* proof which explains; it is the *generalizable* proof" (1978, p. 144, emphasis in the original). As part of our more global research on the use of examples for teaching and learning proof, we focus on generalizing from examples in this poster.

To get insight into this use of examples, we choose to work in the first-order predicate calculus to be able to take over the instantiation processes (Barrier, 2016). Given a deductive theory T, the statement $\forall\, y \in Y, Q(y)$ will be said more general than the statement $\forall\, x \in X, P(x)$ if and only if $\{x \in X, P(x)\} \subset \{y \in Y, Q(y)\}$. We describe and illustrate three proof processes for obtaining $\forall\, y \in Y, Q(y)$ by generalization:

1. by using the statement $\forall\, x \in X, P(x)$ in a proof of $\forall\, y \in Y, Q(y)$,
2. by identifying the generic character (Trouvé, 2023) of (part of) a proof of the statement $\forall\, x \in X, P(x)$, and
3. by using (and possibly changing) a proof of $\forall\, x \in X, P(x)$, for elements of Y.

As exploring the experts' practices is important to think the didactical transposition, we wonder if and how mathematicians generalize from examples. To address these research questions, we analyze the answers of 14 mathematicians to a questionnaire designed to tackle more generally the question of genericity. It leads to the following results. The first process is not mentioned in any answer. In contrast, the second and third processes are respectively cited in 4 and 2 answers. The corresponding answers illustrate the complexity of these processes, both in terms of the diversity of their shape and their possible interaction. 4 of the 14 mathematicians speak about generalizing from examples without involving them in a proving process. For 4 others, the description is not precise enough to decide whether the use of examples for generalizing is involved in a proof process or not, and if so, which process it refers to.

This study enhances our understanding of proof processes in mathematics and stresses their value for learning and teaching proof. In a didactic perspective, we wonder to what extend such processes live in the curriculum and if teachers recognize them as goals of the mathematics class. Besides, we question the possibility to devolve situations involving such processes. If so, what are their specificities? How can (or should) they be implemented? These issues will be discussed during the poster session.

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